Functional Analysis and Operator Theory

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Contents

Chapter 1

Spectral Theory of C^*-algebras

The general theme of this chapter is to study families of Hilbert space operators and spectral theory from a more abstract, algebraic point of view. This approach was pioneered by the school of Gelfand and by von Neumann. Today, this theory has grown into its own subfield of mathematics (math.OA Operator Algebras on arxiv) with rich applications both in mathematics, for example in theory of infinite groups, ergodic theory and random matrix theory, and in physics, for example in quantum statistical mechanics, algebraic quantum field theory or quantum information theory.

The goals for this lecture are much more modest. We will develop the basic theory of so-called commutative C^*-algebras culminating in their characterization as algebras of continuous functions by Gelfand's representation theorem. One central applications is an algebraic recasting of the continuous functional calculus for normal operators on Hilbert space.

1.1 Bounded operators on Hilbert space

In this section we recap some basic properties of bounded operators on Hilbert space with a focus on the interaction between the norm and the operation of taking adjoints. We will see these features reflected in properties of C^*-algebras later.

Several results in spectral theory rely crucially on complex analysis. For this reason, all vector spaces in are assumed to be complex.

Definition 1.1.1 (Bounded operator, operator norm)**.** Let H, K be Hilbert spaces. A linear map $x\colon H \to K$ is called *bounded* if there exists $C > 0$ such that $\|x\xi\| \leq C\|\xi\|$ for all $\xi \in H$. The set of all bounded linear operators from H to K is denoted by $\mathbb{B}(H; K)$. We write $\mathbb{B}(H)$ for $\mathbb{B}(H; H)$.

The *operator norm* of $x \in \mathbb{B}(H;K)$ is defined as

$$\|x\| = \sup_{\|\xi\| \leq 1} \|x\xi\|.$$

Remark 1.1.2. The bounded linear operators between a given pair of Hilbert spaces form a Banach space (with the operator norm).

Let us record some basic properties of the operator norm. Recall that if $x \in \mathbb{B}(H)$, the adjoint x^* is the unique operator in $\mathbb{B}(H)$ that satisfies $\langle \xi, x\eta \rangle = \langle x^*\xi, \eta \rangle$ for all $\xi, \eta \in H$.

Lemma 1.1.3. *Let H be a Hilbert space. The operator norm on $\mathbb{B}(H)$ has the following properties.*

(a) $\|xy\| \leq \|x\|\|y\|$ *for all* $x, y \in \mathbb{B}(H)$.

(b) $\|x^*\| = \|x\|$ *for all* $x \in \mathbb{B}(H)$.

(c) $\|x^*x\| = \|x\|^2$ *for all* $x \in \mathbb{B}(H)$,

Proof. (a) If $\|\xi\| \leq 1$ and $y\xi \neq 0$, then

$$\|xy\xi\| = \left\| x \frac{y\xi}{\|y\xi\|} \right\| \|y\xi\| \leq \|x\|\|y\|.$$

If $y\xi = 0$, the inequality $\|xy\xi\| \leq \|x\|\|y\|$ holds trivially. Taking the supremum over all $\xi \in H$ with $\|\xi\| \leq 1$ yields the claimed inequality.

(b) First note that

$$\|x\|^2 = \sup_{\|\xi\| \leq 1} \|x\xi\|^2 = \sup_{\|\xi\| \leq 1} \langle \xi, x^*x\xi \rangle \leq \|x^*x\|.$$

By (a), we have $\|x^*x\| \leq \|x^*\|\|x\|$. Together with the previous inequality, this implies $\|x\| \leq \|x^*\|$. The reverse inequality follows by exchanging the roles of x and x^*.

(c) In (b) we have already seen that $\|x\|^2 \leq \|x^*x\|$. If we combine this with (a) and apply (b) again, we obtain $\|x\|^2 \leq \|x^*x\| \leq \|x^*\|\|x\| = \|x\|^2$. Thus $\|x\|^2 = \|x^*x\|$. □

Definition 1.1.4. Let H be a Hilbert space. An operator $x \in \mathbb{B}(H)$ is called

- *normal* if $x^*x = xx^*$,
- *self-adjoint* or *symmetric* if $x = x^*$,
- *positive* if there exists $y \in \mathbb{B}(H)$ such that $x = y^*y$,

- a *projection* if $x^* = x^2 = x$,
- *unitary* if $x^*x = xx^* = 1$,
- an *isometry* if $x^*x = 1$,
- a *partial isometry* if x^*x is a projection.

Lemma 1.1.5. *If H is a Hilbert space and $x \in \mathbb{B}(H)$, then* $\ker(x) = \operatorname{ran}(x^*)^\perp$ *and* $\overline{\operatorname{ran} x} = \ker(x^*)^\perp$.

Proof. If $\xi \in \ker x$ and $\eta \in H$, then $\langle \xi, x^*\eta \rangle = 0$, hence $\xi \in \operatorname{ran}(x^*)^\perp$. If $\xi \in \operatorname{ran}(x^*)^\perp$, then

$$\|x\xi\|^2 = \langle \xi, x^*x\xi \rangle = 0,$$

hence $\xi \in \ker x$. The second identity follows by taking the orthogonal complement on both sides of the first identity. □

Proposition 1.1.6. *Let H be a Hilbert space. An operator $x \in \mathbb{B}(H)$ is*

(a) normal if and only if $\|x\xi\| = \|x^*\xi\|$ for all $\xi \in H$,

(b) self-adjoint if and only if $\langle \xi, x\xi \rangle \in \mathbb{R}$ for all $\xi \in H$,

(c) positive if and only if $\langle \xi, x\xi \rangle \geq 0$ for all $\xi \in H$,

(d) a projection if and only if x is the orthogonal projection onto $\operatorname{ran}(x)$,

(e) an isometry if and only if $\|x\xi\| = \|\xi\|$ for all $\xi \in H$,

(f) unitary if and only if it is a surjective isometry,

(g) a partial isometry if and only if it restricts to an isometry from $\ker(x)^\perp$ *to* $\operatorname{ran}(x)$.

Proof. We will use at several places that if $x \in \mathbb{B}(H)$ such that $\langle \xi, x\xi \rangle = 0$ for all $\xi \in H$, then $x = 0$ (exercise).

(a) If x is normal, then $\|x\xi\|^2 = \langle \xi, x^*x\xi \rangle = \langle \xi, xx^*\xi \rangle = \|x^*\xi\|^2$ for all $\xi \in H$. Conversely, $\langle \xi, (x^*x - xx^*)\xi \rangle = \|x\xi\|^2 - \|x^*\xi\|^2 = 0$ for all $\xi \in H$, which implies $x^*x - xx^* = 0$.

(b) If x is self-adjoint, then $\overline{\langle \xi, x\xi \rangle} = \langle x\xi, \xi \rangle = \langle \xi, x\xi \rangle$ for all $\xi \in H$. Conversely, $\langle \xi, (x - x^*)\xi \rangle = \langle \xi, x\xi \rangle - \overline{\langle \xi, x\xi \rangle} = 0$ for all $\xi \in H$, hence $x = x^*$.

(c) If $x = y^*y$, then $\langle \xi, x\xi \rangle = \|y\xi\|^2 \geq 0$ for all $\xi \in H$. Conversely, x is self-adjoint by (b) and $\sigma(x) \subset [0, \infty)$ by the spectral theorem. Thus $x = (x^{1/2})^2$.

(d) We showed that in Mathematical Physics II.

(e) If x is an isometry, then $\|x\xi\|^2 = \langle \xi, x^*x\xi\rangle = \|\xi\|^2$ for all $\xi \in H$. Conversely, $\langle \xi, (x^*x - 1)\xi\rangle = \|x\xi\|^2 - \|\xi\|^2 = 0$ for all $\xi \in H$, hence $x^*x = 1$.

(f) If x is unitary, then x is an invertible isometry with $x^{-1} = x^*$. Thus x is surjective. Conversely, if x is a surjective isometry, then it is invertible and thus $x^{-1} = x^*xx^{-1} = x^*$, which implies $xx^* = 1$.

(g) Let x be a partial isometry. First note that $\ker x \subset \ker(x^*x)$. Conversely, if $\xi \in \ker(x^*x)$, then $\|x\xi\|^2 = \langle \xi, x^*\xi\rangle = 0$, hence $\xi \in \ker(x)$. Thus x^*x is the orthogonal projection onto $\ker(x^*x)^\perp = (\ker x)^\perp$. If $\xi \in (\ker x)^\perp$, then $\|x\xi\|^2 = \langle \xi, x^*x\xi\rangle = \|\xi\|^2$. Hence x is an isometry from $(\ker x)^\perp$ onto $\operatorname{ran} x$.

Conversely, if x is an isometry from $(\ker x)^\perp$ onto $\operatorname{ran} x$ and p the orthogonal projection onto $(\ker x)^\perp$, then

$$\langle \xi, (x^*x)^2\xi\rangle = \langle p\xi, (x^*x)^2 p\xi\rangle = \langle xp\xi, xx^*xp\xi\rangle = \langle p\xi, x^*xp\xi\rangle = \langle \xi, x^*x\xi\rangle$$

for all $\xi \in H$. Thus $(x^*x)^2 = x^*x$. □

Lemma 1.1.7 (Positive square root)**.** *Let H be Hilbert space. For every positive operator $x \in \mathbb{B}(H)$ there exists a unique positive operator $y \in \mathbb{B}(H)$ such that $x = y^2$.*

Proof. Existence: Let $f(\lambda) = \sqrt{\lambda}$ for $\lambda \geq 0$. By the spectral theorem, $f(x)$ is positive and $f(x)^2 = x$.

Uniqueness: Let $y \in \mathbb{B}(H)$ be a positive operator such that $x = y^2$. By the spectral theorem in multiplication operator form, there exists a localizable measure space $(X, \mathcal{A}, \mu)$, a measurable function $\varphi\colon \mathbb{R} \to \mathbb{R}$ and a unitary operator $u\colon L^2(X,\mu) \to H$ such that $y = uM_\varphi u^*$ and $f(y) = uM_{f\circ\varphi}u^*$ for every bounded Borel function $f\colon \sigma(y) \to \mathbb{C}$. Since $y \geq 0$, we have $\varphi \geq 0$ μ-a.e. Thus $y^2 = uM_{\varphi^2}u^*$ and

$$x^{1/2} = (y^2)^{1/2} = uM_{(\varphi^2)^{1/2}}u^* = uM_\varphi u^* = y.$$ □

Proposition 1.1.8 (Polar decomposition)**.** *Let H be a Hilbert space. If $x \in \mathbb{B}(H)$, then there exists a unique pair (v, y) consisting of a partial isometry $v \in \mathbb{B}(H)$ and a positive operator $y \in \mathbb{B}(H)$ such that $x = vy$ and $\ker v = \ker x$.*

Proof. Existence: Let $y = (x^*x)^{1/2}$. By the spectral theorem, $\ker((x^*x)^{1/2}) = \ker(x^*x)$ and clearly $\ker x \subset \ker(x^*x)$. On the other hand, if $\xi \in \ker(x^*x)$, then $\|x\xi\|^2 = \langle \xi, x^*x\xi\rangle = 0$, hence $\xi \in \ker(x^*x)$. Thus $\ker y = \ker x$.

We define

$$v\colon \operatorname{ran} y \to H,\ v(y\xi) = x\xi.$$

Since

$$\|x\xi\|^2 = \langle \xi, x^*x\xi \rangle = \|(x^*x)^{1/2}\xi\|^2 = \|y\xi\|^2,$$

the operator v is well-defined and extends to an isometry from $\overline{\operatorname{ran} y}$ to H. We can extend v to a partial isometry on H by setting $v = 0$ on $(\operatorname{ran} y)^\perp = \ker x$. With this definition, $\ker v = \ker x$ and $x = vy$.

Uniqueness: Let (v', y') be a pair of bounded operators that satisfies the conditions of the proposition. We have $y^2 = x^*x = (y')^2$, hence $y = y'$ by the uniqueness of the square root. Therefore $v = v'$ on $\operatorname{ran} y$, which is dense in $\ker(v)^\perp = \ker(v')^\perp$. Since both operators are continuous, we conclude $v = v'$. □

Remark 1.1.9. The decomposition $x = vy$ from the previous proposition is called the *polar decomposition* of x. As the proof shows, the positive operator y is given by $(x^*x)^{1/2}$. This operator is denoted by $|x|$. This is consistent with functional calculus for self-adjoint operators (see the exercise).

Exercises

1. Let $x \in \mathbb{B}(H)$ such that $\langle \xi, x\xi \rangle = 0$ for all $\xi \in H$. Show that $x = 0$.

2. Let $x \in \mathbb{B}(H)$ be self-adjoint and

$$f\colon \mathbb{R} \to \mathbb{R},\ \lambda \mapsto \begin{cases} 1 & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \\ -1 & \text{if } \lambda < 0 \end{cases}$$
$$g\colon \mathbb{R} \to \mathbb{R},\ \lambda \mapsto |\lambda|.$$

Show that $x = f(x)g(x)$ is the polar decomposition of x. In particular, $(x^*x)^{1/2} = |x|$ in the sense of functional calculus.

1.2 Banach algebras and C^*-algebras

In the next sections, we will take a more abstract look at spectral theory. Recall that if $x \in \mathbb{B}(H)$, the resolvent set $\rho(x)$ is defined as

$$\rho(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \text{ invertible with bounded inverse}\}.$$

In operator theory, one usually checks this condition by showing that $x - \lambda$ is injective (i.e. λ is not an eigenvalue of x) and that $x - \lambda$ is surjective. Boundedness of the inverse is then a consequence of the closed graph theorem.

However, one can also approach the resolvent set more algebraically. A number $\lambda \in \mathbb{C}$ belongs to the resolvent set if and only if there exists $y \in \mathbb{B}(H)$ such that $(x - \lambda)y = y(x - \lambda) = 1$. This formulation only uses the basic algebraic operations in $\mathbb{B}(H)$ (addition, composition of operators and multiplication with scalars) and not the fact that elements of $\mathbb{B}(H)$ are linear maps on a Hilbert space. Thus spectral theory can be studied in the more general context when only these algebraic properties are given. This motivates the following definition.

Definition 1.2.1 (Algebra, invertible elements, spectrum)**.** An *algebra* is a complex vector space A together with a bilinear map $A \times A \to A$, $(a, b) \mapsto ab$, called the multiplication. The algebra A is called *unital* if there exists an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.

If A is a unital algebra, An element $a \in A$ is called *invertible* if there exists $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$.

The *spectrum* of an element $a \in A$ is defined as

$$\sigma_A(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ not invertible}\}.$$

Remark 1.2.2. If the algebra A is unital, then the unit 1 is unique. Likewise, if an element a of a unital algebra is invertible, the inverse a^{-1} is unique.

Example 1.2.3. If H is a Hilbert space, then $\mathbb{B}(H)$ with the usual vector space structure and the multiplication given by operator composition is a unital algebra. The unit is the idenity operator. An operator $x \in \mathbb{B}(H)$ is invertible if and only if it is bijective and the spectrum $\sigma_{\mathbb{B}(H)}(x)$ coincides with the usual spectrum of an operator on a Hilbert space.

Example 1.2.4. If X is a compact Hausdorff space, then $C(X)$ with the usual vector space structure and the multiplication given by pointwise multiplication of functions is a unital algebra. The unit is the constant function 1. A function $f \in C(X)$ is invertible if and only if it has non zeros and the spectrum $\sigma_{C(X)}(f)$ equals $\operatorname{im} f$.

Proposition 1.2.5. *If A is a unital algebra and $a, b \in A$, then $\sigma_A(ab) \cup \{0\} = \sigma_A(ba) \cup \{0\}$.*

Proof. If $\lambda \in \mathbb{C} \setminus (\sigma_A(ab) \cup \{0\})$, let $c = \lambda^{-1}(1 + b(\lambda - ab)^{-1}a)$. We have

$$\begin{aligned}(\lambda - ba)c &= (1 - \lambda^{-1}ba)(1 + b(\lambda - ab)^{-1}a) \\ &= 1 - \lambda^{-1}ba + \lambda^{-1}b(\lambda - ab)(\lambda - ab)^{-1}a \\ &= 1 - \lambda^{-1}ba + \lambda^{-1}ba \\ &= 1.\end{aligned}$$

A similar calculation shows $c(\lambda - ba) = 1$. Thus $\lambda \in \mathbb{C} \setminus (\sigma_A(ba) \cup \{0\})$. □

Remark 1.2.6. The proof of the previous lemma shows $(\lambda - ba)^{-1} = \lambda^{-1}(1 + b(\lambda - ab)^{-1}a)$. It seems like we pulled this formula out of blue air. Formally, it can be justified as follows:

$$\begin{aligned}
(\lambda - ba)^{-1} &= \lambda^{-1}(1 - \lambda^{-1}ba)^{-1} \\
&= \lambda^{-1}\sum_{n=0}^{\infty} \lambda^{-k}(ba)^k \\
&= \lambda^{-1}(1 + \lambda^{-1}b\sum_{k=0}^{\infty} \lambda^{-k}(ab)^k a) \\
&= \lambda^{-1}(1 + \lambda^{-1}b(1 - \lambda^{-1}(ab))^{-1}a) \\
&= \lambda^{-1}(1 + b(\lambda - ab)^{-1}a).
\end{aligned}$$

Note however that in this abstract algebraic setting, we do not even have a notion of convergence so that these manipulations of infinite series are not rigorous.

To be able to speak of convergence etc., that is, to actually do analysis, we need additional structure. A rich class of algebras with a topology is provided by the following definition.

Definition 1.2.7 (Banach algebra)**.** A *Banach algebra* is an algebra A with a norm $\|\cdot\|$ that satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$ and such that A is complete in this norm.

Remark 1.2.8. The submultiplicativity $\|ab\| \leq \|a\|\|b\|$ guarantees that the multiplications is a continuous bilinear map from $A \times A$ to A. Moreover, any norm on an algebra that makes the multiplication continuous can be replaced by an equivalent submultiplicative norm.

Definition 1.2.9 ($*$-algebra, C^*-norm, C^*-algebra)**.** A $*$*-algebra* is an algebra together with a map $A \to A$, $a \mapsto a^*$ with the following properties:

- $(\lambda a + \mu b)^* = \overline{\lambda}a^* + \overline{\mu}b^*$ for all $\lambda, \mu \in \mathbb{C}$, $a, b \in A$,
- $(ab)^* = b^*a^*$ for all $a, b \in A$,
- $(a^*)^* = a$ for all $a \in A$.

A norm $\|\cdot\|$ on a $*$-algebra A is called a C^**-norm* if $\|ab\| \leq \|a\|\|b\|$ and $\|a^*a\| = \|a\|^2$ for all $a, b \in A$. An algebra with a complete C^* norm is called a C^**-algebra*.

Remark 1.2.10. Clearly, every C^*-algebra is a Banach algebra. The converse is far from being true. Not only do we need an additional structure (the involution $a \mapsto a^*$) in the definition of a C^*-algebra, there are also many examples of Banach algebras with a natural involution that do not satisfy the C^* identity for the norm.

Example 1.2.11. The complex-valued polynomials in one variable form a unital algebra $\mathbb{C}[X]$ with involution given by $(\sum_k \alpha_k X^k)^* = \sum_k \overline{\alpha_k} X^k$. There are many C^*-norms on $\mathbb{C}[X]$, but no norm (whether a C^*-norm or not) that makes $\mathbb{C}[X]$ a Banach space.

Example 1.2.12. Let G be a group. Let $\mathbb{C}[G]$ be the vector space with basis G, that is, $\mathbb{C}[G]$ consists of all formal linear combinations $\sum_{g \in G} \alpha_g g$ with finitely many non-zero coefficients α_g. One can define a multiplication on $\mathbb{C}[G]$ as a bilinear extension of the multiplication of G, that is, $(\sum_g \alpha_g g)(\sum_h \beta_h h) = \sum_{\mathfrak{g},h} \alpha_g \beta_h gh$. The algebra $\mathbb{C}[G]$ is called the (complex) *group algebra.*

Moreover, there is an involution on $\mathbb{C}[G]$ defined by

$$\left(\sum_g \alpha_g g \right)^* = \sum_g \overline{\alpha_g} g^{-1}.$$

With this involution, the group algebra becomes a $*$-algebra. The expression

$$\left\| \sum_g \alpha_g g \right\|_{\mathrm{u}} = \sup \left\{ \left\| \sum_g \alpha_g \pi(g) \right\| : \pi \colon G \to \mathbb{U}(H) \text{ group hom.} \right\}$$

defines a C^*-norm on $\mathbb{C}[G]$. Here, $\mathbb{U}(H)$ denotes the group of unitary operators on H. However, this norm is not complete unless G is finite.

Example 1.2.13. If H is a Hilbert space, then $\mathbb{B}(H)$ with the operation of taking adjoints and the operator norm is a C^*-algebra.

Example 1.2.14. If X is a compact Hausdorff space, then $C(X)$ with the complex conjugation as $*$-operation and the supremum norm is a C^*-algebra.

One crucial difference between the last two examples is that while multiplication in $C(X)$ is commutative, operator multiplication in $\mathbb{B}(H)$ is not (unless $\dim H \leq 1$). The goal of this chapter is to show that every commutative unital C^*-algebra is of the form $C(X)$ for some compact Hausdorff space X.

Lemma 1.2.15. *If A is a unital Banach algebra and $a \in A$ with $\|1-a\| < 1$, then a is invertible with $\|a^{-1}\| \leq (1 - \|1-a\|)^{-1}$.*

Proof. Since $\|1-a\| < 1$ and A is complete, the series $\sum_{k=0}^{\infty}(1-a)^k$ converges and the limit has norm bounded above by $(1-\|1-a\|)^{-1}$ (see the exercises). A telescope sum trick shows $a\sum_{k=0}^{\infty}(1-a)^k = \sum_{k=0}^{\infty}(1-a)^k a = 1$. □

Proposition 1.2.16. *If A is a unital Banach algebra and* $\mathrm{Inv}(A)$ *denotes the set of invertible elements of A, then* $\mathrm{Inv}(A)$ *is open in A and $a \mapsto a^{-1}$ is continuous on* $\mathrm{Inv}(A)$.

Proof. If $a \in \mathrm{Inv}(A)$ and $b \in A$ with $\|a-b\| < \|a^{-1}\|^{-1}$, then $\|1-a^{-1}b\| \leq \|a^{-1}\|\|a-b\| < 1$. By the previous lemma, $a^{-1}b$ is invertible and

$$\|(a^{-1}b)^{-1}\| \leq (1-\|a^{-1}\|\|a-b\|)^{-1}.$$

In particular, b is invertible with inverse $b^{-1} = (a^{-1}b)^{-1}a^{-1}$. Therefore, $\mathrm{Inv}(A)$ is open.

Moreover,

$$\begin{aligned} \|a^{-1}-b^{-1}\| &= \|a^{-1}(b-a)b^{-1}\| \\ &\leq \|a^{-1}\|\|b-a\| \underbrace{\|b^{-1}\|}_{\|(a^{-1}b)^{-1}a^{-1}\|} \\ &\leq \|a^{-1}\|^2\|b-a\|\|a^{-1}b\| \\ &\leq \frac{\|a^{-1}\|^2\|b-a\|}{1-\|a^{-1}\|\|b-a\|}. \end{aligned}$$

Thus $b^{-1} \to a^{-1}$ as $b \to a$. □

Proposition 1.2.17. *If A is a non-zero unital Banach algebra and $a \in A$, then $\sigma_A(a)$ is compact, non-empty and contained in $\bar{B}_{\|a\|}(0)$, and*

$$R\colon \mathbb{C}\setminus\sigma_A(a) \to A,\ z \mapsto (z-a)^{-1}$$

is (complex) differentiable.

Proof. If $\lambda \in \mathbb{C}$ with $|\lambda| > \|a\|$, then $\lambda - a = \lambda(1-\lambda^{-1}a)$ is invertible by a previous lemma. Thus $\sigma_A(a) \subset \bar{B}_{\|a\|}(0)$. As the map $\Psi\colon \mathbb{C} \to A$, $\lambda \mapsto a-\lambda$ is continuous and $\mathbb{C}\setminus\sigma_A(a) = \Psi^{-1}(\mathrm{Inv}(A))$, we see that $\sigma_A(a)$ is closed. Therefore $\sigma_A(a)$ is compact.

If $z, w \in \mathbb{C}\setminus\sigma_A(a)$, then

$$\begin{aligned} (a-z)^{-1}-(a-w)^{-1} &= (a-z)^{-1}((a-w)-(a-z))(a-w)^{-1} \\ &= (a-z)^{-1}(z-w)(a-w)^{-1}. \end{aligned}$$

As inversion is continuous, we conclude

$$\lim_{w \to z} \left\| \frac{R(z) - R(w) - (a-z)^{-2}(z-w)}{z-w} \right\| = 0.$$

Hence R is differentiable with $R'(z) = (a-z)^{-2}$.

Suppose that $\sigma_A(a) = \emptyset$ and let $\varphi \in A^*$. If $|z| > \|a\|$, then

$$|\varphi(R(z))| \leq \|\varphi\| \|R(z)\| \leq \|\varphi\| |z|^{-1}(1 - |z|^{-1}\|a\|)^{-1}.$$

Hence $\varphi \circ R$ is a bounded complex differentiable function on $\mathbb{C}$ such that $\lim_{|z| \to \infty} |\varphi(R(z))| = 0$. By Liouville's theorem, $\varphi \circ R = 0$. Since $\varphi \in A^*$ is arbitrary, the Hahn–Banach theorem implies $R = 0$, which is impossible. Thus $\sigma_A(a)$ must be non-empty. □

Remark 1.2.18. Liouville's theorem is one of the results that show that complex differentiable functions behave very differently from real differentiable functions. It states the following: If $f: \mathbb{C} \to \mathbb{C}$ is bounded and complex differentiable, then f is constant.

Theorem 1.2.19 (Gelfand–Mazur)**.** *If A is a non-zero unital Banach algebra in which every non-zero element is invertible, then $A = \mathbb{C}1$.*

Proof. If $a \in A$, then $\sigma_A(a) \neq \emptyset$. Take $z \in \sigma_A(a)$. Since every non-zero element of A is invertible, we conclude $a - z1 = 0$, hence $a = z1$. □

Example 1.2.20. There are unital algebras in which every non-zero element is invertible and which are not isomorphic to $\mathbb{C}$. For example, let $\mathbb{C}(X) = \{P/Q \mid P, Q \in \mathbb{C}[X],\, Q \neq 0\}$. If $P \neq 0$, then P/Q is invertible with inverse $(P/Q)^{-1} = Q/P$. In particular, there is no norm on $\mathbb{C}(X)$ that makes it into a Banach algebra.

Proposition 1.2.21 (Spectral mapping theorem for polynomials)**.** *If A is a unital Banach algebra, $a \in A$ and p a complex polynomial, then $\sigma_A(p(a)) = p(\sigma_A(a))$.*

Proof. The case of a constant polynomial is easy, hence we assume that p is non-constant. For $\lambda \in \mathbb{C}$ there exist $\alpha \neq 0$ and $\mu_1, \ldots, \mu_n \in \mathbb{C}$ such that $p(X) - \lambda = \alpha \prod_{k=1}^n (X - \mu_k)$. Moreover, $p^{-1}(\lambda) = \{\mu_1, \ldots, \mu_k\}$.

We have $\lambda \in \sigma_A(p(a))$ if and only if $p(a) - \lambda$ is not invertible if and only if $a - \mu_k$ is not invertible for some $k \in \{1, \ldots, n\}$ (see the exercises). This in turn is equivalent to $p^{-1}(\lambda) \cap \sigma_A(a) \neq 0$, that is, $\lambda \in p(\sigma_A(a))$. □

Definition 1.2.22 (Spectral radius)**.** If A is a unital algebra and $a \in A$, then the *spectral radius* of a is defined as $r(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$.

Example 1.2.23. If X is a compact Hausdorff space and $f \in C(X)$, then $\sigma_{C(X)}(f) = \operatorname{im} f$ and thus $r(f) = \sup\{|f(x)| : x \in X\} = \|f\|_\infty$.

If A is a non-zero unital Banach algebra, then by the previous results, $r(a) \leq \|a\|$ for every $a \in A$ and the supremum in the definition of $r(a)$ is attained. Moreover, $\sigma_A(ab) \cup \{0\} = \sigma_A(ba) \cup \{0\}$ implies that $r(ab) = r(ba)$. Note that the definition of the spectral radius only uses the algebraic structure of A. For Banach algebras, there is an equivalent characterization in terms of the norms, as we will see next.

Proposition 1.2.24 (Spectral radius formula)**.** *If A is a unital Banach algebra and $a \in A$, then $\|a^n\|^{1/n}$ converges to $r(a)$ as $n \to \infty$.*

Proof. By the spectral mapping theorem, $r(a)^n = r(a^n) \leq \|a^n\|$. Thus $r(a) \leq \liminf_{n\to\infty}\|a^n\|^{1/n}$.

To show $\limsup_{n\to\infty}\|a^n\|^{1/n} \leq r(a)$, let $\Omega = \{z \in \mathbb{C} : |z| > r(a)\}$ and fix $\varphi \in A^*$. As seen previously, the function

$$f\colon \Omega \to \mathbb{C},\ z \mapsto \varphi((a-z)^{-1})$$

is complex differentiable. Thus it has a Laurent series expansion

$$f(z) = \sum_{k=0}^{\infty} \frac{\alpha_k}{z^k}, \qquad z \in \Omega.$$

On the other hand, we know that if $|z| > \|a\|$, then

$$f(z) = z^{-1}\varphi((z^{-1}a - 1)^{-1}) = \sum_{k=0}^{\infty}(-1)^k \frac{\varphi(a^k)}{z^{k+1}}.$$

By the uniqueness of Laurent series expansions, we conclude that $\alpha_0 = 0$ and $\alpha_k = (-1)^{k-1}\varphi(a^{k-1})$ for $k \geq 1$.

Since the Laurent series expansion converges for $|z| > r(a)$, we have $\lim_{k\to\infty} \frac{\varphi(a^k)}{|z|^{k+1}} = 0$ for all $\varphi \in A^*$ and $|z| > r(a)$. Let $T_k\colon A^* \to \mathbb{C}$, $\varphi \mapsto \frac{\varphi(a^k)}{|z|^{k+1}}$. By the Hahn–Banach theorem, $\|T_k\| = \frac{\|a^k\|}{|z|^{k+1}}$. Moreover, by the uniform boundedness principle, there exists $C > 0$ such that $\frac{\|a^k\|}{|z|^{k+1}} \leq C$. Thus

$$\limsup_{k\to\infty}\|a^k\|^{1/k} \leq \limsup_{k\to\infty} C^{1/k}|z|^{1+1/k} = |z|.$$

Taking the infimum over $|z| > r(a)$, we conclude $r(a) \geq \limsup_{k\to\infty}\|a^k\|^{1/k}$. $\square$

Remark 1.2.25. The Laurent series expansion is another result from complex analysis. One version sufficient for our purposes states that for every bounded complex differentiable function $f\colon \mathbb{C} \setminus \bar{B}_R(0) \to \mathbb{C}$ there exists a unique sequence (α_k) in $\mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} \frac{\alpha_k}{z^k}$$

for all $z \in \mathbb{C} \setminus \bar{B}_R(0)$, where the series on the right side converges uniformly.

Exercises

1. Show that there exists no norm that makes $\mathbb{C}[X]$ into a Banach space.
2. Let A be a unital $*$-algebra.
   (a) Show that $1^* = 1$.
   (b) Show that if $a \in A$ is invertible, then a^* is invertible and $(a^*)^{-1} = (a^{-1})^*$.
3. (a) Let E be a Banach space and (x_n) a sequence in E. Show that if $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\lim_{N\to\infty} \sum_{k=0}^{N} x_k$ exists.
   (b) Let A be a Banach algebra and $a \in A$ with $\|a\| < 1$. Show that $\lim_{N\to\infty} \sum_{n=0}^{N} a^n$ exists and has norm bounded above by $(1 - \|a\|)^{-1}$.
4. Let A be a unital $*$-algebra and let $a_1, \ldots, a_n$ be commuting elements of A. Show that $a_1 \ldots a_n$ is invertible if and only if $a_1, \ldots, a_n$ are all invertible.
5. Let A be a unital C^*-algebra. Show that if $a, b \in A$ commute, then $r(ab) \leq r(a)r(b)$.

1.3 The Gelfand transform

Definition 1.3.1 ($*$-homomorphism, character, spectrum)**.** Let A, B be algebras. An *algebra homomorphism* from A to B is a linear map $\varphi\colon A \to B$ such that $\varphi(ab) = \varphi(a)\varphi(b)$for all $a, b \in A$. If A and B are unital, then φ is called *unital* if $\varphi(1) = 1$. If A and B are $*$-algebra, then $\varphi\colon A \to B$ is called a *$*$-homomorphism* if it is an algebra homomorphism and satisfies $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

A non-zero algebra homomorphism from A to $\mathbb{C}$ is called a *character*. If A is a commutative Banach algebra, the set of all characters of A is called the *spectrum* of A and denoted by $\Gamma(A)$.

Remark 1.3.2. If A is a unital commutative Banach algebra and $\varphi\colon A \to \mathbb{C}$ is a character, then $\varphi(1)^2 = \varphi(1^2) = \varphi(1)$, which implies $\varphi(1) \in \{0, 1\}$. If $\varphi(1) = 0$, then $\varphi(a) = \varphi(a)\varphi(1) = 0$ for all $a \in A$, which contradicts the assumption that φ is non-zero. Thus every character on a unital commutative Banach algebra is necessarily unital.

Example 1.3.3. If X and Y are compact Hausdorff spaces and $\varphi\colon X \to Y$ is continuous, then

$$\varphi^*\colon C(Y) \to C(X),\ f \mapsto f \circ \varphi$$

is a unital $*$-homomorphism. In particular, for every $x \in X$ we get a character $\delta_x\colon C(X) \to \mathbb{C}$, $f \mapsto f(x)$ from the continuous map $\varphi\colon \{*\} \to X$, $* \mapsto x$.

Example 1.3.4. Let H be a Hilbert space and $x \in \mathbb{B}(H)$ self-adjoint. Functional calculus

$$C(\sigma(x)) \to \mathbb{B}(H),\ f \mapsto f(x)$$

is a unital $*$-homomorphism. We will see soon that functional calculus exists in the context of abstract C^*-algebras, not only for $\mathbb{B}(H)$.

Definition 1.3.5 (Ideal, maximal ideal)**.** Let A be an algebra. A linear subspace I of A is called a (two-sided) *ideal* of A, denoted by $I \trianglelefteq A$, if $abc \in I$ whenever $a, c \in A$ and $b \in I$. An ideal I of A is called *proper ideal* if $I \neq A$ and *maximal ideal* if it is a proper ideal and for every proper ideal $J \trianglelefteq A$ such that $I \subset J$ one has $I = J$.

Example 1.3.6. If X is a compact Hausdorff space and $Y \subset X$ a closed subset, then $I = \{f \in C(X) \mid f|_Y = 0\}$ is an ideal of $C(X)$. This ideal is maximal if and only if Y is a singleton.

Example 1.3.7. If H is a finite-dimensional Hilbert space, then the only ideals of $\mathbb{B}(H)$ are the trivial ones: $\{0\}$ and $\mathbb{B}(H)$. If H is an infinite-dimensional separable Hilbert space, then $\mathbb{B}(H)$ has a unique non-trivial closed ideal, called the ideal of compact operators. There are many non-trivial ideals of $\mathbb{B}(H)$ that are not closed. We will study such objects in more detail in the second part of this course.

Remark 1.3.8. If A is unital and $I \trianglelefteq A$ contains an invertible element a, then $1 = aa^{-1} \in I$ and thus $b = b1 \in I$ for every $b \in A$. Hence $I = A$.

If A is an algebra and $I \trianglelefteq A$, the quotient space A/I has not only the structure of a vector space, but is again an algebra with the multiplication $(a + I)(b + I) = ab + I$. If A is unital, then A/I is again unital with unit $1 + I$

Lemma 1.3.9. *Let A be a unital Banach algebra. If $I \trianglelefteq A$ is closed, then*

$$A/I \to [0,\infty),\ a + I \mapsto \inf_{b \in I} \|a - b\|$$

is a norm that makes A/I into a Banach algebra and the quotient map

$$q\colon A \to A/I,\ a \mapsto a + I$$

is a contractive unital algebra homomorphism.

Proof. It is easy to see that the map is a norm and $\|a + I\| \le \|a\|$ for $a \in A$. To see that the norm is submultiplicative, let $a_1, a_2 \in A$. We have

$$\begin{aligned}
\inf_{b \in I} \|a_1 a_2 - b\| &\le \inf_{b_1, b_2 \in I} \|a_1 a_2 - \underbrace{(a_1 b_2 + a_2 b_1 - b_1 b_2)}_{I}\| \\
&= \inf_{b_1, b_2 \in I} \|(a_1 - b_1)(a_2 - b_2)\| \\
&\le \inf_{b_1, b_2 \in I} \|a_1 - b_1\| \|a_2 - b_2\| \\
&= (\inf_{b_1 \in I} \|a_1 - b_1\|)(\inf_{b_2 \in I} \|a_2 - b_2\|).
\end{aligned}$$

It remains to show that A/I with this norm is complete. If $(a_n + I)$ is a sequence in A/I such that $\sum_{n=0}^{\infty} \|a_n + I\| < \infty$, there exist $b_n \in A$ such that $a_n - b_n \in I$ and $\sum_{n=0}^{\infty} \|b_n\| < \infty$. Since A is complete, there exists $b \in A$ such that $\lim_{N \to \infty} \sum_{n=0}^{N} b_n = b$. Therefore,

$$\left\| b + I - \sum_{n=0}^{N} a_n + I \right\| = \left\| b + I - \sum_{n=0}^{N} b_n + I \right\| \le \left\| b - \sum_{n=0}^{N} b_n \right\| \to 0. \qquad \square$$

We will use the following two lemmas from commutative algebra.

Lemma 1.3.10. *Let A be a unital commutative algebra. An ideal $I \trianglelefteq A$ is maximal if and only if A/I is a field.*

Lemma 1.3.11. *Let A be a unital commutative algebra. If $\varphi\colon A \to \mathbb{C}$ is a non-zero algebra homomorphism, then $\ker \varphi \trianglelefteq A$ and*

$$\psi\colon A/\ker \varphi \to \mathbb{C},\ a + \ker \varphi \mapsto \varphi(a)$$

is a bijective algebra homomorphism.

Lemma 1.3.12. *If A is a unital commutative Banach algebra, then every maximal ideal of A is closed and every character of A is contractive.*

Proof. If $I \trianglelefteq A$ is a maximal ideal, then I does not contain any invertible element. In particular, $\|a - 1\| \geq 1$ for all $a \in I$. Thus $\bar{I}$ is again a proper ideal. Maximality of I implies $I = \bar{I}$.

By the previous two lemmas, if $\varphi \in \Gamma(A)$, then $\ker \varphi$ is a maximal ideal of A, hence closed by the first paragraph. Thus $\varphi = \psi \circ q$ with the maps ψ and q from previous lemmas. Moreover, $A/\ker \varphi = \{\lambda + \ker \varphi \mid \lambda \in \mathbb{C}\}$ and

$$\|\lambda + \ker \varphi\| = |\lambda| \inf_{a \in I} \|a - 1\| \geq |\lambda| = |\psi(\lambda)|.$$

Hence ψ is contractive. As q is also contractive, we conclude that φ is contractive. □

We want to give the spectrum $\Gamma(A)$ a topology. By definition, $\Gamma(A)$ is a subset of the dual space A^*. In fact, we have seen that $\Gamma(A)$ is contained in the unit ball of A^*. Since A is a Banach space, the dual space A^* comes equipped with a norm. However, this topology is not suitable for our purposes (it has too few compact sets). Instead, we use the following topology.

Definition 1.3.13 (Weak* topology)**.** Let E be a Banach space. The *weak* topology* on E^* is the coarsest topology that makes the maps $\varphi \mapsto \varphi(\xi)$ continuous for all $\xi \in E$.

Remark 1.3.14. There is also a more explicit description of the open sets in weak* topology: A subset U of the dual space E^* is *weak* open* if for every $\varphi \in U$ there exist $\varepsilon > 0$ and $\xi_1, \ldots, \xi_n \in E$ such that

$$\{\psi \in E^* : |\varphi(x_k) - \psi(x_k)| < \varepsilon \text{ for } 1 \leq k \leq n\} \subset U.$$

However, it is often more convenient to work with the abstract characterization.

Theorem 1.3.15 (Banach–Alaoglu)**.** *If E is a Banach space, then the unit ball of E^* is weak* compact.*

We will not prove this result in this course. However, there is an outline of the proof in the case when E is separable in the exercises.

Proposition 1.3.16. *If A is a unital commutative Banach algebra, the spectrum $\Gamma(A)$ is weak* compact.*

Proof. We have already seen that $\Gamma(A)$ is contained in the unit ball of A^*. By the Banach–Alaoglu theorem, it remains to show that $\Gamma(A)$ is weak* closed in A^*. We have $\Gamma(A) = \{\varphi \in A^* \mid \varphi(1) = 1\} \cap \bigcap_{a,b \in A} \{\varphi \in A^* \mid \varphi(ab) = \varphi(a)\varphi(b)\}$. By the definition of the weak* topology, the maps $\varphi \mapsto \varphi(1)$ and $\varphi \mapsto \varphi(ab) - \varphi(a)\varphi(b)$ for $a, b \in A$ are continuous. Thus $\Gamma(A)$ is weak* closed as intersection of weak* closed sets. □

The next result motivates the terminology spectrum for the character space of a commutative C^*-algebra.

Lemma 1.3.17. *Let A be a unital commutative Banach algebra generated by $a \in A$, that is, a is not contained in a proper Banach closed unital subalgebra of A. The map*

$$\Gamma(A) \to \sigma_A(a),\ \varphi \mapsto \varphi(a)$$

is a homeomorphism.

Proof. First we have to show that $\varphi(a) \in \sigma_A(a)$ for all $\varphi \in \Gamma(A)$. Indeed, if $a - \varphi(a)$ were invertible, then

$$1 = \varphi(1) = \varphi((a - \varphi(a))(a - \varphi(a))^{-1}) = (\varphi(a) - \varphi(a))\varphi((a - \varphi(a))^{-1}) = 0,$$

a contradiction.

To see that the map is injective, let $\varphi, \psi \in \Gamma(A)$ with $\varphi(a) = \psi(a)$ and let $B = \{b \in A \mid \varphi(b) = \psi(b)\}$. Since φ and ψ are continuous, B is closed in A. Moreover, since φ and ψ are unital algebra homomorphisms, B is a unital subalgebra of A. As $a \in B$ and a generates A as Banach algebra, we conclude $B = A$. Hence $\varphi = \psi$.

To see that the map is surjective, let $\lambda \in \sigma_A(a)$ and let I be smallest closed ideal containing $a - \lambda$. Let $q\colon A \to A/I$ be the quotient map and $B = q^{-1}(\mathbb{C} + I)$. Note that $a + I = \lambda + I$, hence $a \in B$. Moreover, since the quotient map is a contractive unital algebra homomorphism, B is a closed unital subalgebra of A. As a generates A, we conclude $B = A$. Thus for every $b \in A$ there exists a unique $\varphi(b) \in \mathbb{C}$ such that $b + I = \varphi(b) + I$. It is not hard to see that $\varphi \in \Gamma(A)$ and $\varphi(a) = \lambda$.

By definition of the weak* topology, the map $\varphi \mapsto \varphi(a)$ is continuous. Since $\Gamma(A)$ is weak* compact, it is a homeomorphism. □

Definition 1.3.18 (Gelfand transform)**.** Let A be a unital commutative Banach algebra. For $a \in A$ let

$$\hat{a}\colon \Gamma(A) \to \mathbb{C},\ \varphi \mapsto \varphi(a).$$

The *Gelfand transform* is the map

$$\Gamma\colon A \to C(\Gamma(A)),\ a \mapsto \hat{a}.$$

Lemma 1.3.19. *Let A be a unital commutative Banach algebra. An element $a \in A$ is contained in a maximal ideal if and only if it is not invertible.*

Proof. If a is invertible and I an ideal containing a, then $1 = a^{-1}a \in I$, hence I cannot be proper and in particular not maximal.

Conversely, assume that a is not invertible and let $I = \{ba \mid b \in A\}$. Since A is commutative, I is an ideal, and since A is unital, $a \in I$. If $1 \in I$, there would exists $b \in A$ such that $ba = 1$, in contradiction to our assumption that a is not invertible. Hence I is a proper ideal. Let $\mathcal{I}$ be the set of all proper ideals containing I, ordered by inclusion. By Zorn's lemma, $\mathcal{I}$ has a maximal element J (exercise). By definition, J is a maximal ideal containing a. □

Proposition 1.3.20. *Let A be a unital commutative Banach algebra. The Gelfand transform is a contractive unital algebra homomorphism and for every $a \in A$, the Gelfand transform $\Gamma(a)$ is invertible in $C(\Gamma(A))$ if and only if a is invertible in A.*

Proof. We have already seen that $\|\varphi\| \leq 1$ for every $\varphi \in \Gamma(A)$. Thus

$$\|\hat{a}\| = \sup_{\varphi \in \Gamma(A)} |\hat{a}(\varphi)| = \sup_{\varphi \in \Gamma(A)} |\varphi(a)| \leq \|a\|.$$

It is easy to see that the Gelfand transform is an algebra homomorphism. If $a \in A$ is invertible, then $\Gamma(a)\Gamma(a^{-1}) = \Gamma(1) = 1$, hence $\Gamma(a)$ is invertible. Conversely, if $a \in A$ is not invertible, then a is contained in a maximal ideal by the previous lemma. Thus A/I is a Banach algebra in which every element is invertible, hence $A/I \cong \mathbb{C}$ by the Banach–Mazur theorem. The quotient map $q\colon A \to A/I$ is a character and $q(a) = 0$. Thus $\hat{a}(q) = 0$, which means that $\hat{a}$ is not invertible. □

Corollary 1.3.21. *If A is a unital Banach algebra, then $\sigma(\Gamma(a)) = \sigma(a)$ and $\|\Gamma(a)\| = r(\Gamma(a)) = r(a)$ for all $a \in A$.*

Exercises

1. Recall that a topological space is called *separable* if it has a countable dense subset. Every subset of a separable metric space is again separable (this fails for general topological spaces). Let E be a separable Banach space.

   (a) Show that if $\{\xi_k \mid k \in \mathbb{N}\}$ is a dense subset of the unit ball of E, then
   $$d(\varphi, \psi) = \sum_{k=0}^{\infty} 2^{-k} |\varphi(\xi_k) - \psi(\xi_k)|$$
   is a metric on the unit ball of E^*.

(b) Show that the metric from (a) induces the weak* topology on the unit ball of E^*.

(c) Show that every sequence in the unit ball of E^* has a subsequence that converges with respect to d.

2. Let A be a unital commutative algebra and $I \trianglelefteq A$ a proper ideal. Show that the set of proper ideals containing I is partially ordered by inclusion and every chain has a maximal element.

3. In this exercise we construct the *Stone–Čech compactification* of the natural numbers. Let ℓ^∞ denote the space of all bounded complex sequences with the supremum norm. Clearly, ℓ^∞ is a unital commutative C^*-algebra. We denote its spectrum by $\beta\mathbb{N}$.

   (a) For $n \in \mathbb{N}$ let $\delta_n \colon \ell^\infty \to \mathbb{C}$, $x \mapsto x_n$. Show that $\{\delta_n \mid n \in \mathbb{N}\}$ is dense in $\beta\mathbb{N}$.

   (b) Show that $\{\delta_n\}$ is open in $\beta\mathbb{N}$ for every $n \in \mathbb{N}$.

   (c) Show that $\beta\mathbb{N}$ has the following universal property: For every compact Hausdorff space K and every map $f \colon \mathbb{N} \to K$ there exists a unique continuous map $\tilde{f} \colon \beta\mathbb{N} \to K$ such that $\tilde{f}(\delta_n) = f(n)$ for all $n \in \mathbb{N}$.

4. In this exercise we revisit the Banach limits from the appendix using so-called *ultralimits*. Recall from the previous exercise that $\beta\mathbb{N} = \Gamma(\ell^\infty)$. For $x \in \ell^\infty$ and $\omega \in \beta\mathbb{N} \setminus \{\delta_n \mid n \in \mathbb{N}\}$, it is customary to write $\lim_{n\to\omega} x_n$ for $\omega(x)$.

   (a) Show that if x is convergent, then $\lim_{n\to\omega} x_n = \lim_{n\to\infty} x_n$.

   (b) Show that
$$\mathrm{LIM} \colon \ell^\infty \to \mathbb{C},\ x \mapsto \lim_{n\to\omega} \frac{1}{n} \sum_{k=1}^{n} x_k$$
   is a Banach limit.

1.4 Continuous functional calculus

Let A be a (unital) C^*-algebra. The definition of normal, self-adjoint, positive etc. elements of A is the same as the algebraic definition for bounded operators. We write A_{sa} for the set of self-adjoint elements and A_+ for the set of positive elements of A. If $a, b \in A_{\mathrm{sa}}$, we define $a \le b$ if $b - a \in A_+$.

Lemma 1.4.1. *Let A be a C^*-algebra. If $a, b \in A_{\mathrm{sa}}$ with $a \leq b$ and $c \in A$, then $c^*ac \leq c^*bc$.*

Proof. If $d \in A$ such that $b - a = d^*d$, then $c^*(b-a)c = (dc)^*(dc) \geq 0$. □

Proposition 1.4.2. *If A is a C^*-algebra and $a \in A$ is normal, then $\|a\| = r(a)$.*

Proof. If $a \in A$ is self-adjoint, then $\|a\|^2 = \|a^*a\| = \|a^2\|$. By induction one sees that $\|a\|^{2^n} = \|a^{2^n}\|$ for all $n \in \mathbb{N}$ and therefore $\|a\| = \lim_{n\to\infty} \|a^{2^n}\|^{2^{-n}} = r(a)$.

If a is normal, then $r(a^*a) \leq r(a^*)r(a) = r(a)^2$ as shown in an exercise before. Thus

$$\|a\|^2 = \|a^*a\| = r(a^*a) \leq r(a)^2 \leq \|a\|^2.$$

□

Corollary 1.4.3. *Every unital $*$-homomorphism between unital C^*-algebras is contractive and every unital $*$-isomorphism is isometric.*

Proof. If A, B are unital C^*-algebras and $\Phi\colon A \to B$ is a unital $*$-homomorphism, then $\sigma_B(\Phi(a)) \subset \sigma_A(a)$ for all $a \in A$. Thus

$$\|\Phi(a)\|^2 = \|\Phi(a^*a)\| = r(\Phi(a^*a)) \leq r(a^*a) = \|a\|^2.$$

If Φ is a unital $*$-isomorphism, then $\sigma_B(\Phi(a)) = \sigma_A(a)$ for all $a \in A$ and the inequality in the previous equation becomes an equality. □

Corollary 1.4.4. *On a given unital $*$-algebra there is at most one norm that makes it a C^*-algebra.*

Proof. The identity map is a unital $*$-isomorphism, hence isometric by the previous corollary. □

Lemma 1.4.5. *Let A be a unital C^*-algebra. If $a \in A$ is self-adjoint, then $\sigma_A(a) \subset \mathbb{R}$.*

Proof. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + i\beta \in \sigma_A(a)$. For $t \in \mathbb{R}$ let $b = a - \alpha + it$. We have $i(\beta + t) \in \sigma_A(b)$ and b is normal. Thus

$$(\beta + t)^2 \leq r(b)^2 = \|b^*b\| = \|(a-\alpha)^2 + t^2\| \leq \|a - \alpha\|^2 + t^2,$$

which implies $\beta^2 + 2t\beta \leq \|a - \alpha\|^2$. If $\beta \neq 0$, the supremum of the left side over $t \in \mathbb{R}$ is ∞. Therefore $\beta = 0$. □

Proposition 1.4.6 (Invariance of the spectrum)**.** *Let A be a unital C^*-algebra and $B \subset A$ a unital C^*-subalgebra. If $b \in B$, then $\sigma_B(b) = \sigma_A(b)$.*

Proof. Note that $\mathrm{Inv}(B) \subset \mathrm{Inv}(A)$. Let $b \in B$ be self-adjoint and not invertible in B. By the previous lemma, $b - i/n \in \mathrm{Inv}(B)$ for $n \in \mathbb{N}$. If $(\|(b-i/n)^{-1}\|)$ were bounded, say by a constant $C > 0$, then we would have

$$\left\|1 - (b - i/n)^{-1} b\right\| = \left\|(b - i/n)^{-1} (b - i/n - b)\right\| \leq \frac{1}{n} \left\|(b - i/n)^{-1}\right\| \to 0,$$

which implies that for n large enough, $(b - i/n)^{-1} b$ is invertible in B and hence so is b, in contradiction to our assumption. As inversion is continuous on $\mathrm{Inv}(A)$, we see that b is not invertible in A.

For general $b \in B$ we have $b \in \mathrm{Inv}(B)$ if and only if $b^*b \in \mathrm{Inv}(B)$ if and only if $b^*b \in \mathrm{Inv}(A)$ if and only if $b \in \mathrm{Inv}(A)$. In particular, $\sigma_A(b) = \sigma_B(b)$ for all $b \in B$. □

End Lecture May 15

Theorem 1.4.7 (Gelfand representation theorem)**.** *If A is a unital commutative C^*-algebra, then the Gelfand transform $\Gamma\colon A \to C(\Gamma(A))$ is an isometric unital $*$-isomorphism.*

Proof. If $a \in A$ is self-adjoint, then $\operatorname{im}\Gamma(a) = \sigma(\Gamma(a)) = \sigma(a) \subset \mathbb{R}$. Thus $\Gamma(a)$ is self-adjoint. In general, we can write $a \in A$ as $a = b + ic$ with $b = \frac{1}{2}(a + a^*)$ and $c = \frac{1}{2i}(a - a^*)$ self-adjoint and $\Gamma(a) = \Gamma(b) + i\Gamma(c)$ with $\Gamma(b)$, $\Gamma(c)$ self-adjoint. Thus $\Gamma(a^*) = \overline{\Gamma(a)}$ and so Γ is a $*$-homomorphism.

As discussed before, $\varphi(1) = 1$ for all $\varphi \in \Gamma(A)$, hence Γ is unital.

We have seen before that $\|\Gamma(a)\| = r(\Gamma(a)) = r(a)$ for all $a \in A$. Thus

$$\|\Gamma(a)\|^2 = \|\Gamma(a^*a)\| = r(a^*a) = \|a^*a\| = \|a\|^2$$

for all $a \in A$, which means that Γ is isometric.

As a consequence, the image of Γ is a closed unital $*$-subalgebra of $C(\Gamma(A))$. By definition, the range of Γ separates the points of $\Gamma(A)$. It follows from the Stone–Weierstraß theorem that Γ is surjective. □

Corollary 1.4.8. *Every unital commutative C^*-algebra is $*$-isomorphic to $C(X)$ for some compact Hausdorff space X.*

If A is a unital C^*-algebra and $a \in A$, then the unital C^*-algebra B generated by a is commutative if and only if a is normal. In this case, we have already seen that $\hat{a}$ is a homeomorphism from $\Gamma(B)$ onto $\sigma(a)$.

Definition 1.4.9 (Continuous functional calculus)**.** Let A be a unital C^*-algebra and $a \in A$ normal. Let B denote the unital C^*-algebra generated by a and $\Gamma\colon C(\Gamma(B)) \to B$ the Gelfand transform. For $f \in C(\sigma(a))$ we define $f(a) = \Gamma^{-1}(f \circ \hat{a}) \in B \subset A$. The map

$$C(\sigma(a)) \to A,\ f \mapsto f(a)$$

is called the *continuous functional calculus*.

Theorem 1.4.10 (Continuous funtional calculus). *Let A be a unital C^*-algebra and $a \in A$ normal. The functional calculus satisfies the following properties:*

(a) If $f(z) = \sum_{k=0}^n \alpha_k z^k$ for $z \in \sigma(a)$, then $f(a) = \sum_{k=0}^n \alpha_k a^k$.

(b) If $\lambda \in \mathbb{C} \setminus \sigma(a)$ and $f(z) = (z-\lambda)^{-1}$, then $f(a) = (a-\lambda)^{-1}$.

(c) If $f \in C(\sigma(a))$, then $\sigma(f(a)) = f(\sigma(a))$ and $\|f(a)\| = \|f\|_\infty$.

(d) If B is unital C^*-algebra and $\Phi\colon A \to B$ is a unital $*$-homomorphism, then $\Phi(f(a)) = f(\Phi(a))$ for all $f \in C(\sigma(a))$

(e) If (a_n) is a sequence of normal elements in A that converges to a and Ω is a compact neighborhood of $\sigma(a)$, then $\sigma(a_n) \subset \Omega$ eventually and $f(a_n) \to f(a)$ for every $f \in C(\Omega)$.

Proof. (a) If $f(z) = 1$, then $f(a) = \Gamma^{-1}(1 \circ \hat{a}) = \Gamma^{-1}(1) = 1$, and if $f(z) = z$, then $f(a) = \Gamma^{-1}(\hat{a}) = a$. For general polynomials, the claim follows from the fact that continuous functional calculus is an algebra homomorphism.

(b) Let $g(z) = z - \lambda$. Since continuous functional calculus is a unital algebra homomorphism, we have $1 = (fg)(a) = f(a)g(a) = g(a)f(a)$. By (a), $g(a) = a - \lambda$. Thus $f(a) = (a-\lambda)^{-1}$.

(c) follows directly from the fact that Γ is an isometric $*$-isomorphism.

(d) is clear if f is a polynomial. Arbitrary $f \in C(\sigma(a))$ can be approximated by polynomials in supremum norm by the Stone–Weierstraß theorem, and then the result follows from the continuity of the Gelfand transform.

(e) Since $\sigma(a)$ is compact and Ω is a neighborhood of $\sigma(a)$, there exists $\varepsilon > 0$ such that $d(\lambda, \sigma(a)) \geq \varepsilon$ for all $\lambda \in \mathbb{C} \setminus \Omega$. By (b) and (c),

$$\|(a-\lambda)^{-1}\| = \sup_{z \in \sigma(a)} |(z-\lambda)^{-1}| = \frac{1}{d(\lambda, \sigma(a))} \leq \varepsilon^{-1}.$$

Hence, if $\|b - a\| < \varepsilon$, then $\|(b-\lambda) - (a-\lambda)\| < \varepsilon \leq \|(a-\lambda)^{-1}\|^{-1}$, which implies that $b - \lambda$ is invertible as we have shown when we proved that the invertible elements form an open subset. In particular, $\mathbb{C} \setminus \Omega \subset \rho(a_n)$ for n sufficiently large.

To see that $f(a_n) \to f(a)$, let $\varepsilon > 0$. By the Stone–Weierstraß theorem, there exists a polynomial $g \in C(\Omega)$ such that $\|f - g\|_\infty \leq \frac{\varepsilon}{2}$. Since multiplication in A is continuous, $g(a_n) \to g(a)$. Thus

$$\begin{aligned}\|f(a_n) - f(a)\| &\leq \|f(a_n) - g(a_n)\| + \|g(a_n) - g(a)\| + \|g(a) - f(a)\| \\ &\leq 2\|f - g\|_\infty + \|g(a_n) - g(a)\| \\ &\leq \varepsilon + \|g(a_n) - g(a)\|.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude $\|f(a_n) - f(a)\| \to 0$. □

1.5 Applications of Functional Calculus

Definition 1.5.1 (Real, imaginary, positive, negative part)**.** Let A be a C^*-algebra and $a \in A$. The *real* and *imaginary part* of a are defined as $\operatorname{Re} a = \frac{1}{2}(a + a^*)$ and $\operatorname{Im} a = \frac{1}{2i}(a - a^*)$. If a is self-adjoint, then its *positive* and *negative part* are defined as $a_+ = \frac{1}{2}(a + |a|)$ and $a_- = \frac{1}{2}(a - |a|)$.

Remark 1.5.2. Note that $a = \operatorname{Re} a + i \operatorname{Im} a$. In particular, every element of a C^*-algebra is a linear combination of two self-adjoint elements. Moreover, $\|\operatorname{Re} a\|, \|\operatorname{Im} a\| \leq \frac{1}{2}(\|a\| + \|a^*\|) = \|a\|$.

The positive and negative part of a self-adjoint element can equivalently be defined in terms of functional calculus (i. e. applying the function $\lambda \mapsto \lambda_\pm$ to x). It follows immediately that $\sigma(x_\pm) \subset [0, \infty)$, $x_+ x_- = x_- x_+ = 0$ and $x = x_+ - x_-$.

Lemma 1.5.3. *If A is a unital C^*-algebra and $a, b \in A$ are self-adjoint with $\sigma(a), \sigma(b) \subset [0, \infty)$, then $\sigma(a + b) \subset [0, \infty)$.*

Proof. First note that $\sigma(\|a\| - a) \subset [0, \|a\|]$ and similarly for b. By the spectral radius formula, $\|\|a\| - a\| = r(\|a\| - a) \leq \|a\|$ and $\|\|b\| - b\| \leq \|b\|$. Thus

$$\begin{aligned}\sup_{\lambda \in \sigma(a+b)} (\|a\| + \|b\| - \lambda) &= r(\|a\| + \|b\| - (a + b)) \\ &\leq \|\|a\| - a\| + \|\|b\| - b\| \\ &\leq \|a\| + \|b\|.\end{aligned}$$

Hence $\sigma(a + b) \subset [0, \infty)$. □

Proposition 1.5.4. *Let A be a unital C^*-algebra. A normal element $a \in A$ is*

(a) self-adjoint if and only if $\sigma(a) \subset \mathbb{R}$,

(b) positive if and only if $\sigma(a) \subset [0, \infty)$,

(c) unitary if and only if $\sigma(a) \subset \{z \in \mathbb{C} : |z| = 1\}$,

(d) a projection if and only if $\sigma(a) \subset \{0, 1\}$.

Proof. (a), (c) and (d) follow immediately from functional calculus. We only show (a) here as a demonstration. Let $f\colon \sigma(a) \to \mathbb{C}$, $\lambda \mapsto \lambda$. By functional calculus, $\sigma(a) = \sigma(f) = \sigma(a)$ and $a^* = \bar{f}(a)$. Thus $a = a^*$ if and only if $f = \bar{f}$ if and only if $\sigma(a) = \sigma(f) \subset \mathbb{R}$.

(b) If $\sigma(a) \subset [0, \infty)$, then $a = (a^{1/2})^* a^{1/2}$ by functional calculus. Assume conversely that $a = b^* b$ for some $b \in A$. We need to show that $\sigma(a) \subset [0, \infty)$ or, equivalently, $a_- = 0$.

Let $c = ba_-$ and note that $c^* c = a_- b^* b a_- = a_- a a_- = -a_-^3$. Hence $\sigma(cc^*) \subset \sigma(c^* c) \cup \{0\} \subset (-\infty, 0]$. We have $c^* c + cc^* = 2(\operatorname{Re} c)^2 + 2(\operatorname{Im} c)^2$ and $\sigma(2(\operatorname{Re} c)^2 + 2(\operatorname{Im} c)^2) \subset [0, \infty)$ by the previous lemma. Moreover, since $\sigma(-cc^*) \subset [0, \infty)$, we also have $\sigma(c^* c) = \sigma(2(\operatorname{Re} c)^2 + 2(\operatorname{Im} c)^2 - cc^*) \subset [0, \infty]$ by the previous lemma. Therefore $\sigma(-a_-^3) = \sigma(c^* c) = 0$. As a_- is self-adjoint, this implies $\|a_-\|^3 = \|a_-^3\| = r(a_-^3) = 0$. □

Corollary 1.5.5. *An element v of a unital C^*-algebra is a partial isometry if and only if v^* is a partial isometry.*

Proof. By definition, v is a partial isometry if and only if $v^* v$ is a projection. Since $v^* v$ is self-adjoint, this is equivalent to $\sigma(v^* v) \subset \{0, 1\}$ by the previous proposition. As $\sigma(vv^*) \subset \sigma(v^* v) \cup \{0\}$, the conclusion follows. □

Corollary 1.5.6. *Let A be a unital C^*-algebra. The set A_+ of positive elements of A is closed and if $a, b \in A_+$ and $\lambda, \mu \geq 0$, then $\lambda a + \mu b \in A_+$. Moreover, if $a \in A$ is self-adjoint, then $-\|a\| \leq a \leq \|a\|$.*

Proof. By the previous proposition, $A_+ = \{a \in A_{\mathrm{sa}} \mid \sigma(a) \subset [0, \infty)\}$. If $a_n \in A_+$ and $a_n \to a$, then $0 = (a_n)_- \to a_-$ by continuity of functional calculus. Thus $a_- = 0$, which implies $a \in A_+$. Thus A_+ is closed. The other statements are easy consequences of the previous results. □

Proposition 1.5.7. *Every element of a unital C^*-algebra is a linear combination of four unitaries.*

Proof. Every element a of a C^*-algebra is a linear combination of two self-adjoint elements (its real and imaginary part). By rescaling, we may further assume that these self-adjoint elements have norm at most 1. If $a \in A$ is self-adjoint with $\|a\| \leq 1$, then $u = a + i(1 - a^2)^{1/2}$ is unitary:

$$uu^* = u^* u = (a + i(1 - a^2)^{1/2})(a - i(1 - a^2)^{1/2}) = a^2 + 1 - a^2 = 1.$$

Moreover, $u + u^* = a$. □

Theorem 1.5.8 (Operator monotonicity of the square root)**.** *Let A be a unital C^*-algebra. If $a, b \in A$ are positive and $a \leq b$, then $a^{1/2} \leq b^{1/2}$. Moreover, if a and b are additionally invertible, then $b^{-1} \leq a^{-1}$.*

Proof. We first assume that $a, b \in A$ are positive and invertible. Recall that $c^* A_+ c \subset A_+$ for $c \in A$. As $a \leq b$, we have $b^{-1/2} a b^{-1/2} \leq 1$, thus

$r(a^{1/2}b^{-1}a^{1/2}) = r(b^{-1/2}ab^{-1/2}) \leq 1$, which implies $a^{1/2}b^{-1}a^{1/2} \leq 1$. Hence $b^{-1} \leq a^{-1}$.

Moreover, $\|a^{1/2}b^{-1/2}\|^2 = \|b^{-1/2}ab^{-1/2}\|^2 \leq 1$, which implies

$$\begin{aligned} b^{-1/4}a^{1/2}b^{-1/4} &\leq r(b^{-1/4}a^{1/2}b^{-1/4}) \\ &= r(a^{1/4}b^{-1/2}a^{1/4}) \\ &= r(a^{1/2}b^{-1/2}) \\ &\leq \|a^{1/2}b^{-1/2}\| \\ &\leq 1. \end{aligned}$$

Therefore $a^{1/2} \leq b^{1/2}$.

In general, if $a, b \in A_+$ with $a \leq b$ are not necessarily invertible and $\varepsilon > 0$, we have $a+\varepsilon \leq b+\varepsilon$ and $a+\varepsilon$, $b+\varepsilon$ are positive and invertible (since $\sigma(a+\varepsilon), \sigma(b+\varepsilon) \subset [\varepsilon, \infty)$). By the previous paragraph, $(a+\varepsilon)^{1/2} \leq (b+\varepsilon)^{1/2}$. By continuity of functional calculus, $(a+\varepsilon)^{1/2} \to a^{1/2}$ and $(b+\varepsilon)^{1/2} \to b^{1/2}$ as $\varepsilon \to 0$. □

Remark 1.5.9. If $I \subset \mathbb{R}$ is an interval, a function $f\colon I \to \mathbb{R}$ is called *operator monotone* if $f(a) \subset f(b)$ whenever a, b are self-adjoint elements of a unital C^*-algebra with $a \leq b$. Since $\mathbb{C}$ is a unital C^*-algebra with self-adjoint part $\mathbb{R}$, every operator monotone function is monotone. The converse is not true: For example, $\lambda \mapsto \lambda^2$ is not operator monotone on $[0, \infty)$. The previous result shows that $\lambda \mapsto \sqrt{\lambda}$ and $\lambda \mapsto -1/\lambda$ are operator monotone on $[0, \infty)$.

Definition 1.5.10 (Absolute value)**.** If A is a unital C^*-algebra and $a \in A$, then the *absolute value* $|a|$ of a is defined as $|a| = (a^*a)^{1/2}$.

Corollary 1.5.11. *If A is a unital C^*-algebra and $a, b \in A$, then $|ab| \leq \|a\||b|$.*

Proof. Since $a^*a \leq \|a^*a\| = \|a\|^2$, we have $b^*a^*ab \leq \|a\|^2 b^*b$. The claim now follows from the operator monotonicity of the square root function. □

1.6 Bonus: Group C^*-algebras

Let G be a group. As we have seen at the very beginning of this course, the formal finite linear combinations of elements of G (more rigorously, the free complex vector space over G) form a unital $*$-algebra $\mathbb{C}[G]$ with the operations

$$\left(\sum_g \alpha_g g\right)\left(\sum_h \beta_h h\right) = \sum_{g,h} \alpha_g \beta_h gh,$$
$$\left(\sum_g \alpha_g g\right)^* = \sum_g \overline{\alpha_g} g^{-1}.$$

We also claimed that

$$\left\|\sum_g \alpha_g g\right\|_{\mathrm{u}} = \sup\left\{\left\|\sum_g \alpha_g \pi(g)\right\| : \pi\colon G \to \mathbb{U}(H) \text{ group hom.}\right\}$$

defines a C^*-norm on $\mathbb{C}[G]$, where $\mathbb{U}(H)$ denotes the group of unitary operators on H.

The C^* norm property is indeed not hard to verify given that the operator norm on $\mathbb{B}(H)$ is a C^* norm. What that takes some more considerations is the fact that the supremum on the right side is always finite and the resulting semi-norm is positive definite.

Lemma 1.6.1. *If G is a group, then $\|\cdot\|_{\mathrm{u}}$ is a norm on $\mathbb{C}[G]$.*

Proof. First note that if $\pi\colon G \to \mathbb{U}(H)$ is a group homomorphism, then

$$\|\pi(g)\|^2 = \|\pi(g)^{-1}\pi(g)\| = \|\pi(e)\| = \|1\| \leq 1.$$

Thus

$$\left\|\sum_g \alpha_g \pi(g)\right\| \leq \sum_g |\alpha_g|.$$

In particular, $\left\|\sum_g \alpha_g g\right\|_{\mathrm{u}} < \infty$.

To show that $\|\cdot\|_{\mathrm{u}}$ is positive definite, define

$$\lambda_g\colon \ell^2(G) \to \ell^2(G),\ (\lambda_g f)(h) = f(g^{-1}h)$$

for $g \in G$. Clearly, λ_g is a bijective isometry (with inverse $\lambda_{g^{-1}}$), hence a unitary. Moreover, $\lambda_e = 1$ and $\lambda_g\lambda_h = \lambda_{gh}$ for $g, h \in G$ are easy to see. Thus $\lambda\colon G \to \mathbb{U}(\ell^2(G))$ is a group homomorphism.

We have

$$\sum_g \alpha_g \lambda_g \mathbb{1}_e = \sum_g \alpha_g \mathbb{1}_g.$$

In particular, $\|\sum_g \alpha_g \lambda_g\| = 0$ if and only if $\alpha_g = 0$ for all $g \in G$. Therefore $\|\cdot\|_{\mathrm{u}}$ is positive definite. □

Definition 1.6.2 (Reduced and full group C^*-algebra)**.** Let G be a group. The representation $\lambda\colon G \to \mathbb{U}(\ell^2(G))$ from the proof of the previous lemma is called the *left regular representation* of G. The closure of $\{\sum_g \alpha_g \lambda_g \mid \alpha \in c_c(G)\}$ is called the *reduced group C^*-algebra* and denoted by $C_r^*(G)$.

The completion of $\mathbb{C}[G]$ with respect to $\|\cdot\|_{\mathrm{u}}$ is called the *full group C^*-algebra* and denoted by $C^*(G)$.

By definition, whenever $\pi\colon G \to \mathbb{U}(H)$ is a group homomorphism, then π can be extended to a contractive linear map from $C^*(G)$ to $\mathbb{B}(H)$, still denoted by the same letter π. It is not hard to see that this extension is a unital $*$-homomorphism. In particular, there is a surjective unital $*$-homomorphism $\lambda\colon C^*(G) \to C_r^*(G)$ for every group G. In general, λ is not injective.

Definition 1.6.3 (Amenable group)**.** A group G is called amenable if the map $\lambda\colon C^*(G) \to C_r^*(G)$ is a injective.

Example 1.6.4. Every finite group is amenable. In this case, $C^*(G) = \mathbb{C}[G]$ and $\lambda|_{\mathbb{C}[G]}$ is injective, as we have seen before.

The name "amenable" is a word play on the word "mean" in the sense of the following definition. Invariant means are the more common way to define amenable groups, but this definition is equivalent to ours, as we will see soon.

Definition 1.6.5 (Invariant mean)**.** Let G be a group. An *left-invariant mean* is a map $\mu\colon \ell^\infty(G) \to \mathbb{C}$ such that

- $\mu(1) = 1$,
- $\mu(f) \geq 0$ if $f \geq 0$,
- $\mu(f(g^{-1}\cdot)) = \mu(f)$ for all $f \in \ell^\infty(G)$ and $g \in G$.

Theorem 1.6.6. *For a group G, the following properties are equivalent:*

(i) G is amenable.

(ii) $C_r^*(G)$ has a character.

(iii) $\ell^\infty(G)$ has a left-invariant mean.

Proof. The proof requires some techniques we have not covered in this course. Some of them can be found in Chaper 3. We only sketch (i) $\implies$ (iii) and (ii) $\implies$ (iii) here. For a complete proof, see Brown, Ozawa. *C^*-Algebras and Finite-Dimensional Approximations*, Theorem 2.6.8.

(i) $\Longrightarrow$ (ii): Let $\pi\colon G \to S^1$, $g \mapsto 1$ be the trivial representation. By the definition of $C^*(G)$, the map π can be extended to a unital $*$-homomorphism $\varphi\colon C^*(G) \to \mathbb{C}$. If G is amenable, then λ is a $*$-isomomorphism and $\varphi \circ \lambda^{-1}\colon C_r^*(G) \to \mathbb{C}$ is a character.

(ii) $\Longrightarrow$ (iii): Let $\varphi\colon C_r^*(G) \to \mathbb{C}$ be a character. We can extend φ to a bounded linear functional $\psi\colon \mathbb{B}(\ell^2(G)) \to \mathbb{C}$ with $\psi(1) = 1$ and $\psi(x) \geq 0$ for all $x \geq 0$ (see Lemma 3.1.4). Since $\psi|_{C_r^*(G)}$ is a $*$-homomorphism, we have

$$\psi(\lambda_g x \lambda_g^*) = \varphi(\lambda_g)\psi(x)\overline{\varphi(\lambda_g)} = \psi(x)$$

for all $x \in \mathbb{B}(\ell^2(G))$ and $g \in G$ (see Lemma 3.1.5).

For $f \in \ell^\infty(G)$ consider the multiplication operator M_f on $\ell^2(G)$. We have $\lambda_g M_f \lambda_g^* = M_{f(g^{-1}\,\cdot)}$. Thus

$$\mu\colon \ell^\infty(G) \to \mathbb{C},\ f \mapsto \psi(M_f)$$

is a left-invariant mean. □

Corollary 1.6.7. *Every abelian group is amenable.*

Proof. If G is abelian, then $C_r^*(G)$ is commutative and thus has a character by Gelfand theory. □

Example 1.6.8. Let $\mathbb{F}_2$ be the free group on two generators. As a set, $\mathbb{F}_2$ consists of all finite words with letters a, a^{-1}, b, b^{-1} such that no two adjacent letters are inverse to each other. The identity element in this group is the empty word and the group multiplication is given by concatenation (with cancellation of adjacent inverse letters). For example, $(ab^{-1}ab)(b^{-1}aa) = ab^{-1}aaa$.

The group $\mathbb{F}_2$ is not amenable (exercise).

If G is an abelian group, then $C^*(G)$, which is canonically isomorphic to $C_r^*(G)$ by the previous corollary, is a unital commutative C^*-algebra. By the Gelfand representation theorem, $C^*(G) = C(\Gamma(C^*(G)))$. Let us determine the spectrum of $C^*(G)$.

Definition 1.6.9 (Pontryagin dual)**.** Let G be an abelian group. The *Pontryagin dual* $\hat{G}$ of G is the set of all group homomorphisms from G to S^1.

Note that $\hat{G}$ becomes itself a group when endowed with the multiplication $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$ for $\chi_1, \chi_1 \in \hat{G}$ and $g \in G$.

Lemma 1.6.10. *If G is an abelian group, then the map*

$$\Gamma(C^*(G)) \to \hat{G},\ \varphi \mapsto \varphi|_G$$

is bijective.

Proof. First recall that $\varphi \in \Gamma(C^*(G))$ is a unital $*$-homomorphism. In particular, if $g \in G$, then $|\varphi(g)|^2 = \varphi(g^{-1}g) = \varphi(e) = 1$. Thus $\varphi(G) \subset S^1$.

If $\chi \in \hat{G}$, then χ is a group homomorphism from G to $\mathbb{U}(\mathbb{C})$. Hence it can be extended to a unital $*$-homomorphism from $C^*(G)$ to $\mathbb{C}$. This settles surjectivity.

To see that the map is injective, note that if $\operatorname{span} G$ is dense in $C^*(G)$. Hence if two bounded linear maps on $C^*(G)$ coincide on G, they coincide on $C^*(G)$. □

Remark 1.6.11. We already know that the spectrum of a unital C^*-algebra is a compact Hausdorff space. The previous lemma allows us to transport the topology on $\Gamma(C^*(G))$ to $\hat{G}$. This makes $\hat{G}$ into what is called a compact topological group (see the exercises).

Corollary 1.6.12. *If G is an abelian group, then $C^*(G)$ is $*$-isomorphic to $C(\hat{G})$.*

Example 1.6.13. Let $G = \mathbb{Z}$. Since G is a free group, the map $\hat{G} \to S^1$, $\chi \mapsto \chi(1)$ is a bijection. Moreover, if $\chi \in \hat{G}$ and $n \in \mathbb{Z}$, then $\chi(n) = \chi(1)^n$. Hence $\chi \mapsto \chi(n)$ is continuous for all $n \in \mathbb{Z}$ if and only if $\chi \mapsto \chi(1)$ is continuous. Thus $\hat{G} \to S^1$, $\chi \mapsto \chi(1)$ is a homeomorphism. It follows that $C^*(\mathbb{Z}) \cong C(S^1)$.

More generally, one can show that for $d \in \mathbb{N}$, the group C^*-algebra $C^*(\mathbb{Z}^d)$ is $*$-isomorphic to $C(\mathbb{T}^d)$, where $\mathbb{T}^d = (S^1)^d$ is the d-dimensional torus.

Example 1.6.14. Let $n \in \mathbb{N}$ and let $C_n = \{z \in S^1 \mid z^n = 1\}$ the set of n-th roots of unity. If $G = \mathbb{Z}/n\mathbb{Z}$, the map $\hat{G} \to C_n$, $\chi \mapsto \chi(1)$ is a bijection and both sets carry the discrete topology. Thus $C^*(\mathbb{Z}/n\mathbb{Z}) \cong C(C_n) \cong \mathbb{C}^n$.

Exercises

1. Let G be an abelian group.

   (a) Endow $\hat{G}$ with the coarsest topology that makes the maps $\hat{G} \to S^1$, $\chi \mapsto \chi(g)$ continuous for all $g \in G$. Show that the map

   $$\Gamma(C^*(G)) \to \hat{G},\ \varphi \mapsto \varphi|_G$$

   is a homeomorphism.

   (b) Show that the maps

   $$\hat{G} \times \hat{G} \to \hat{G}, (\chi_1, \chi_2) \mapsto \chi_1\chi_2$$
   $$\hat{G} \to \hat{G}, \chi \mapsto \chi^{-1}$$

   are continuous. Here $\hat{G} \times \hat{G}$ is endowed with the product topology.

2. Inside the free group $\mathbb{F}_2$, let A^+, A^- be the set of words beginning with a, a^{-1}, respectively, and likewise define B^+, B^-. Further, let $C = \{1, b, b^2, \dots\} \subset \mathbb{F}_2$.

   (a) Show that

$$\begin{aligned}\mathbb{F}_2 &= A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \cup C) \\ &= A^+ \sqcup aA^- \\ &= b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C).\end{aligned}$$

   (b) Show that $\mathbb{F}_2$ is not amenable (Hint: Use the characterization with left-invariant means).

Chapter 2

Schatten Classes and Compact Operators on Hilbert Space

In this chapter we will study some two-sided ideals of the bounded operators on a Hilbert space, known as Schatten classes and compact operators.

2.1 Trace-class operators

The Schatten classes are derived from the study of the trace of an operator. In finite dimensions, every operator has a well-defined trace that is independent of the chosen basis. In infinite dimensions, this is no longer true. The trace-class operators is exactly the space of operators for which we can still define a reasonable notion of trace.

But first, let us start with a smaller class of operators for which there are no convergence problems whatsoever in the definition of the trace.

Definition 2.1.1 (Finite-rank operators). Let H be a Hilbert space. An operator $x \in \mathbb{B}(H)$ is called *finite rank* if $\operatorname{ran} x$ is finite-dimensional. The *rank* of x is $\dim \operatorname{ran} x$. The space of all finite rank operators on H is denoted by $\mathbb{F}(H)$.

As $\overline{\operatorname{ran} x} = \ker(x^*)^\perp$ and $\overline{\operatorname{ran}(x^*)} = \overline{\operatorname{ran}(x^*|_{\ker(x^*)^\perp})}$, the adjoint of a finite-rank operator is again a finite-rank operator. Furthemore, if x has finite rank, then $\ker(x)^\perp = \overline{\operatorname{ran}(x^*)}$ is also finite-dimensional.

Definition 2.1.2 (Bra ket notation). Let H be a Hilbert space. For $\xi \in H$ we define

$$|\xi\rangle : \mathbb{C} \to H,\ \lambda \mapsto \lambda\xi$$

and $\langle\xi| = |\xi\rangle^*$.

Remark 2.1.3. If $\eta \in H$ and $\lambda \in \mathbb{C}$, then

$$\overline{\langle \xi | \, \eta} \lambda = \langle \eta, |\xi\rangle \, \lambda \rangle = \langle \eta, \lambda \xi \rangle = \langle \eta, \xi \rangle \lambda.$$

Thus $\langle \xi | \, \eta = \langle \xi, \eta \rangle$.

Lemma 2.1.4. *If H is a Hilbert space, then $\mathbb{F}(H) = \operatorname{span}\{|\xi\rangle \langle \eta| : \xi, \eta \in H\}$ and $\mathbb{F}(H)$ is a two-sided ideal of $\mathbb{B}(H)$.*

Proof. As $\operatorname{ran}(|\xi\rangle \langle \eta|) \subset \mathbb{C}\xi$, we have $|\xi\rangle \langle \eta| \in \mathbb{F}(H)$. Conversely, if $x \in \mathbb{F}(H)$, let $(\xi_j)_{j=1}^m$ be an ONB of $\operatorname{ran} x$ and $(\eta_k)_{k=1}^n$ an ONB of $\ker(x^*)^\perp$. For arbitrary $\zeta \in H$ we have

$$x\zeta = \sum_{k=1}^n \langle \eta_k, \zeta \rangle x \eta_k = \sum_{j=1}^m \sum_{k=1}^n \langle \eta_k, \zeta \rangle \langle \xi_j, x\eta_k \rangle \xi_j = \sum_{j=1}^m \sum_{k=1}^n \langle \xi_j, x\eta_k \rangle \, |\xi_j\rangle \, \langle \eta_k| \, \zeta.$$

Moreover, if $\xi, \eta \in H$ and $x, y \in \mathbb{B}(H)$, then $x \, |\xi\rangle \, \langle \eta| \, y = |x\xi\rangle \, \langle y^*\eta| \in \mathbb{F}(H)$. Hence $\mathbb{F}(H)$ is a two-sided ideal of $\mathbb{B}(H)$. □

Definition 2.1.5 (Trace)**.** Let H be a Hilbert space with orthonormal basis $(e_j)_{j \in J}$. The *trace* of $x \in \mathbb{B}(H)_+$ is defined as

$$\operatorname{tr}(x) = \sum_{j \in J} \langle e_j, x e_j \rangle \in [0, \infty].$$

Remark 2.1.6. Recall that if J is uncountable, the series value in the definition of the trace is defined to be

$$\sup_{F \subset J \text{ finite}} \sum_{j \in F} \langle e_j, x e_j \rangle.$$

Remark 2.1.7. Clearly the trace is monotone in the sense that if $x, y \in \mathbb{B}(H)_+$ with $x \leq y$, then $\operatorname{tr}(x) \leq \operatorname{tr}(y)$.

Example 2.1.8. Let $p \in \mathbb{B}(H)$ be a projection and $(e_j)_{j \in J}$ an ONB of H for which there exists $I \subset J$ such that$(e_i)_{i \in I}$ is an ONB of pH. We have

$$\operatorname{tr}(p) = \sum_{j \in J} \langle e_j, p e_j \rangle = \sum_{i \in I} \|e_i\|^2 = |I|.$$

In particular, $\operatorname{tr}(p) < \infty$ if and only if $\dim(pH) < \infty$, in which case $\operatorname{tr}(p)$ equals the rank of p.

We will see later that $\operatorname{tr}(p)$ is independent of the choice of ONB.

Lemma 2.1.9. *If H is a Hilbert space with ONB $(e_j)_{j \in J}$ and $x \in \mathbb{B}(H)$, then $\operatorname{tr}(x^*x) = \operatorname{tr}(xx^*)$.*

Proof. By Parseval's identity and Fubini's theorem for series, we have

$$\begin{aligned}\sum_{j\in J}\langle e_j, x^*xe_j\rangle &= \sum_{j\in J}\sum_{i\in J}\langle xe_j, e_i\rangle\langle e_i, xe_j\rangle \\ &= \sum_{i\in J}\sum_{j\in J}\langle xe_j, e_i\rangle\langle e_i, xe_j\rangle \\ &= \sum_{i\in J}\langle e_i, xx^*e_i\rangle. \qquad \square\end{aligned}$$

Lemma 2.1.10. *Let H be a Hilbert space with ONB $(e_j)_{j\in J}$. If $x \in \mathbb{B}(H)$ is positive and $u \in \mathbb{B}(H)$ is unitary, then $\operatorname{tr}(uxu^*) = \operatorname{tr}(x)$. In particular, $\operatorname{tr}(x)$ does not depend on the chosen ONB.*

Proof. By the previous lemma,

$$\operatorname{tr}(uxu^*) = \operatorname{tr}((ux^{1/2})(ux^{1/2})^*) = \operatorname{tr}((ux^{1/2})^*(ux^{1/2})) = \operatorname{tr}(x). \qquad \square$$

Definition 2.1.11 (Trace-class operator, trace norm)**.** Let H be a Hilbert space. An operator $x \in \mathbb{B}(H)$ is called *trace class* if $\operatorname{tr}(|x|) < \infty$. The space of all trace-class operators on H is denoted by $L^1(\mathbb{B}(H))$. The *trace norm* $\|\cdot\|_1$ on $L^1(\mathbb{B}(H))$ is defined by $\|x\|_1 = \operatorname{tr}(|x|)$.

Lemma 2.1.12. *Let H be a Hilbert space. If $x \in \mathbb{B}(H)$ has polar decomposition $x = v|x|$ and $\xi \in H$, then*

$$2|\langle \xi, x\xi\rangle| \leq \langle \xi, |x|\xi\rangle + \langle v^*\xi, |x|v^*\xi\rangle.$$

Proof. If $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, then

$$0 \leq \|(|x|^{1/2} - \lambda|x|^{1/2}v^*)\xi\|^2 = \langle \xi, |x|\xi\rangle + \langle v^*\xi, |x|v^*\xi\rangle - 2\operatorname{Re}\lambda\langle \xi, |x|v^*\xi\rangle.$$

Note that $|x|v^* = (v|x|)^* = x^*$. If we choose λ such that $\lambda\langle x\xi, \xi\rangle = |\langle x\xi, \xi\rangle|$, we obtain the claimed inequality. $\square$

Lemma 2.1.13. *Let H be a Hilbert space with orthonormal basis $(e_j)_{j\in J}$. If $x \in L^1(\mathbb{B}(H))$, then the series*

$$\sum_{j\in J}\langle e_j, xe_j\rangle$$

converges absolutely and $\sum_{j\in J}|\langle e_j, xe_j\rangle| \leq \|x\|_1$.

Proof. Let $x = v|x|$ be the polar decomposition of x. By the previous lemma,

$$2\sum_{j\in J}|\langle e_j, xe_j\rangle| \leq \operatorname{tr}(|x|) + \operatorname{tr}(v|x|v^*).$$

By a lemma above, $\operatorname{tr}(v|x|v^*) = \operatorname{tr}(|x|^{1/2}v^*v|x|^{1/2})$. Since v^*v is a projection,

$$\begin{aligned}\operatorname{tr}(|x|^{1/2}v^*v|x|^{1/2}) &= \sum_{j\in J}\langle |x|^{1/2}e_j, v^*v|x|^{1/2}e_j\rangle \\ &\leq \sum_{j\in J}\langle |x|^{1/2}e_j, |x|^{1/2}e_j\rangle \\ &= \operatorname{tr}(|x|).\end{aligned}$$

□

Definition 2.1.14 (Trace)**.** Let H be a Hilbert space with ONB $(e_j)_{j\in J}$. The *trace* on $L^1(\mathbb{B}(H))$ is defined as

$$\operatorname{tr}\colon L^1(\mathbb{B}(H)) \to \mathbb{C},\, x \mapsto \sum_{j\in J}\langle e_j, xe_j\rangle.$$

Remark 2.1.15. With this definition, the previous lemma can be reformulated as $|\operatorname{tr}(x)| \leq \operatorname{tr}(|x|) = \|x\|_1$ for $x \in L^1(\mathbb{B}(H))$.

Proposition 2.1.16. *Let H be a Hilbert space with ONB $(e_j)_{j\in J}$.*

(a) The trace-class operators form a linear subspace of $\mathbb{B}(H)$ and the trace norm $\|\cdot\|_1$ is a norm on $L^1(\mathbb{B}(H))$.

(b) $L^1(\mathbb{B}(H)) = \operatorname{span}(L^1(\mathbb{B}(H)) \cap \mathbb{B}(H)_+)$.

(c) The trace is independent of the chosen ONB.

(d) If $x \in L^1(\mathbb{B}(H))$, then $x^* \in L^1(\mathbb{B}(H))$.

Proof. (a) Let $x, y \in L^1(\mathbb{B}(H))$ and let $x + y = w|x + y|$ be the polar decomposition of $x + y$. Note that $|w^*x| \leq \|w\||x| \leq |x|$ and similar for w^*y. Thus $\operatorname{tr}(|w^*x|), \operatorname{tr}(|w^*y|) < \infty$. The previous lemma implies that

$$\sum_j |\langle e_j, w^*xe_j\rangle| \leq \operatorname{tr}(|w^*x|) \leq \operatorname{tr}(|x|)$$

and similar for w^*y. Thus

$$\sum_{j\in J}\langle e_j, |x+y|e_j\rangle = \sum_{j\in J}(\langle e_j, w^*xe_j\rangle + \langle e_j, w^*ye_j\rangle) \leq \operatorname{tr}(|x|) + \operatorname{tr}(|y|).$$

This means that $x+y \in L^1(\mathbb{B}(H))$ and $\|x+y\|_1 \leq \|x\|_1+\|y\|_1$. The remaining properties of a linear subspace and a norm are easy to show.

(b) If $x, y \in \mathbb{B}(H)$, then one can check that (exercise)

$$x|y| = \frac{1}{4}\sum_{k=0}^{3} i^k (x+i^k)|y|(x+i^k)^*.$$

For each $k \in \{0, \ldots, 3\}$ we have

$$\operatorname{tr}((x+i^k)|y|(x+i^k)^*) = \operatorname{tr}(|y|^{1/2}(x+i^k)^*(x+i^k)|y|^{1/2}) \leq \|x+i^k\|^2 \operatorname{tr}(|y|).$$

In particular, if $y \in L^1(\mathbb{B}(H))$, then $(x+i^k)|y|(x+i^k)^* \in L^1(\mathbb{B}(H))$ for all $k \in \{0, \ldots, 3\}$ and $x|y|$ is a linear combination of positive trace-class operators. Thus, if x is the partial isometry in the polar decomposition of y, then $x|y| = y$ is a linear combination of positive trace-class operators. That settles (b).

(c) follows now from (b) and the fact that the trace of positive trace-class operators is independent of the choice of the ONB.

(d) If $x \in L^1(\mathbb{B}(H))$, then x is a linear combination positive trace-class operators by (b). Thus x^* is also a linear combination of positive-trace-class operator, hence $x^* \in L^1(\mathbb{B}(H))$ by (a). □

Theorem 2.1.17. *If H is a Hilbert space, then $L^1(\mathbb{B}(H))$ is an ideal of $\mathbb{B}(H))$ and if $x, z \in \mathbb{B}(H)$, $y \in L^1(\mathbb{B}(H))$, then*

$$\begin{aligned} \|x\| &\leq \|x\|_1, \\ \operatorname{tr}(xy) &= \operatorname{tr}(yx), \\ \|y^*\|_1 &= \|y\|_1, \\ \|xyz\|_1 &\leq \|x\|\|y\|_1\|z\|. \end{aligned}$$

Proof. If $\xi \in H$ with $\|\xi\| = 1$, we can extend it to an ONB of H. Since the trace is independent of the ONB, we deduce

$$\|x\xi\|^2 = \langle \xi, x^*x\xi \rangle \leq \operatorname{tr}(x^*x) \leq \|x\|\|x\|_1.$$

Taking the supremum over all $\xi \in H$ with $\|\xi\| = 1$, we obtain $\|x\| \leq \|x\|_1$.

If $u \in \mathbb{B}(H)$ is unitary and the trace is independent of the ONB, we have

$$\operatorname{tr}(yu) = \sum_{j\in J} \langle e_j, yue_j \rangle = \sum_{j \in J} \langle ue_j, uyue_j \rangle = \operatorname{tr}(uy).$$

As x is a linear combination of four unitaries, we can use the linearity of the trace to get $\operatorname{tr}(yx) = \operatorname{tr}(xy)$.

Let $y = v|y|$ be the polar decomposition of y and $y^* = w|y^*|$ the polar decomposition of y^*. If we combine these two, we obtain $|y^*| = w^*|y|v^*$. Therefore,

$$\mathrm{tr}(|y^*|) = \mathrm{tr}(w^*|y|v^*) = \mathrm{tr}(v^*w^*|y|) \leq \mathrm{tr}(|v^*w^*|y||) \leq \mathrm{tr}(|y|).$$

The converse inequality follows by exchanging the roles of y and y^*.

For the last inequality, $|xyz| \leq \|x\||yz|$ implies $\|xyz\|_1 \leq \|x\|\|yz\|_1$. By the previous step, $\|yz\|_1 = \|z^*y^*\|_1$ and thus $\|yz\|_1 \leq \|z^*\|\|y^*\|_1 = \|z\|\|y\|_1$. □

Proposition 2.1.18. *The space of trace-class operators $L^1(\mathbb{B}(H))$ with the trace norm is a Banach space.*

Proof. Let (x_n) be a Cauchy sequence in $L^1(\mathbb{B}(H))$. Since $\|\cdot\| \leq \|\cdot\|_1$, the sequence (x_n) is also Cauchy for the operator norm. Hence there exists $x \in \mathbb{B}(H)$ such that $x_n \to x$ in operator norm. Then also $x_n^* \to x^*$ in operator norm and thus $|x_n| \to |x|$ by continuity of functional calculus.

Let (e_j) be an ONB of H. For every finite subset F of J we have

$$\sum_{j\in F}\langle e_j, |x|e_j\rangle = \lim_{n\to\infty}\sum_{j\in F}\langle e_j, |x_n|e_j\rangle \leq \liminf_{n\to\infty}\|x_n\|_1.$$

Thus $\mathrm{tr}(|x|) \leq \liminf_{n\to\infty}\|x_n\|_1$. In particular, $x \in L^1(\mathbb{B}(H))$.

It remains to show that $\|x_n - x\|_1 \to 0$. By an analogous argument, $\|x_n - x\|_1 \leq \liminf_{m\to\infty}\|x_n - x_m\|_1$. Since (x_n) is Cauchy in $L^1(\mathbb{B}(H))$, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for $m, n \geq N$. Hence, if $n \geq N$, then

$$\|x_n - x\|_1 \leq \liminf_{m\to\infty}\|x_n - x_m\|_1 \leq \sup_{m\geq N}\|x_n - x_m\| \leq \varepsilon.$$ □

Theorem 2.1.19. *Let H be a Hilbert space. For every $x \in \mathbb{B}(H)$ the map*

$$\psi_x\colon L^1(\mathbb{B}(H)) \to \mathbb{C},\ y \mapsto \mathrm{tr}(xy)$$

is a bounded linear functional and

$$\psi\colon \mathbb{B}(H) \to L^1(\mathbb{B}(H))^*,\ x \mapsto \psi_x$$

is an isometric isomorphism.

Proof. If $y \in L^1(\mathbb{B}(H))$, then $|\mathrm{tr}(xy)| \leq \|xy\|_1 \leq \|x\|\|y\|_1$. Thus $\psi_x \in L^1(\mathbb{B}(H))^*$ and $\|\psi_x\| \leq \|x\|$. Conversely,

$$\|x\| = \sup_{\|\xi\|,\|\eta\|\leq 1}|\langle \xi, x\eta\rangle| = \sup_{\|\xi\|,\|\eta\|\leq 1}|\mathrm{tr}(x\,|\xi\rangle\,\langle\eta|)| = \sup_{\|\xi\|,\|\eta\|\leq 1}|\psi_x(|\xi\rangle\,\langle\eta|)|.$$

Moreover,

$$\||\xi\rangle\langle\eta|\|_1 = \mathrm{tr}((|\eta\rangle\langle\xi|\,|\xi\rangle\langle\eta|)^{1/2}) = \mathrm{tr}(|\eta\rangle\langle\eta|) = 1.$$

Therefore, $\|x\| \le \|\psi_x\|$.

If $\varphi \in L^1(\mathbb{B}(H))^*$ and $\xi, \eta \in H$, then

$$|\varphi(|\xi\rangle\langle\eta|)| \le \|\varphi\|\||\xi\rangle\langle\eta|\|_1 \le \|\varphi\|\|\xi\|\|\eta\|.$$

By the Riesz representation theorem, there exists $x \in \mathbb{B}(H)$ such that

$$\varphi(|\xi\rangle\langle\eta|) = \langle\eta, x\xi\rangle = \psi_x(|\xi\rangle\langle\eta|)$$

for all $\xi, \eta \in H$. Since the finite-rank operators are dense in $L^1(\mathbb{B}(H))$ (exercise) and operators of the form $|\xi\rangle\langle\eta|$ with $\xi, \eta \in H$ span the finite-rank operators, we conclude $\varphi = \psi_x$. □

Exercises

1. Show that if A is a unital $*$-algebra and $a, b \in A$, then

$$ab = \frac{1}{4}\sum_{k=0}^{3} i^k (a + i^k) b (a + i^k)^*$$

2. Show that the finite-rank operators are dense in $L^1(\mathbb{B}(H))$.

2.2 Hilbert–Schmidt operators

Definition 2.2.1 (Hilbert–Schmidt operator)**.** Let H be a Hilbert space. An operator $x \in \mathbb{B}(H)$ is called *Hilbert–Schmidt operator* if $\mathrm{tr}(|x|^2) < \infty$. The space of Hilbert–Schmidt operators on H is denoted by $L^2(\mathbb{B}(H))$.

Proposition 2.2.2. *Let H be a Hilbert space.*

(a) If $x \in L^2(\mathbb{B}(H))$, then $x^* \in L^2(\mathbb{B}(H))$.

(b) $L^2(\mathbb{B}(H))$ is a linear subspace of $\mathbb{B}(H)$.

(c) If $x, y \in L^2(\mathbb{B}(H))$, then $xy, yx \in L^1(\mathbb{B}(H))$ and $\mathrm{tr}(xy) = \mathrm{tr}(yx)$.

(d) If $x \in L^2(\mathbb{B}(H))$ and $y \in \mathbb{B}(H))$, then $xy, yx \in L^2(\mathbb{B}(H))$ and $\mathrm{tr}(|xy|^2)$, $\mathrm{tr}(|yx|^2) \le \|y\|^2\,\mathrm{tr}(|x|^2)$.

Proof. (a) If $x \in \mathbb{B}(H)$, then $\operatorname{tr}(|x|^2) = \operatorname{tr}(x^*x) = \operatorname{tr}(xx^*) = \operatorname{tr}(|x^*|^2)$. Thus $x \in L^2(\mathbb{B}(H))$ if and only if $x^* \in L^2(\mathbb{B}(H))$.

(b) If $x, y \in L^2(\mathbb{B}(H))$, then $|x+y|^2 \leq |x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2$. Thus $\operatorname{tr}(|x+y|^2) \leq 2\operatorname{tr}(|x|^2) + 2\operatorname{tr}(|y|^2)$, which implies $x + y \in L^2(\mathbb{B}(H))$. The remaining properties of a linear subspace are easy to verify.

(c) If $x, y \in L^2(\mathbb{B}(H))$, then the polarization identity implies

$$xy = \frac{1}{4}\sum_{k=0}^{3} i^{-k}|x^* + i^k y|^2.$$

By (a) and (b), $|x^* + i^k y| \in L^2(\mathbb{B}(H))$. Thus $xy \in L^1(\mathbb{B}(H))$ and

$$\operatorname{tr}(xy) = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\operatorname{tr}(|x^* + i^k y|^2) = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\operatorname{tr}(|y^* + i^k x|^2) = \operatorname{tr}(yx).$$

(d) Since $|yx|^2 = x^*y^*yx \leq \|y\|^2|x|^2$, we have $\operatorname{tr}(|yx|^2) \leq \|y\|^2\operatorname{tr}(|x|^2)$. In particular, $yx \in L^2(\mathbb{B}(H))$. For xy we can use that $xy = (y^*x)^*$ and $\operatorname{tr}(z^*z) = \operatorname{tr}(zz^*)$ for all $z \in \mathbb{B}(H)$ to arrive at the same conclusion (exercise). □

Definition 2.2.3 (Hilbert–Schmidt inner product)**.** Let H be a Hilbert space. The *Hilbert–Schmidt inner product* on $L^2(\mathbb{B}(H))$ is defined as

$$\langle\cdot,\cdot\rangle_{\mathrm{HS}} \colon L^2(\mathbb{B}(H)) \times L^2(\mathbb{B}(H)) \to \mathbb{C},\ (x, y) \mapsto \operatorname{tr}(x^*y).$$

Just as the trace-class operators, the Hilbert–Schmidt operators form a complete normed space.

Lemma 2.2.4. *If H is a Hilbert space, then $L^2(\mathbb{B}(H))$ with the Hilbert–Schmidt inner product is a Hilbert space.*

Proof. The proof is analogous to the proof of completeness of $L^1(\mathbb{B}(H))$. But since this is such a standard argument, let us repeat it here to internalize it. If (x_n) is a Cauchy sequence in $L^2(\mathbb{B}(H))$, then it is also Cauchy with respect to the operator norm, hence there exists $x \in \mathbb{B}(H)$ such that $x_n \to x$ in operator norm.

If $(e_j)_{j\in J}$ is an ONB of H and $F \subset J$ is finite, then

$$\sum_{j\in F}\|xe_j\|^2 = \lim_{n\to\infty}\sum_{j\in F}\|x_n e_j\|^2 \leq \liminf_{n\to\infty}\|x_n\|_2^2.$$

Thus $x \in L^2(\mathbb{B}(H))$ and $\|x\|_2^2 \leq \liminf_{n\to\infty}\|x_n\|_2^2$.

It remains to show that $\|x_n - x\|_2 \to 0$. By an analogous argument, $\|x_n - x\|_2 \leq \liminf_{m\to\infty}\|x_n - x_m\|_1$. Since (x_n) is Cauchy in $L^2(\mathbb{B}(H))$, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\|x_n - x_m\| < \varepsilon$ for $m, n \geq N$. Hence, if $n \geq N$, then

$$\|x_n - x\|_2 \leq \liminf_{m\to\infty} \|x_n - x_m\|_2 \leq \sup_{m\geq N} \|x_n - x_m\| \leq \varepsilon. \qquad \square$$

Proposition 2.2.5 (Noncommutative Hölder inequality)**.** *Let H be a Hilbert space. If $x, y \in L^2(\mathbb{B}(H))$, then $xy \in L^1(\mathbb{B}(H))$ and*

$$\|xy\|_1 \leq \|x\|_2 \|y\|_2.$$

Proof. Let $xy = v|xy|$ be the polar decomposition of xy. We have $|xy| = v^*(xy) = (x^*v)^*y$. As seen before, $x^*v \in L^2(\mathbb{B}(H))$ and $\|x^*v\|_2 \leq \|x^*\|_2 = \|x\|_2$. By the Cauchy–Schwarz inequality,

$$\operatorname{tr}(|xy|) = \operatorname{tr}((x^*v)^*y) \leq \|x^*v\|_2 \|y\|_2 \leq \|x\|_2 \|y\|_2. \qquad \square$$

Every Hilbert space is isometrically isomorphic to an L^2 space over some measure space. In this setting, Hilbert–Schmidt operators have a very explicit representation in terms of kernel operators. We will only state this for the unit interval with the Lebesgue measure here. It is true for arbitrary σ-finite measure spaces, but requires some more measure theoretical tools (see the exercises). A proof can be found in Simon's book *Operator Theory* (Theorem 3.8.5) or in Peterson's notes (Theorem 2.2.3).

Proposition 2.2.6. *If $k \in L^2([0,1]^2)$, then*

$$T_k \colon L^2([0,1]) \to L^2([0,1]),\ (T_k f)(x) = \int_{[0,1]} k(x,y) f(y)\, dy$$

defines a Hilbert–Schmidt operator and $T_k^* = T_{k^*}$, where $k^*(x,y) = \overline{k(y,x)}$.

Moreover, the map

$$T \colon L^2([0,1]^2) \to L^2(\mathbb{B}(L^2([0,1]^2))),\ k \mapsto T_k$$

is an isometric isomorphism.

Exercises

1. Fill in the details of the proof of Proposition 2.2.2 (d).
2. Let $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be σ-finite measure spaces and let $\mathcal{F} \subset \mathcal{P}(X \times Y)$ be the σ-algebra generated by all sets of the form $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

(a) Show that there exists a unique measure $\mu \times \nu$ on $\mathcal{F}$ such that $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$.

(b) For $f \in L^2(X, \mu)$ and $g \in L^2(Y, \nu)$ let

$$f \otimes g \colon X \times Y \to \mathbb{C},\ (x, y) \mapsto f(x)g(y).$$

Show that $f \otimes g \in L^2(X \times Y, \mu \times \nu)$ and that $\operatorname{span}\{f \otimes g \mid f \in L^2(X, \mu),\ g \in L^2(Y, \nu)\}$ is dense in $L^2(X \times Y, \mu \times \nu)$.

(c) Show that if $(\xi_i)_{i \in I}$ is an ONB of $L^2(X, \mu)$ and $(\eta_j)_{j \in J}$ is an ONB of $L^2(Y, \nu)$, then $(\xi_i \otimes \eta_j)_{(i,j) \in I \times J}$ is an ONB of $L^2(X \times Y, \mu \times \nu)$.

(d) Show that Proposition 2.2.6 remains valid if $L^2([0, 1])$ is replaced by $L^2(X, \mu)$.

2.3 Schatten classes

Already the notation suggests that the trace-class operators and Hilbert–Schmidt operators are (noncommutative) analogs of the Lebesgue spaces L^1 and L^2. More generally, the Schatten classes are noncommutative analogs of L^p spaces for general exponents $p \in [1, \infty)$. We mostly state the results without proofs here. A more detailed account can be found in most functional analysis textbooks, for example Section 3.7 of Barry Simon's book on Operator Theory (A Comprehensive Course in Analysis, Part 4), which also comes with some interesting historial remarks.

Definition 2.3.1 (Schatten p-class)**.** Let H be a Hilbert space and $p \in [1, \infty)$. The Schatten p-class $L^p(\mathbb{B}(H))$ is defined as $L^p(\mathbb{B}(H)) = \{x \in \mathbb{B}(H) : \operatorname{tr}(|x|^p) < \infty\}$. The Schatten p norm on $L^p(\mathbb{B}(H))$ is defined by $\|x\|_p = \operatorname{tr}(|x|^p)^{1/p}$ for $x \in L^p(\mathbb{B}(H))$.

Lemma 2.3.2. *Let H be a Hilbert space. If $1 \leq p \leq q < \infty$, then $L^p(\mathbb{B}(H)) \subset L^q(\mathbb{B}(H))$.*

Proof. If $x \in L^p(\mathbb{B}(H))$, then $|x|^q = |x|^{p/2}|x|^{q-p}|x|^{p/2} \leq \||x|^{q-p}\||x|^p$. Thus $\operatorname{tr}(|x|^q) \leq \||x|^{q-p}\| \operatorname{tr}(|x|^p) < \infty$. □

Proposition 2.3.3. *Let H be a Hilbert space and $p \in [1, \infty)$. If $x \in L^p(\mathbb{B}(H))$ is self-adjoint, then H admits an orthonormal basis $(e_j)_{j \in J}$ consisting of eigenvectors of x, and if $(\lambda_j)_{j \in J}$ are the associated eigenvalues, then*

$$\|x\|_p = \left(\sum_{j \in J} |\lambda_j|^p\right)^{1/p}.$$

Proof. Let e be the spectral measure of x so that $x = \int_{\mathbb{R}} \lambda \, de(\lambda)$. For every $n \in \mathbb{N}$ we have $\operatorname{tr}(e(\mathbb{R}\setminus(-1/n, 1/n))) \leq n^p \operatorname{tr}(|x|^p) < \infty$. Thus $H_n = \operatorname{ran} e(\mathbb{R}\setminus(-1/n, 1/n))$ is finite-dimensional. Moreover, $H_n \subset H_{n+1}$ and $xH_n \subset H_n$.

As we already know that self-adjoint operators on finite-dimensional Hilbert spaces have an ONB consisting of eigenvectors, we can inductively define an orthonormal system $(e_i)_{i \in I}$ consisting of eigenvectors of x such that $H_n \subset \operatorname{span}\{e_i \mid i \in I\}$.

By the definition of a projection-valued measure, $\bigcup_{n=1}^{\infty} H_n$ is dense in $e(\{0\})^{\perp}$. Thus we can complete $(e_i)_{i \in I}$ to an ONB of H by adding an ONB of $e(\{0\})H$. As $e(\{0\})H = \ker x$, its elements are all eigenvectors of x (to the eigenvalue 0). Therefore, e_j is an eigenvector of x for every $j \in J$.

If we let $(\lambda_j)_{j \in J}$ denote the corresponding eigenvalues, then

$$\operatorname{tr}(|x|^p)^{1/p} = \left(\sum_{j \in J} \langle e_j, |x|^p e_j \rangle \right)^{1/p} = \left(\sum_{j \in J} |\lambda_j|^p \right)^{1/p}. \qquad \square$$

Remark 2.3.4. Using an ONB consisting of eigenvectors, it is easy to see that $\operatorname{tr}(x) = \sum_{j \in J} \lambda_j$ for $x \in L^1(\mathbb{B}(H))$ self-adjoint. The same formula is valid for arbitrary $x \in L^1(\mathbb{B}(H))$. This result is known as Lidskii's theorem.

Remark 2.3.5. From the previous proposition, it is not hard to deduce that $L^p(\mathbb{B}(H))$ consists of those $x \in \mathbb{B}(H)$ for which H admits an ONB consisting of eigenvectors of $|x|$ and the corresponding family of eigenvalues $(\lambda_j)_{j \in J}$ belongs to $\ell^p(J)$. Similarly, one can define subspaces of $\mathbb{B}(H)$ by replacing ℓ^p by any function space on J. One example (for the function space c_0) will be treated in section after the next.

Corollary 2.3.6. *Let H be a Hilbert space and $1 \leq p < \infty$. If $x \in L^p(\mathbb{B}(H)$, then $x^* \in L^p(\mathbb{B}(H))$ and $\|x\|_p = \|x^*\|_p$.*

Proposition 2.3.7 (Noncommutative Hölder inequality)**.** *Let H be a Hilbert space and $p, q \in (1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If $x \in L^p(\mathbb{B}(H))$ and $y \in L^q(\mathbb{B}(H))$, then $xy \in L^1(\mathbb{B}(H))$ and*

$$\operatorname{tr}(|xy|) \leq \|x\|_p \|y\|_q.$$

2.4 Interlude: Net convergence

Let X be a topological space. If the topology on X is induced by a metric, then all topological notions can be characterized by convergence of sequences: The closure $\overline{A}$ of $A \subset X$ is given by $\{x \in X \mid \exists (x_n) \text{ sequence in } A \colon x_n \to x\}$, a map f from X into a topological space Y is continuous if and only if $x_n \to x$ implies $f(x_n) \to f(x)$ etc.

This is not true for general topological spaces. One reason is that countable families such as sequences are not able to detect all open sets if there are too many of them. That is one reason to introduce nets indexed by possibly uncountable sets.

Definition 2.4.1 (Directed set, net)**.** A *directed set* is a pair $(I, \prec)$ consisting of a set I and a relation $\prec$ on I such that

- $i \prec i$ for all $i \in I$,
- $i \prec j$ and $j \prec k$ implies $i \prec k$ for all $i, j, k \in I$,
- for all $i, j \in I$ there exists $k \in I$ such that $i \prec k$ and $j \prec k$.

If X is a set, a *net* $(x_i)_{i \in I}$ in X is a map from a directed set I to X.

Example 2.4.2. The natural number with their usual ordering form a directed set. Thus every sequence is a net.

Example 2.4.3. If J is any set, then $\mathcal{P}(J)$ with the ordering $\subset$ forms a directed set. The same is true for the set $\{F \in \mathcal{P}(J) \mid F \text{ finite}\}$. The first two properties of a directed set are clear. For the third, it suffices to notice that if $A, B \subset J$, then $A \subset A \cup B$ and $B \subset A \cup B$.

Note that $\mathcal{P}(J)$ can be uncountable and $\subset$ is usually not a total order: There can be subsets A, B of J such that neither $A \subset B$ nor $B \subset A$.

Definition 2.4.4 (Net convergence)**.** Let X be a topological space and $x \in X$. A net $(x_i)_{i \in I}$ in X *converges* to x, denoted by $x_i \to x$, if for every neighborhood U of x there exists $i_0 \in I$ such that $x_i \in U$ whenever $i_0 \prec i$.

Example 2.4.5. Let $(\alpha_j)_{j \in J}$ be a family in $[0, \infty)$. The net $(\sum_{j \in F} \alpha_j)_{F \subset J \text{ finite}}$ converges to $\alpha \in [0, \infty)$ if and only if $\sup_{F \subset J \text{ finite}} \sum_{j \in F} \alpha_j = \alpha$.

Example 2.4.6. Let H be a Hilbert space with ONB $(e_j)_{j \in J}$. For every $\xi \in H$ the net $(\sum_{j \in F} \langle e_j, \xi \rangle e_j)_{F \subset J \text{ finite}}$ converges to ξ.

Contrary to sequences, nets are sufficient to characterize closure, continuity etc. in arbitrary topological spaces.

Lemma 2.4.7. *Let X be a topological space. For every $A \subset X$, the closure of A is given by $\overline{A} = \{x \in X \mid \exists \text{ net } (x_i)_{i \in I} \text{ in } A\colon x_i \to x\}$.*

Proof. If $x \in \overline{A}$, then $U \cap A \neq \emptyset$ for every open neighborhood of x. Let $\mathcal{U}(x)$ be the set of open neighborhoods of x, ordered by $U \prec V$ if $V \subset U$, and for every $U \in \mathcal{U}(x)$ let $x_U \in U \cap A$. Then $(x_U)_{U \in \mathcal{U}(x)}$ is a net in A that converges to x.

Conversely, if $x \in X \setminus \overline{A}$, then $X \setminus \overline{A}$ is an open neighborhood of x that does not contain any points in A. Thus no net in A can converge to x. □

Remark 2.4.8. Since a subset A of X is closed if and only if $\overline{A} = A$, limits of nets characterize the closed and thus also the open sets of X. In other words, a topology is completely determined by the limits of nets.

Proposition 2.4.9. *Let X and Y be topological spaces. For a map $f \colon X \to Y$, the following assertions are equivalent:*

(i) f is continuous.

(ii) If $(x_i)_{i \in I}$ is a net in X that converges to $x \in X$, then $f(x_i) \to f(x)$.

(iii) $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset X$.

Proof. (i) $\implies$ (ii): Let V be an open neighborhood of $f(x)$. Since f is continuous, $f^{-1}(V)$ is open and contains x. Hence there exists $i_0 \in I$ such that $x_i \in f^{-1}(V)$ for $i_0 \prec i$. Thus $f(x_i) \in V$ for $i_0 \prec i$. This means that $f(x_i) \to f(x)$.

(ii) $\implies$ (iii): If $x \in \overline{A}$, then by the previous lemma there exists a net $(x_i)_{i \in I}$ in A such that $x_i \to x$. By (ii), $f(x_i) \to f(x)$. Again by the previous lemma, this implies $f(x) \in \overline{f(A)}$.

(iii) $\implies$ (i): Let $C \subset Y$ be closed. By (iii), $f(\overline{f^{-1}(C)}) \subset C$. Thus $\overline{f^{-1}(C)} \subset f^{-1}(C)$. Hence $f^{-1}(C)$ is closed. □

One can also characterize compact subsets in terms of net convergence in a way that is similar to the sequence characterization in metric spaces. To do so, we need the definition of subnets, which is a little subtle.

Definition 2.4.10 (Subnet)**.** Let I, J be directed sets. A map $\beta \colon J \to I$ is called *order-preserving* if $j_1 \prec j_2$ implies $\beta(j_1) \prec \beta(j_2)$ and it is called *cofinal* if for every $i \in I$ there exists $j \in J$ such that $i \prec \beta(j)$. If X is a set and $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ are nets in X, then $(y_j)_{j \in J}$ is called a *subnet* of $(x_i)_{i \in I}$ if there exists an order-preserving cofinal map $\beta \colon J \to I$ such that $y_j = x_{\beta(j)}$ for all $j \in J$.

Remark 2.4.11. Note that we allow for a subnet to be indexed by a different directed set. In particular, a subnet of a sequence is not necessarily a subsequence (or a sequence at all).

Example 2.4.12. The sequence $(1, 2, 2, 3, 3, 3, \dots)$ is a subnet of $(1, 2, 3, \dots)$ even though it is clearly not a subsequence.

Example 2.4.13. Consider $\mathbb{N}^2$ with the order $(m_1, n_1) \prec (m_2, n_2)$ if $m_1 \leq m_2$ and $n_1 \leq n_2$. It is not hard to see that $\mathbb{N}^2$ with this order is a directed set. The map $\beta\colon \mathbb{N}^2 \to \mathbb{N}$, $(m, n) \mapsto m + n$ is order-preserving and cofinal. Hence $(m + n)_{(m,n)\in\mathbb{N}^2}$ is a subnet of $(n)_{n\in\mathbb{N}}$ even though it is not a sequence.

Theorem 2.4.14. *For a topological space X, the following properties are equivalent:*

(i) X is compact.

(ii) If $(C_i)_{i\in I}$ is a family of closed subsets of X such that $\bigcap_{i\in F} C_i \neq \emptyset$ for all $F \subset I$ finite, then $\bigcap_{i\in I} C_i \neq \emptyset$.

(iii) Every net in X has a convergent subnet.

Proof. (i) $\Longrightarrow$ (ii): If $\bigcap_{i\in I} C_i$ were empty, then $(X \setminus C_i)_{i\in I}$ would be an open cover of X. However, $\bigcup_{i\in F} X \setminus C_i = X \setminus \bigcap_{i\in F} C_i \neq X$ for every finite $F \subset I$. Hence $(X \setminus C_i)_{i\in I}$ has no finite subcover, contradicting the compactness of X.

(ii) $\Longrightarrow$ (iii): Let $(x_i)_{i\in I}$ be a net in X. For $i \in I$ define $C_i = \overline{\{x_j \mid i \prec j\}}$. If $F \subset I$ is finite, then there exists $j \in I$ such that $i \prec j$ for all $i \in F$. Hence $x_j \in \bigcap_{i\in F} C_i$. By (ii), there exists $x \in \bigcap_{i\in I} C_i$.

Let J be the set of all pair (U, i) where U is an open neighborhood of x and $i \in I$ such that $x_i \in U$. We order J by defining $(U_1, i_1) \prec (U_2, i_2)$ if $U_2 \subset U_1$ and $i_1 \prec i_2$. It is not hard to verify that J with this order is a directed set and the map $\beta\colon J \to I$, $(U, i) \mapsto i$ is order-preserving and cofinal. Hence $(x_i)_{(U,i)\in J}$ is a subnet of $(x_i)_{i\in I}$.

If V is an open neighborhood of x, then there exists $i_0 \in I$ such that $x_{i_0} \in V$ by definition of x. Moreover, if $(V, i_0) \prec (U, i)$, then $x_i \in U \subset V$. Therefore, the net $(x_i)_{(U,i)\in J}$ converges to x.

(iii) $\Longrightarrow$ (i): Suppose there exists an open cover $(U_i)_{i\in I}$ without finite subcover. Let $\mathcal{F}$ be the set of finite subsets of I, ordered by inclusion. As discussed before, $\mathcal{F}$ with this order is a directed set. For every $F \in \mathcal{F}$ there exists $x_F \in X \setminus \bigcup_{F\in\mathcal{F}} U_F$.

We claim that $(x_F)_{F\in\mathcal{F}}$ has no convergent subnet. Indeed, for every $x \in X$ there exists $i \in I$ such that U_i is an open neighborhood of x. Whenever $\{i\} \prec F$, then $x_F \notin U_i$. Thus $(x_i)_{i\in I}$ cannot have a convergent subnet. $\square$

2.5 Compact operators

Definition 2.5.1 (Weak topology)**.** Let H be a Hilbert space. The *weak topology* on H is the coarsest topology that makes the maps

$$H \to \mathbb{C},\, \xi \mapsto \langle \eta, \xi \rangle$$

continuous for all $\eta \in H$.

Remark 2.5.2. We know from the Riesz representation theorem that the map $H \to H^*$, $\xi \mapsto \langle \xi|$ is a norm-preserving bijection. It is then easy to see that this map is a homeomorphism when H is endowed with the weak topology and H^* with the weak* topology. In particular, the unit ball of H is compact in the weak topology as a consequence of the Banach–Alaoglu theorem.

Remark 2.5.3. In the light of the previous remark, we have the following characterization of the weak topology on H: A subset U of H is weakly open if and only if for every $\xi \in U$ there exist $\eta_1, \dots, \eta_n \in H$ and $\varepsilon > 0$ such that

$$\{\zeta \in H : |\langle \eta_k, \xi - \zeta \rangle| < \varepsilon \text{ for } 1 \leq k \leq n\} \subset U.$$

Lemma 2.5.4. *Let H be a Hilbert space. A net $(\xi_i)_{i \in I}$ converges weakly to $\xi \in H$ if and only if $\langle \eta, \xi_i \rangle \to \langle \eta, \xi \rangle$ for every $\eta \in H$.*

Proof. If $\xi_i \to \xi$ weakly, then $\langle \eta, \xi_i \rangle \to \langle \eta, \xi \rangle$ for all $\eta \in H$ since $\langle \eta|$ is weakly continuous for every $\eta \in H$ by definition.

For the converse implication let U be a weakly open neighborhood of ξ. By the previous remark, we may assume that there exist $\eta_1, \dots, \eta_n \in H$ and $\varepsilon > 0$ such that

$$U = \{\zeta \in H : |\langle \eta_k, \xi - \zeta \rangle| < \varepsilon \text{ for } 1 \leq k \leq n\}.$$

Since $\langle \eta_k, \xi_i \rangle \to \langle \eta_k, \xi \rangle$ for $1 \leq k \leq n$ by assumption, there exist $i_1, \dots, i_n \in I$ such that $|\langle \eta_k, \xi \rangle - \langle \eta_k, \xi_i \rangle| < \varepsilon$ for $i_k \prec i$, $1 \leq k \leq n$. By definition of a directed set, there exists $j \in I$ such that $i_1, \dots, i_n \prec j$. Thus $\xi_i \in U$ for $j \prec i$. Hence $(\xi_i)_{i \in I}$ converges weakly to ξ. □

Lemma 2.5.5. *Let H be a Hilbert space. Every $x \in \mathbb{B}(H)$ is continuous for the weak topology.*

Proof. If $\xi_i \to \xi$ weakly, then $\langle \eta, x\xi_i \rangle = \langle x^*\eta, \xi_i \rangle \to \langle x^*\eta, \xi \rangle = \langle \eta, x\xi \rangle$ for all $\eta \in H$, hence $x\xi_i \to x\xi$ weakly. □

Lemma 2.5.6. *Let H be a Hilbert space. If $K \subset H$ is a finite-dimensional subspace, then the weak topology restricted to K coincides with the norm topology.*

Proof. Let $(e_j)_{j\in J}$ be an ONB of H for which there exists $F \subset J$ finite such that $(e_j)_{j\in F}$ is an ONB of K. If $(\xi_i)_{i\in I}$ is a net in K and $\xi \in H$ such that $\xi_i \to \xi$ weakly, we have $\langle e_j, \xi\rangle = \lim_i \langle e_j, \xi_i\rangle$ for all $j \in J$. In particular, $\langle e_j, \xi\rangle = 0$ for $j \in J \setminus F$, which means that $\xi \in K$. Thus K is weakly closed.

Moreover,

$$\|\xi_i - \xi\|^2 = \sum_{j\in F} |\langle e_j, \xi_i - \xi\rangle|^2 \to 0.$$

Hence the identity map from K with the weak topology to K with the norm topology is continuous. The inverse is continuous by definition of the weak topology. Therefore, the weak topology coincides with the norm topology on K. □

Proposition 2.5.7. *Let H be a Hilbert space and let $(H)_1$ denote the closed unit ball of H. For $x \in \mathbb{B}(H)$, the following are equivalent:*

(i) $x \in \overline{\mathbb{F}(H)}^{\|\cdot\|}$.

(ii) $x|_{(H)_1}$ *is continuous from the weak topology to the norm topology.*

(iii) $x(H)_1$ *is compact in the norm topology.*

(iv) $x(H)_1$ *has compact closure in the norm topology.*

Proof. (i) $\Longrightarrow$ (ii): If $x \in \mathbb{F}(H)$, then the image of the unit ball is contained in a finite-dimensional subspace. By the previous lemmas, x is continuous for the weak topology and the weak topology on $x(H)_1$ coincides with the norm topology.

If $x \in \overline{\mathbb{F}(H)}^{\|\cdot\|}$, then $x|_{(H)_1}$ is a uniform limit of continuous functions (from the weak to the norm topology), hence itself continuous (exercise).

(ii) $\Longrightarrow$ (iii): As remarked above, $(H)_1$ is compact in the weak topology. (ii) implies that $x(H)_1$ is compact in the norm topology as image of a compact set under a continuous map.

(iii) $\Longrightarrow$ (iv): This is trivial.

(iv) $\Longrightarrow$ (i): Let $(e_i)_{i\in I}$ be an ONB of H. For $F \subset I$ finite let $P_F = \sum_{i\in F} |e_i\rangle\,\langle e_i|$. By definition of an ONB, for every $\eta \in H$ the net $(P_F\eta)_{F\subset I \text{ finite}}$ converges to η.

Suppose that $(P_F x)_{F\subset I \text{ finite}}$ does not converge to x in operator norm. Otherwise passing to a subnet, we may assume that there exist $\xi_F \in (H)_1$ and $\varepsilon > 0$ such that $\|P_F x\xi_F - x\xi_F\| \geq \varepsilon$ for all $F \subset I$ finite. Moreover, since $x(H)_1$ has compact closure by (iv), we may further assume that $x\xi_F \to \eta$ for some $\eta \in H$ (otherwise we can once more pass to a subnet).

We then obtain

$$\begin{aligned}\varepsilon &\leq \|P_F x\xi_F - x\xi_F\| \\ &\leq \|(1-P_F)(x\xi_F - \eta)\| + \|(1-P_F)\eta\| \\ &\leq \|x\xi_F - \eta\| + \|\eta - P_F\eta\|.\end{aligned}$$

The right side converges to zero, which gives a contradiction. □

Definition 2.5.8 (Compact operator)**.** Let H be a Hilbert space. An operator $x \in \mathbb{B}(H)$ that satisfies any of the four equivalent conditions from the previous lemma is called *compact*. The space of all compact operators on H is denoted by $\mathbb{K}(H)$.

Remark 2.5.9. It follows from (i) in the previous lemma that $\mathbb{K}(H)$ is a closed ideal of $\mathbb{B}(H)$.

Lemma 2.5.10. *Let H Hilbert space. The identity operator in H is compact if and only if H is finite-dimensional.*

Proof. If H is finite-dimensional, then $\mathbb{F}(H) = \mathbb{B}(H)$, hence every operator on H is compact. Conversely, if H is infinite-dimensional and $x \in \mathbb{F}(H)$, then there exists $\xi \in (\operatorname{ran} x)^\perp$ with $\|\xi\| = 1$ and we have

$$\|x - 1\| \geq \|(x-1)\xi\| = \|\xi\| = 1.$$

Thus $1 \notin \overline{\mathbb{F}(H)}^{\|\cdot\|}$. □

Lemma 2.5.11. *Let H be a Hilbert space. For any $p \in [1, \infty)$ the Schatten p-class $L^p(\mathbb{B}(H))$ is contained in $\mathbb{K}(H)$.*

Proof. Let $x \in L^p(\mathbb{B}(H))$ and let $x = u|x|$ be its polar decomposition. Since $\mathbb{K}(H)$ is a closed ideal, it suffices to show that $|x| \in \mathbb{K}(H)$. We know that there exists an ONB $(e_j)_{j\in J}$ of H and a family $(\lambda_j)_{j\in J}$ in $\ell^p(J)$ such that $\langle \xi, |x|\eta\rangle = \sum_{j\in J} \lambda_j \langle \xi, e_j\rangle\langle e_j, \eta\rangle$ for all $\xi, \eta \in H$ (exercise). In particular, for every $n \in \mathbb{N}$, the set $J_n = \{\lambda_j \mid j \in J,\, \lambda_j \geq 1/n\}$ is finite.

Let $x_n = \sum_{j\in J_n} \lambda_j \,|e_j\rangle\,\langle e_j|$. Clearly, $x_n \in \mathbb{F}(H)$. Moreover,

$$\begin{aligned}|\langle \xi, (|x| - x_n)\eta\rangle| &\leq \sum_{j\in J\setminus J_n} \lambda_j |\langle \xi, e_j\rangle||\langle e_j, \eta\rangle| \\ &\leq \frac{1}{n}\left(\sum_{j\in J} |\langle \xi, e_j\rangle|^2\right)^{1/2}\left(\sum_{j\in J} |\langle e_j, \eta\rangle|^2\right)^{1/2} \\ &= \frac{1}{n}\|\xi\|\|\eta\|.\end{aligned}$$

Hence $\||x| - x_n\| < \frac{1}{n}$. Therefore, $|x| \in \overline{\mathbb{F}(H)}^{\|\cdot\|} = \mathbb{K}(H)$. □

Proposition 2.5.12. *Let H be a Hilbert space. For every $x \in L^1(\mathbb{B}(H))$ the map*

$$\psi_x \colon \mathbb{K}(H) \to \mathbb{C},\ y \mapsto \operatorname{tr}(xy)$$

is a bounded linear functional and

$$\psi \colon L^1(\mathbb{B}(H)) \to \mathbb{K}(H)^*,\ x \mapsto \psi_x$$

is an isometric isomorphism.

Exercises

1. Justify the characterization of the weak topology given in Remark 2.5.3.

2. Let H be a Hilbert space, $p \in [1, \infty)$ and $x \in L^p(\mathbb{B}(H))$ self-adjoint. We have already seen that H admits an ONB $(e_j)_{j\in J}$ consisting of eigenfunctions of x and the family $(\lambda_j)_{j\in J}$ of the corresponding eigenvalues belongs to $\ell^p(J)$. Show that

$$\langle \xi, x\eta\rangle = \sum_{j\in J} \lambda_j \langle \xi, e_j\rangle\langle e_j, \eta\rangle$$

   for all $\xi, \eta \in H$, where the series on the right side is to be interpreted as net limit of the finite partial sums.

3. Let X be a topological space, E a normed space and $(f_i)_{i\in I}$ a net of continuous functions from X to E. Show that if $f \colon X \to E$ such that $\sup_{x\in X}\|f_i(x) - f(x)\| \to 0$, then f is continuous.

Chapter 3

Bonus: The structure of noncommutative C^*-algebras

We have seen in the first chapter that the commutative (unital) C^*-algebras are all of the form $C(X)$ for some compact Hausdorff space X. The situation is quite different for noncommutative C^*-algebras. in this chapter we will give characterization (of a different flavor) of all unital C^*-algebras.

3.1 States and the GNS construction

Lemma 3.1.1. *Let A be a unital C^*-algebra. For $\varphi \in A^*$, each pair of the following properties implies the third.*

(i) $\varphi(1) = 1$.

(ii) $\varphi(A_+) \subset [0, \infty)$.

(iii) $\|\varphi\| = 1$.

Proof. (i)+(ii) $\implies$ (iii): By (i), $\|\varphi\| \geq 1$. If $a \in A_+$, then $a \leq \|a\|$, hence $\varphi(a) \leq \varphi(\|a\|1) = \|a\|$ by (i) and (ii). Note that the map $(a, b) \mapsto \varphi(a^*b)$ is a positive sesquilinear form by (ii). By Cauchy–Schwarz, if $a \in A$, then $|\varphi(a)|^2 \leq \varphi(a^*a)\varphi(1) \leq \|a^*a\| = \|a\|^2$. Thus $\|\varphi\| \leq 1$.

(ii)+(iii) $\implies$ (i): By (ii), $\varphi(1) \in [0, \infty)$, and by (iii), $|\varphi(1)| \leq 1$. As in the previous part, we can apply Cauchy–Schwarz and positivity to see that $|\varphi(a)|^2 \leq \varphi(a^*a)\varphi(1) \leq \|a\|^2\varphi(1)^2$. Hence $\|\varphi\| \leq \varphi(1)$. By (iii), $\varphi(1) \geq 1$.

(i)+(iii) $\implies$ (ii): Let $a \in A_+$. We first show that $\varphi(a) \in \mathbb{R}$. Let $\alpha = \operatorname{Re}\varphi(a)$ and $\beta = \operatorname{Im}\varphi(a)$. For all $t \in \mathbb{R}$ we have

$$\alpha^2 + (\beta + t)^2 = |\varphi(a + it)|^2 \leq \|a + it\|^2 = \|a\|^2 + t^2,$$

where we used that $\|a+it\|^2 = \|\Gamma(a+it)\|^2 = \|\Gamma(a)+it\|^2 = \|\Gamma(a)\|^2 + t^2$ since $\Gamma(a)$ is real-valued. As a consequence,

$$2\beta t + t^2 \leq \alpha^2 + (\beta+t)^2 \leq \|a\|^2 + t^2,$$

which implies $2\beta t \leq \|a\|$. This inequality can only hold for all $t \in \mathbb{R}$ if $\beta = 0$.

To show that $\varphi(a) \geq 0$, first note that $0 \leq a \leq \|a\|$, which implies $0 \leq \|a\| - a \leq \|a\|$, hence $\|\|a\| - a\| \leq \|a\|$. Thus

$$\varphi(a) = \varphi(\|a\| - (\|a\| - a)) = \|a\| - \varphi(\|a\| - a) \geq \|a\| - \|\|a\| - a\| \geq 0. \quad \square$$

Definition 3.1.2 (State)**.** Let A be a unital C^*-algebra. A *state* on A is a functional that satisfies any two of the three properties from the previous lemma. The set of all states on A is denoted by $S(A)$ and called the *state space* of A.

Remark 3.1.3. As we used in the proof of the previous lemma, if $\varphi \in S(A)$, then $(a,b) \mapsto \varphi(a^*b)$ is a positive sesquilinear form (not quite an inner product because it can fail to be positive definite). This is crucial for the GNS construction, which we will study next.

Lemma 3.1.4. *Let A be a unital C^*-algebra and B a unital C^*-subalgebra. Every state on B can be extended to a state on A.*

Proof. If $\varphi \in S(B)$, then it can be extended to a linear function $\psi \in A^*$ with $\|\psi\| = \|\varphi\| = 1$ by the Hahn–Banach theorem. As $\psi(1) = \varphi(1) = 1$, the functional ψ is a state on A. $\square$

Lemma 3.1.5. *Let A be a unital C^*-algebra and $\varphi \in S(A)$. If $a \in A$ such that $\varphi(a^*a) = \varphi(aa^*) = |\varphi(a)|^2$, then $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(ba) = \varphi(b)\varphi(a)$ for all $a, b \in A$.*

Proof. Let $a, b \in A$. By Cauchy–Schwarz,

$$\begin{aligned} |\varphi(ab) - \varphi(a)\varphi(b)| &= |\varphi((a^* - \varphi(a^*))^* b)| \\ &\leq \varphi((a - \varphi(a))(a^* - \varphi(a^*)))^{1/2} \varphi(b^*b)^{1/2}. \end{aligned}$$

If $\varphi(aa^*) = |\varphi(a)|^2$, then

$$\varphi((a - \varphi(a))(a^* - \varphi(a^*))) = 2\varphi(aa^*) - 2\varphi(a)\varphi(a^*) = 0.$$

Thus $\varphi(ab) = \varphi(a)\varphi(b)$. The proof of the second claim is analogous. $\square$

Lemma 3.1.6. *Let A be a unital C^*-algebra. If $\varphi \in S(A)$, then $N_\varphi = \{a \in A \mid \varphi(a^*a) = 0\}$ is a closed linear subspace of A, $aN_\varphi \subset N_\varphi$ for all $a \in A$ and*

$$\langle \cdot, \cdot \rangle_\varphi \colon A/N_\varphi \times A/N_\varphi \to \mathbb{C}, (a + N_\varphi, b + N_\varphi) \mapsto \varphi(a^*b)$$

is an inner product.

Proof. If $a, b \in N_\varphi$, then $|a+b|^2 \leq |a+b|^2 + |a-b|^2 \leq 2|a|^2 + 2|b|^2$, hence $\varphi(|a+b|^2) \leq 2\varphi(|a|^2) + 2\varphi(|b|^2) = 0$, which means $a + b \in N_\varphi$. The other properties of a closed linear subspace are easy to show. If $a \in A$ and $b \in N_\varphi$, then $|ab|^2 \leq \|a\|^2|b|^2$, hence $\varphi(|ab|^2) \leq \|a\|^2\varphi(|b|^2) = 0$, which means $ab \in N_\varphi$.

The only property of an inner product that is not quite obvious is that $\langle \cdot, \cdot \rangle_\varphi$ is well-defined, i.e. $\varphi(a_1^*b_1) = \varphi(a_2^*b_2)$ if $a_1 - a_2 \in N_\varphi$ and $b_1 - b_2 \in N_\varphi$. To see this, we use the Cauchy–Schwarz inequality:

$$\begin{aligned} |\varphi(a_1^*b_1) - \varphi(a_2^*b_2)| &\leq |\varphi((a_1 - a_2)^*b_1)| + |\varphi(a_2^*(b_1 - b_2))| \\ &\leq \varphi(|a_1 - a_2|^2)^{1/2}\varphi(|b_1|^2)^{1/2} + \varphi(|a_2|^2)^{1/2}\varphi(|b_1 - b_2|^2)^{1/2} \\ &= 0. \end{aligned}$$

□

The inner product space in the previous lemma is in general not complete. However, one can always extend the inner product to a bigger space which is complete an contains the original space as a dense subspace. This process is known as completion (see the exercises).

Definition 3.1.7 (GNS Hilbert space, vacuum vector)**.** Let A be a unital C^*-algebra and $\varphi \in S(A)$. The completion of A/N_φ with respect to the inner product $\langle \cdot, \cdot \rangle_\varphi$ is called the *GNS Hilbert space* of φ and denoted by H_φ. The image of $1 + N_\varphi$ inside H_φ is denoted by Ω_φ.

Lemma 3.1.8. *Let A be a unital C^*-algebra and $\varphi \in S(A)$. For every $a \in A$ there exists a unique operator $\pi_\varphi(a) \in \mathbb{B}(H_\varphi)$ such that $\pi_\varphi(b + N_\varphi) = ab + N_\varphi$. Moreover, the map $\pi_\varphi \colon A \to \mathbb{B}(H_\varphi)$ is a unital $*$-homomorphism.*

Proof. If $a, b \in A$, then

$$\langle ab + N_\varphi, ab + N_\varphi \rangle_\varphi = \varphi(b^*a^*ab) \leq \|a\|^2\varphi(b^*b) = \|a\|^2\langle b + N_\varphi, b + N_\varphi \rangle_\varphi.$$

Hence, if $(b_n + N_\varphi)$ is a Cauchy sequence in A/N_φ, then $(ab_n + N_\varphi)$ is also a Cauchy sequence. Hence we can define $\pi_\varphi(a)\xi = \lim_{n\to\infty}(ab_n + N_\varphi)$ whenever $\xi \in H_\varphi$ and $b_n + N_\varphi \to \xi$. It is not hard to check that this definition is independent of the chosen sequence $(b_n + N_\varphi)$ and $\pi_\varphi(a)$ thus defined is a linear operator.

Boundedness follows directly from the estimate in the previous paragraph and uniqueness is a direct consequence of the density of A/N_φ in H_φ. The fact that π_φ is a unital $*$-homomorphism is easy to check. □

Definition 3.1.9 (GNS representation)**.** Let A be a unital C^*-algebra and $\varphi \in S(A)$. The $*$-homomorphism π_φ defined in the previous lemma is called the *GNS representation* induced by φ.

Example 3.1.10. Let X be a compact Hausdorff space and $x \in X$. Clearly, $\varphi\colon C(X) \to \mathbb{C}$, $f \mapsto f(x)$ is a state on $C(X)$. In this case, $N_\varphi = \{f \in C(X) \mid f(x) = 0\}$ and $C(X)/N_\varphi \to \mathbb{C}$, $f + N_\varphi \mapsto f(x)$ is an isomorphism. Thus $H_\varphi \cong \mathbb{C}$ (in this case, there is no need for a completion) and $\pi_\varphi = \varphi$. Note that this representation is far from being injective.

Example 3.1.11. Let $K \subset \mathbb{C}$ be compact and $\varphi\colon C(K) \to \mathbb{C}$, $f \mapsto \int_K f\,d\mu$, where μ is the normalized Lebesgue measure on K. In this case $N_\varphi = \{0\}$, $H_\varphi = L^2(K,\mu)$, $\Omega_\varphi = 1$ and $\pi_\varphi(f)g = fg$. This representation of $C(K)$ on $L^2(K,\mu)$ is injective.

Example 3.1.12. Let H be a Hilbert space. If $\rho \in L^1(\mathbb{B}(H))$ is positive with $\mathrm{tr}(\rho) = 1$, then $\varphi\colon \mathbb{B}(H) \to \mathbb{C}$, $x \mapsto \mathrm{tr}(x\rho)$ is a state.

If $\rho = |\xi\rangle\langle\xi|$ for $\xi \in H$ with $\|\xi\| = 1$, then $N_\varphi = \{x \in \mathbb{B}(H) \mid \xi \in \ker x\}$ and $U\colon \mathbb{B}(H)/N_\varphi \to H$, $x + N_\varphi \mapsto x\xi$ is an isometry for $\langle\,\cdot\,,\cdot\,\rangle_\varphi$. Thus $H_\varphi \cong H$ via U (again, there is no need for a completion) and $\pi_\varphi(x) = U^*xU$.

On the other hand, if $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis of H and $\rho = \sum_{n=1}^\infty \lambda_n |e_n\rangle\langle e_n|$ with $\lambda_n > 0$ for all $n \in \mathbb{N}$, then $N_\varphi = \{0\}$ and $U\colon \mathbb{B}(H) \to L^2(\mathbb{B}(H))$, $x \mapsto x\rho^{1/2}$ extends to a unitary from H_φ to $L^2(\mathbb{B}(H))$. Moreover, $U\pi_\varphi(x)U^*y = xy$ for $x \in \mathbb{B}(H)$ and $y \in L^2(\mathbb{B}(H))$.

Exercises

1. Let V be an inner product space and let $\mathcal{C}(V)$ be the set of all Cauchy sequences in V with elementwise addition and scalar multiplication.

   (a) Show that $\mathcal{N} = \{(x_n) \in \mathcal{C}(V) : \langle x_n, x_n\rangle \to 0\}$ is a linear subspace of $\mathcal{C}(V)$.

   (b) Let $H = \mathcal{C}(V)/\mathcal{N}$. Show that
   $$\langle\cdot,\cdot\rangle_H\colon H \times H \to \mathbb{C},\ ((x_n) + \mathcal{N}, (y_n) + \mathcal{N}) \mapsto \lim_{n\to\infty} \langle x_n, y_n\rangle$$
   is an inner product and H is complete in this inner product.

   (c) Show that the equivalence classes of constant sequences form a dense linear subspace of H.

So far we have constructed a Hilbert space H and an isometric embedding $i\colon V \to H$, namely the map that takes an element $\xi \in H$ to $(\xi) + \mathcal{N}$, such that $i(V)$ is dense in H.

(a) Show that if H_1, H_2 are Hilbert spaces and $i_1\colon V \to H_1$, $i_2\colon V \to H_2$ are isometric embeddings with dense image, then there exists a unique unitary $u\colon H_1 \to H_2$ such that $u \circ i_1 = i_2$.

In this sense, the pair (H, i) constructed in the first part is essentially unique. It is called the completion of the inner product space V.

2. Uniqueness of the GNS representation: Let A be a C^*-algebra and $\varphi \in S(A)$. Show that if H is a Hilbert space, $\Omega \in H$ is a unit vector and $\pi\colon A \to \mathbb{B}(H)$ is a unital $*$-homomorphism such that $\{\pi(a)\Omega \mid a \in A\}$ is dense in H and $\langle \Omega, \pi(a)\Omega\rangle = \varphi(a)$ for all $a \in A$, then there exists a unitary $U\colon H_\varphi \to H$ such that $U\Omega_\varphi = \Omega$ and $U\pi_\varphi(a) = \pi(a)U$ for all $a \in A$.

3.2 The Gelfand–Naimark–Segal theorem

In this section we prove the main structure theorem for noncommutative (unital) C^*-algebras due to Gelfand, Naimark and Segal, which states that every C^*-algebra can be faithfully represented on a Hilbert space. In this sense, the class of abstract C^*-algebras (defined through the axioms at the beginning of this course) is the same as the class of concrete C^*-algebras (closed self-adjoint subalgebras of the bounded operators on a Hilbert space), up to isomorphism. This result was one of the big achievements of early C^*-algebra theory.

The GNS construction provides a technique to produce a Hilbert space representation from a state on a C^*-algebra. As the examples in the previous section show, this representation may fail to be injective. The next lemma shows that there are always sufficiently many states on a C^*-algebra, which is the crucial step to build a faithful representation from the full state space.

Lemma 3.2.1. *Let A be a unital C^*-algebra. For every $a \in A_+$ there exists $\varphi \in S(A)$ such that $\varphi(a) = \|a\|$.*

Proof. Let B be the unital C^*-algebra generated by a. Since a is normal, B is commutative. Thus $\Gamma(B)$ is a compact Hausdorff space and $\hat{a}\colon \Gamma(B) \to \sigma(a)$ is continuous. Let $\chi \in \Gamma(B)$ such that $\chi(a) = \hat{a}(\chi) = \|\hat{a}\| = \|a\|$.

Since $\chi \in \Gamma(B)$, we have $\chi(1) = 1$ and $\chi(b^*b) = |\chi(b)|^2 \geq 0$ for all $b \in B$. By the previous lemma, $\|\chi\| = 1$. By the Hahn–Banach theorem, there exists

$\varphi \in A^*$ with $\|\varphi\| = 1$ that extends $\chi \circ \Gamma$. In particular, $\varphi(1) = 1$, which implies that $\varphi \in S(A)$, and $\varphi(a) = \|a\|$. $\square$

Theorem 3.2.2 (Gelfand–Naimark–Segal)**.** *Every unital C^*-algebra is $*$-isomorphic to a closed unital $*$-subalgebra of $\mathbb{B}(H)$ for some Hilbert space H.*

Proof. Let A be a unital C^*-algebra. For $\varphi \in S(A)$ let consider the GNS Hilbert space H_φ and the GNS representation $\pi_\varphi \colon A \to \mathbb{B}(H)$. We let $H = \bigoplus_{\varphi \in S(A)} H_\varphi$ and define $\pi \colon A \to \mathbb{B}(H)$ by $\pi(a)(\xi_\varphi)_{\varphi \in S(A)} = (\pi_\varphi(a)\xi_\varphi)_{\varphi \in S(A)}$. It is not hard to check that π is a unital $*$-homorphism.

To see that π is injective, let $a \in \ker \pi$. By the previous lemma, there exists $\varphi \in S(A)$ such that $\varphi(a^*a) = \|a\|^2$. Let $\xi_\psi = \Omega_\varphi$ if $\psi = \varphi$ and $\xi_\psi = 0$ otherwise. Clearly, $(\xi_\psi)_{\psi \in S(A)}$ belongs to H and

$$\|\pi(a)(\xi_\psi)_{\psi \in S(A)}\|^2 = \sum_{\psi \in S(A)} \|\pi_\psi(a)\xi_\psi\|^2 = \|\pi_\varphi(a)\Omega_\varphi\|^2 = \varphi(a^*a).$$

Hence, $a \in \ker \pi$ implies $\|a\| = 0$. $\square$

Example 3.2.3 (Calkin algebra). Let H be a separable infinite-dimensional Hilbert space. Since $\mathbb{K}(H) \trianglelefteq \mathbb{B}(H)$, the quotient $\mathbb{B}(H)/\mathbb{K}(H)$ is a unital algebra. Moreover, the $*$-operation descends to to an involution of $\mathbb{B}(H)/\mathbb{K}(H)$. The quotient norm

$$\|x + \mathbb{K}(H)\| = \inf\{\|y\| : x - y \in \mathbb{K}(H)\}$$

is makes $\mathbb{B}(H)/\mathbb{K}(H)$ into a C^*-algebra (but it takes some more advanced tools to prove this).

It is not obvious that $\mathbb{B}(H)/\mathbb{K}(H)$ can be faithfully represented on any Hilbert space. In fact, $\mathbb{B}(H)/\mathbb{K}(H)$ cannot be faithfully represented on a *separable* Hilbert space: Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H, let $(S_i)_{i\in I}$ be an uncountable family of infinite subsets of $\mathbb{N}$ with the property that $S_i \cap S_j$ is finite for $i \neq j$ and let p_i be the orthogonal projection onto $\overline{\operatorname{span}\{e_n \mid n \in S_i\}}$. Since p_ip_j is the projection onto $\overline{\operatorname{span}\{e_n \mid n \in S_i \cap S_j\}}$, which is finite-dimensional, we have $p_ip_j + \mathbb{K}(H) = \mathbb{K}(H)$. in other words, $(p_i + \mathbb{K}(H))_{i\in I}$ is an uncountable family of non-zero orthogonal projections in $(\mathbb{B}(H)/\mathbb{K}(H))$.

However, if K is separable, then $\mathbb{B}(K)$ cannot contain an uncountable family of non-zero orthogonal projections: If one takes an orthonormal basis of the range of each projection, then their union is an uncountable orthonormal set in K. But since K is separable, each orthonormal set is at most countable.

Definition 3.2.4 (Pure state, irreducible representation)**.** Let A be a unital C^*-algebra. A state φ on A is called *pure* if for every pair of states ψ_1, ψ_2 on A and $\lambda \in (0,1)$ such that $\varphi = \lambda\psi_1 + (1-\lambda)\psi_2$ one has $\psi_1 = \psi_2 = \varphi$.

Let H be a Hilbert space and $\pi\colon A \to \mathbb{B}(H)$ a unital $*$-homomorphism. A subspace K of H is called *invariant* if $\pi(a)K \subset K$ for all $a \in A$. The representation π is called *irreducible* if the only closed invariant subspaces are $\{0\}$ and H.

Remark 3.2.5. Let C be a convex subset of a vector space. An element $x \in C$ is called an *extreme point* of C if whenever $x = \lambda y + (1-\lambda)z$ for $y, z \in C$ and $\lambda \in (0,1)$, then $x = y = z$. In other words, there is no line segment in C passing through x. Hence the pure states on a unital C^*-algebra A are exactly the extreme points of the state space.

By the Banach–Alaoglu theorem, the state space $S(A)$ is compact in the weak* topology. Moreover, the Krein–Milman theorem (which we will also not prove in this course) states that if C is a closed weak* compact subset of the dual space of a Banach space, then the convex combinations of the extreme points of C are dense in C. Therefore, the convex combinations of pure states on A are dense in $S(A)$.

Lemma 3.2.6 (Schur)**.** *Let A be a unital C^*-algebra and H a Hilbert space. A unital $*$-homomorphism $\pi\colon A \to \mathbb{B}(H)$ is an irreducible representation if and only if every $x \in \mathbb{B}(H)$ that commutes with $\pi(a)$ for all $a \in A$ is a scalar multiple of the identity.*

Proof. Let $\pi(A)' = \{x \in \mathbb{B}(H) \mid x\pi(a) = \pi(a)x \text{ for all } a \in A\}$. First assume that $\pi(A)' = \mathbb{C}1$. If $K \subset H$ is a closed invariant subspace, let p denote the orthogonal projection onto K. If If $a \in A$ is self-adjoint, then $\pi_\varphi(a)p = p\pi_\varphi(a)p$ and thus $p\pi_\varphi(a) = p\pi_\varphi(a)p = \pi_\varphi(a)p$ by taking adjoints. Since A is spanned by its self-adjoint elements, $\pi_\varphi(a)p = p\pi_\varphi(a)$ for all $a \in A$, that is, $p \in \pi(A)'$. As $\pi(A)' = \mathbb{C}1$, we must either have $p = 0$ (and thus $K = 0$) or $p = 1$ (and thus $K = H$).

Assume conversely that π is irreducible. If $x \in \pi(A)'$, then

$$x^*\pi(a) = (\pi(a^*)x)^* = (x\pi(a^*))^* = \pi(a)x^*$$

for all $a \in A$, hence $x^* \in \pi(A)'$. As $\pi(A)'$ is clearly a linear subspace, it follows that $\pi(A)'$ is spanned by its self-adjoint element.

Now let $x \in \pi(A)'$ be self-adjoint and let e denote its spectral measure so that $x = \int \lambda\, de(\lambda)$. If $u \in A$ is unitary, it is not hard to check that $\pi(u)e(\cdot)\pi(u^*)$ is again a spectral measure and $\int f(\lambda)\, d(\pi(u)e(\lambda)\pi(u)^*) = \pi(u)f(x)\pi(u)^*$ for all bounded Borel functions $f\colon \mathbb{R} \to \mathbb{R}$. In particular, as $x \in \pi(A)'$, we have $x = \pi(u)x\pi(u)^* = \int \lambda\, d(\pi(u)e(\lambda)\pi(u)^*)$. by the uniqueness of the spectral measure, we conclude $\pi(u)e(S) = e(S)\pi(u)$ for all Borel sets $S \subset \mathbb{R}$.

Since A is spanned by its unitary elements, it follows that $e(S) \in \pi(A)'$ for all Borel sets $S \subset \mathbb{R}$. Thus $e(S)H$ is a closed invariant subspace for all Borel sets $S \subset \mathbb{R}$. As π is irreducible, $e(S) = 0$ or $e(S) = 1$ for all Borel sets $S \subset \mathbb{R}$. With $\lambda = \sup\{\mu \in \mathbb{R} \mid e((-\infty, \mu]) = 0\}$ we obtain $x = \lambda 1$ (exercise) □

Proposition 3.2.7. *Let A be a unital C^*-algebra. A state φ on A is pure if and only if the GNS representation π_φ is irreducible.*

Proof. First assume that φ is pure. Let $K \subset H_\varphi$ be a closed invariant subspace and let p denote the orthogonal projection onto K. As seen in the proof of the previous lemma, $p \in \pi(A)'$.

Let $\xi = p\Omega_\varphi$, $\eta = (1-p)\Omega_\varphi$. If $\xi, \eta \neq 0$, then

$$\varphi(a) = \langle \Omega_\varphi, \pi_\varphi(a)\Omega_\varphi \rangle = \|\xi\|^2 \frac{\langle \xi, \pi_\varphi(a)\xi \rangle}{\|\xi\|^2} + \|\eta\|^2 \frac{\langle \eta, \pi_\varphi(a)\eta \rangle}{\|\eta\|^2}$$

for all $a \in A$. Since φ is a pure state,

$$\|\pi_\varphi(a)\Omega_\varphi\|^2 = \varphi(a^*a) = \frac{\|\pi_\varphi(a)\xi\|^2}{\|\xi\|^2} = \frac{\|p\pi_\varphi(a)\Omega_\varphi\|^2}{\|p\Omega_\varphi\|^2}.$$

Since $\pi_\varphi(A)\Omega_\varphi$ is dense in H_φ, we obtain $\|p\Omega_\varphi\|^2\|\zeta\|^2 = \|p\zeta\|^2$ for all $\zeta \in H_\varphi$, which can only hold if $p = 0$ or $p = 1$. But this contradicts $\xi, \eta \neq 0$.

If say $\xi = 0$, then $0 = \pi_\varphi(a)p\Omega_\varphi = p\pi_\varphi(a)\Omega_\varphi$ for all $a \in A$, hence $p = 0$ and thus $K = 0$. In the other case $\eta = 0$, an analogous argument shows $K = H$.

For the converse implication assume that π_φ is irreducible. Let $\psi_1, \psi_2 \in S(A)$ and $\lambda \in (0,1)$ such that $\varphi = \lambda\psi_1 + (1-\lambda)\psi_2$. Consider the map

$$U\colon \pi_\varphi(A) \to H_{\psi_1} \oplus H_{\psi_2},\ \pi_\varphi(a)\Omega_\varphi \mapsto (\sqrt{\lambda}\pi_{\psi_1}(a)\Omega_{\psi_1}, \sqrt{1-\lambda}\pi_{\psi_2}(a)\Omega_{\psi_2}).$$

It is not too hard to see that U is well-defined and extends to an isometry on H_φ such that $U\pi_\varphi(a) = (\pi_{\psi_1}(a) \oplus \pi_{\psi_2}(a))U$ for all $a \in A$ (exercise).

Let p denote the orthogonal projection from $H_{\psi_1} \oplus H_{\psi_2}$ onto H_{ψ_1}. Clearly, $p(\pi_{\psi_1}(a) \oplus \pi_{\psi_2}(a)) = (\pi_{\psi_1}(a) \oplus \pi_{\psi_2}(a))p$ for all $a \in A$.

Thus $U^*pU \in \pi_\varphi(A)'$. Since π_φ is assumed to be irreducible, there exists $\mu \geq 0$ such that $U^*pU = \mu 1$ by the previous lemma. Moreover,

$$\mu\varphi(a) = \langle \Omega_\varphi, U^*pU\pi_\varphi(a)\Omega_\varphi \rangle = \lambda\langle \Omega_{\psi_1}, \pi_{\psi_1}(a)\Omega_{\psi_1} \rangle = \lambda\psi_1(a)$$

for all $a \in A$. For $a = 1$ we obtain $\mu = \lambda$ and therefore $\varphi = \psi_1$. □

Corollary 3.2.8. *Let A be a unital commutative C^*-algebra and H a Hilbert space.*

(a) A representation $\pi\colon A \to \mathbb{B}(H)$ is irreducible if and only if $\dim H = 1$.

(b) A state on A is pure if and only if it is a character.

Proof. (a) If $\dim H = 1$, the only subspaces are $\{0\}$ and H. Conversely, if π is irreducible, then $\pi(A) \subset \pi(A)' = \mathbb{C}1$ since A is commutative. But every operator on H commutes with multiples of the idenity, hence $\mathbb{B}(H) = \pi(A)' = \mathbb{C}1$, which is only the case of $\dim H = 1$.

(b) If φ is a character, then $\varphi((a - \varphi(a))^*(a - \varphi(a))) = 0$, hence $a\Omega_\varphi = \varphi(a)\Omega_\varphi$. Thus $H_\varphi = \mathbb{C}\Omega_\varphi$ is 1-dimensional. By (a), π_φ is irreducible, which implies that φ is pure by the previous proposition.

If φ is a pure state, then $H_\varphi = \mathbb{C}\Omega_\varphi$ by (a) and the previous proposition, hence $\pi_\varphi(a)\Omega_\varphi = \varphi(a)\Omega_\varphi$ for all $a \in A$. Since π_φ is multiplicative, φ is a character. □

Exercises

1. Let $(H_j)_{j\in J}$ be a family of Hilbert spaces and let

$$\bigoplus_{j\in J} H_j = \{(\xi_j)_{j\in J} \mid \xi_j \in H_j,\ (\|\xi_j\|)_{j\in J} \in \ell^2(J)\}.$$

Show that

$$\langle\cdot,\cdot\rangle\colon \bigoplus_{j\in J} H_j \times \bigoplus_{j\in J} H_j \to \mathbb{C},\ ((\xi_j),(\eta_j)) \mapsto \sum_{j\in J}\langle \xi_j, \eta_j\rangle$$

defines an inner product that makes $\bigoplus_{j\in J} H_j$ into a Hilbert space.

2. Let H be a Hilbert space and $x \in \mathbb{B}(H)$ self-adjoint. Show that if $\mathbb{1}_S(x) = 0$ or $1_S(x) = 1$ for every Borel set $S \subset \mathbb{R}$, then

$$\sup\{\mu \in \mathbb{R} \mid \mathbb{1}_{(-\infty,\mu]}(x) = 0\} = \inf\{\mu \in \mathbb{R} \mid \mathbb{1}_{(\mu,\infty)}(x) = 1\},$$

and if we denote this common value by λ, then $x = \lambda 1$.

3. Show that every finite-dimensional unital C^*-algebra is $*$-isomorphic to $M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ for some $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$.

Appendix A

The Hahn–Banach theorem

The Hahn–Banach theorem is one of the most important results in functional analysis. If provides one with "sufficiently many" bounded linear functionals on a Banach space in order to distinguish its elements. This is an almost universally useful fact.

In contrast to the rest of this course, we consider both real and complex vector spaces in this section. We denote by $\mathbb{K}$ a base field that is either $\mathbb{C}$ or $\mathbb{R}$.

To start with, recall the following result from set theory.

Lemma A.0.1 (Zorn)**.** *If $\mathcal{P}$ is a non-empty partially ordered set such that every chain in $\mathcal{P}$ has an upper bound, then $\mathcal{P}$ has a maximal element.*

Remark A.0.2. The terminology used in Zorn's lemma is defined as follows. A relation $\prec$ on a set $\mathcal{P}$ is called a *partial order* if it is

- transitive, that is, $x \prec y$ and $y \prec z$ implies $x \prec z$ for all $x, y, z \in \mathcal{P}$,
- reflexive, that is, $x \prec x$ for all $x \in \mathcal{P}$,
- anti-symmetric, that is, $x \prec y$ and $y \prec x$ implies $x = y$ for all $x, y \in \mathcal{P}$.

A subset $\mathcal{C}$ of $\mathcal{P}$ is called a *chain* if for every pair $(x, y) \in \mathcal{P}^2$ the relation $x \prec y$ or $y \prec x$ holds. An *upper bound* for $\mathcal{C}$ is an element $z \in \mathcal{P}$ such that $x \prec z$ for all $x \in \mathcal{C}$. An element $z \in \mathcal{P}$ is a *maximal element* of $\mathcal{P}$ if for every $x \in \mathcal{P}$ such that $z \prec x$ it holds that $x = z$.

We do not prove this result here. In fact, in the usual set-theoretic foundations of mathematics, Zorn's lemma is equivalent to the axiom of choice, so we may as well consider it as one of our axioms for the purpose of this course.

Definition A.0.3 (Sublinear functional, semi-norm)**.** Let E be a vector space over $\mathbb{K}$. A *sublinear functional* on E is a map $p\colon E \to \mathbb{R}$ that satisfies

- $p(\lambda\xi) = \lambda p(\xi)$ for $\lambda \geq 0$, $\xi \in E$,
- $p(\xi + \eta) \leq p(\xi) + p(\eta)$ for $\xi, \eta \in E$.

A *semi-norm* on E is a map $p\colon E \to [0, \infty)$ that satisfies

- $p(\lambda\xi) = |\lambda| p(\xi)$ for $\lambda \in \mathbb{K}$, $\xi \in E$,
- $p(\xi + \eta) \leq p(\xi) + p(\eta)$ for $\xi, \eta \in E$.

Theorem A.0.4 (Hahn–Banach, sublinear functional version)**.** *Let E be a real vector space, F a linear subspace of E and $p\colon E \to \mathbb{R}$ a sublinear functional. If $f\colon F \to \mathbb{R}$ is a linear functional such that $f(\xi) \leq p(\xi)$ for all $\xi \in F$, then there exists a linear extension $\tilde{f}$ of f to E that satisfies $\tilde{f}(\xi) \leq p(\xi)$ for all $\xi \in E$.*

Proof. Let $\mathcal{P}$ the set of all pairs (G, g) consisting of a linear subspace G of E that contains F and a linear functional $g\colon G \to \mathbb{R}$ that extends f and satisfies $f(\xi) \leq p(\xi)$ for all $\xi \in G$. Since $(F, f) \in \mathcal{P}$, the set $\mathcal{P}$ is non-empty. We define a partial order on $\mathcal{P}$ by setting $(G_1, g_1) \prec (G_2, g_2)$ if $G_1 \subset G_2$ and $g_2|_{G_1} = g_1$.

If $\mathcal{C} \subset \mathcal{P}$ is a chain, let $\hat{G} = \bigcup_{G:(G,g)\in\mathcal{C}} G$ and define $\hat{g}\colon \hat{G} \to \mathbb{R}$ by $\hat{g}(\xi) = g(\xi)$ if $x \in G$ and $(G, g) \in \mathcal{C}$. The chain property of $\mathcal{C}$ ensures that $\hat{G}$ is a subspace and $\hat{g}$ is well-defined. Moreover, $(\hat{G}, \hat{g}) \in \mathcal{P}$ and $(G, g) \prec (\hat{G}, \hat{g})$ for every $(G, g) \in \mathcal{C}$ follow directly from the construction. Thus $\mathcal{C}$ has an upper bound.

By Zorn's lemma, $\mathcal{P}$ has a maximal element $(\tilde{F}, \tilde{f})$. To finish the proof, we need to show that $\tilde{F} = E$. Suppose that this is not the case. Let $\zeta \in E \setminus \tilde{F}$ and. We want to define $h\colon \operatorname{span}(\tilde{F} \cup \{\zeta\}) \to \mathbb{R}$ such that $(\operatorname{span}(\tilde{F} \cup \{\zeta\}), h) \in \mathcal{P}$ and $(\tilde{F}, \tilde{f}) \prec (\operatorname{span}(\tilde{F} \cup \{\zeta\}), h)$.

Since p is sublinear, we have

$$\tilde{f}(\xi) + \tilde{f}(\eta) \leq p(\xi + \eta) \leq p(\xi + \zeta) + p(\eta - \zeta)$$

for all $\xi, \eta \in \tilde{F}$, hence

$$m = \sup_{\eta \in \tilde{F}} (\tilde{f}(\eta) - p(\eta - \zeta)) \leq \inf_{\xi \in \tilde{F}} (p(\xi + \zeta) - \tilde{f}(\xi)) = M.$$

Let $\alpha \in [m, M]$ and define

$$h\colon \operatorname{span}\{\tilde{F} \cup \{\zeta\}\} \to \mathbb{R},\ h(\xi + \lambda\zeta) = \tilde{f}(\xi) + \lambda\alpha$$

for $\xi \in \tilde{F}$ and $\lambda \in \mathbb{R}$. Since $\tilde{F}$ and ζ are linearly independent, h is well-defined, and it is obviously an extension of $\tilde{f}$. It remains to show that h is dominated by p.

If $\lambda > 0$, then since $\alpha \leq M$, we have

$$\begin{aligned} h(\xi + \lambda\zeta) &= \tilde{f}(\xi) + \lambda\alpha \\ &\leq \tilde{f}(\xi) + \lambda \inf_{\eta \in \tilde{F}} (p(\eta + \zeta) - \tilde{f}(\eta)) \\ &\leq \tilde{f}(\xi) + \lambda \left(p\left(\frac{\xi}{\lambda} + \zeta\right) - \tilde{f}\left(\frac{\xi}{\lambda}\right)\right) \\ &= p(\xi + \lambda\zeta). \end{aligned}$$

If $\lambda < 0$, we reach the same conclusion using $\alpha \geq m$ instead. $\square$

Corollary A.0.5 (Hahn–Banach, semi-norm version)**.** *Let E be a vector space over $\mathbb{K}$, F a linear subspace of E and $p\colon E \to [0, \infty)$ a semi-norm. If $f\colon F \to \mathbb{K}$ is a linear functional such that $|f(\xi)| \leq p(\xi)$ for all $\xi \in F$, then there exists a linear extension $\tilde{f}$ of f to E that satisfies $|\tilde{f}(\xi)| \leq p(\xi)$ for all $\xi \in E$.*

Proof. Case $\mathbb{K} = \mathbb{R}$: Every seminorm is a sublinear functional. By the sublinear functional version of the Hahn–Banach theorem, the functional f can be extended to a linear functional $\tilde{f}$ on E such that $\tilde{f}(\xi) \leq p(\xi)$ for all $\xi \in E$. At the same time, $-\tilde{f}(\xi) = \tilde{f}(-\xi) \leq p(-\xi) = p(\xi)$. Thus $|\tilde{f}(\xi)| \leq p(\xi)$.

Case $\mathbb{K} = \mathbb{C}$: Let $g = \operatorname{Re} f$. This functional is real-linear, hence it can be extended to a real-linear functional $\tilde{g}$ on E such that $|\tilde{g}(\xi)| \leq p(\xi)$ for all $\xi \in E$ by the first part. Let $\tilde{f}(\xi) = \tilde{g}(\xi) - i\tilde{g}(i\xi)$ for $\xi \in E$. If $\xi \in F$, then $-ig(i\xi) = -i \operatorname{Re}(ig(\xi)) = i \operatorname{Im} \xi$, hence $\tilde{f}(\xi) = f(\xi)$ for $\xi \in F$.

By definition, $\tilde{f}$ is real-linear. However,

$$\tilde{f}(i\xi) = \tilde{g}(i\xi) + i\tilde{g}(\xi) = i\tilde{f}(\xi).$$

Together with real linearity, this implies that $\tilde{f}$ is in fact complex-linear. To show that $\tilde{f}$ is dominated by p, let $\xi \in E$ and $z \in \mathbb{C}$ with $|z| = 1$ such that $|\tilde{f}(\xi)| = z\tilde{f}(\xi)$. We have

$$|\tilde{f}(\xi)| = z\tilde{f}(\xi) = \operatorname{Re} \tilde{f}(z\xi) = \tilde{g}(z\xi) \leq p(z\xi) = p(\xi).$$ $\square$

Corollary A.0.6 (Hahn-Banach, bounded functional version)**.** *Let E be a normed space and F a linear subspace of E. If $f\colon F \to \mathbb{K}$ is a bounded linear functional, then there exists a bounded linear extension $\tilde{f}$ of f to E with $\|\tilde{f}\| = \|f\|$.*

Proof. Apply the previous result with the seminorm $p(\xi) = \|f\|\|\xi\|$. $\square$

Corollary A.0.7. *Let E be a non-zero normed space. For every $\xi \in E$ there exists $f \in E^*$ with $\|f\| = 1$ such that $f(\xi) = \|\xi\|$.*

Proof. Let $F = \operatorname{span}\{\xi\}$ and define $g\colon F \to \mathbb{K}$, $g(\lambda\xi) = \lambda\|\xi\|$. Clearly, $\|g\| = 1$ and $g(\xi) = \|\xi\|$. By the previous corollary, g can be extended to a linear functional on E with the same norm. $\square$

Exercises

1. In this exercise we construct so-called Banach limits.

   (a) Let ℓ^∞ denote the space of bounded sequences in $\mathbb{K}$. Show that

   $$p\colon \ell^\infty \to \mathbb{K},\ x \mapsto \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k$$

   is a sublinear functional on ℓ^∞.

   (b) Show that there exists a bounded linear functional $\mathrm{LIM}\colon \ell^\infty \to \mathbb{K}$ with the following properties:

   - If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\mathrm{LIM}(x) \geq 0$.
   - If S denotes the shift operator on ℓ^∞, i.e., $(Sx)_n = x_{n+1}$, then $\mathrm{LIM}(Sx) = \mathrm{LIM}(x)$ for all $x \in \ell^\infty$.
   - If x is a convergent sequence, then $\mathrm{LIM}(x) = \lim_{n\to\infty} x_n$.

   Any such functional is called a *Banach limit*.

Appendix B

The Stone–Weierstraß theorem

Let X be a set and $\mathcal{F}$ a family of functions on X. We say that $\mathcal{F}$ separates the points of X if whenever $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{F}$ with $f(x) \neq f(y)$.

Theorem B.0.1 (Stone–Weierstraß). *Let X be a compact Hausdorff space. If $A \subset C(X)$ is a unital $*$-subalgebra that separates the points of X, then A is dense in $C(X)$ (with respect to the supremum norm).*

Proof. We first note that continuity of the operations implies that $\overline{A}$ is again a unital $*$-subalgebra of $C(X)$.

Step 1: If $f \in A$ and $f \geq 0$, then $\sqrt{f} \in \overline{A}$.

By rescaling, we can assume that $0 \leq f \leq 1$. Let $g = 1 - f$. The binomial series convergence uniformly on the unit disk, hence

$$\sqrt{f(x)} = \sqrt{1 - g(x)} = \lim_{n \to \infty} \sum_{k=0}^{n} \binom{\frac{1}{2}}{k} (-1)^k g(x)^k$$

uniformly in $x \in X$. Therefore, $\sqrt{f} \in \overline{A}$.

Step 2: If $f, g \in A$ are real-valued, then $\min\{f, g\}, \max\{f, g\} \in \overline{A}$.

This follows immediately from the following identities:

$$\min\{f, g\} = \frac{f + g + \sqrt{(f - g)^2}}{2}, \quad \max\{f, g\} = \frac{f + g + \sqrt{(f - g)^2}}{2}.$$

Step 3: If $f \in C(X)$ is real-valued and $\varepsilon > 0$, then there exists $g \in A$ such that $\|f - g\|_\infty < \varepsilon$.

For $x, y \in X$ with $x \neq y$ choose $h \in A$ such that $h(x) \neq h(y)$, which exists since A separates the points of X. Otherwise replacing h by $\operatorname{Re} h$ or $\operatorname{Im} h$, we can assume that h is real-valued.

Let

$$f_{x,y} : X \to \mathbb{R},\ z \mapsto f(y) + (f(x) - f(y)) \frac{h(z) - h(y)}{h(x) - h(y)},$$

which belongs to A since it is a linear combination of h and constant functions.

Further let

$$U_{x,y} = \{z \in X \mid f_{x,y}(z) < f(z) + \varepsilon/2\}.$$

Since f and $f_{x,y}$ are continuous, the sets $U_{x,y}$, $x, y \in X$, are open. Moreover, since $f_{x,y}(y) = f(y)$, we have $y \in U_{x,y}$. Therefore, $(U_{x,y})_{y\in X}$ is an open cover of X for every $x \in X$.

As X is compact, there exists $n \in \mathbb{N}$ and $y_1, \ldots, y_n \in X$ such that $X = \bigcup_{k=1}^{n} U_{x,y_k}$. Let $f_x = \min_{1\leq k\leq n} f_{x,y_k}$, which belongs to $\bar{A}$ by the Step 2. Furthermore, let

$$V_x = \{z \in X \mid h_x(z) > f(z) - \varepsilon/2\}.$$

Since $f_{x,y}(x) = f(x)$ for all $y \in X$, we also have $f_x(x) = f(x)$ and thus $x \in V_x$. Using once again compactness of X, we get $m \in \mathbb{N}$ and $x_1, \ldots, x_m \in X$ such that $X = \bigcup_{j=1}^{m} V_{x_j}$. Let $\tilde{g} = \max_{1\leq j\leq m} f_{x_j}$, which belongs to $\overline{A}$ by Step 2.

By construction, $f - \varepsilon/2 < \tilde{g} < f + \varepsilon/2$, hence $\|f - \tilde{g}\|_\infty < \varepsilon/2$. Take $g \in A$ with $\|g - \tilde{g}\|_\infty < \varepsilon/2$. By the triangle inequality, $\|f - g\|_\infty < \varepsilon$.

Step 4: A is dense in $C(X)$.

For arbitrary $f \in C(X)$ and $\varepsilon > 0$, we find $g, h \in A$ such that $\|\mathrm{Re}\, f - g\|_\infty < \varepsilon/2$ and $\|\mathrm{Im}\, f - h\|_\infty < \varepsilon/2$. By the triangle inequality, $\|f - (g + ih)\|_\infty < \varepsilon/2$. □

The great generality of the Stone–Weierstraß theorem makes it applicable in a variety of situations.

Corollary B.0.2 (Weierstraß). *Polynomial functions are dense in* $C([0, 1])$.

Proof. Clearly, polynomial functions on a subset of $\mathbb{R}$ form a unital $*$-algebra. Moreover, linear functions already separate points. □

Corollary B.0.3. *Let* $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. *Functions of the form*

$$S^1 \to \mathbb{C},\ z \mapsto \sum_{k=-N}^{N} a_k z^k$$

with $N \in \mathbb{N}$ *and* $a_k \in \mathbb{C}$ *are dense in* $C(S^1)$.