Functional Analysis and Operator Theory

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Contents

Chapter 1

Spectral Theory of C^*-algebras

1.1 Bounded operators on Hilbert space

Definition 1.1.1 (Bounded operator, operator norm)**.** Let H, K be Hilbert spaces. A linear map $x\colon H \to K$ is called *bounded* if there exists $C > 0$ such that $\|x\xi\| \leq C\|\xi\|$ for all $\xi \in H$. The set of all bounded linear operators from H to K is denoted by $\mathbb{B}(H; K)$. We write $\mathbb{B}(H)$ for $\mathbb{B}(H; H)$.

The *operator norm* of $x \in \mathbb{B}(H; K)$ is defined as

$$\|x\| = \sup_{\|\xi\| \leq 1} \|x\xi\|.$$

Remark 1.1.2. The bounded linear operators between a given pair of Hilbert spaces form a Banach space (with the operator norm).

Let us record some basic properties of the operator norm. Recall that if $x \in \mathbb{B}(H)$, the adjoint x^* is the unique operator in $\mathbb{B}(H)$ that satisfies $\langle \xi, x\eta \rangle = \langle x^*\xi, \eta \rangle$ for all $\xi, \eta \in H$.

Lemma 1.1.3. *Let H be a Hilbert space. The operator norm on $\mathbb{B}(H)$ has the following properties.*

(a) $\|xy\| \leq \|x\|\|y\|$ *for all* $x, y \in \mathbb{B}(H)$.

(b) $\|x^*\| = \|x\|$ *for all* $x \in \mathbb{B}(H)$.

(c) $\|x^*x\| = \|x\|^2$ *for all* $x \in \mathbb{B}(H)$,

Proof. (a) If $\|\xi\| \leq 1$ and $y\xi \neq 0$, then

$$\|xy\xi\| = \left\| x \frac{y\xi}{\|y\xi\|} \right\| \|y\xi\| \leq \|x\|\|y\|.$$

If $y\xi = 0$, the inequality $\|xy\xi\| \leq \|x\|\|y\|$ holds trivially. Taking the supremum over all $\xi \in H$ with $\|\xi\| \leq 1$ yields the claimed inequality.

(b) First note that

$$\|x\|^2 = \sup_{\|\xi\|\leq 1} \|x\xi\|^2 = \sup_{\|\xi\|\leq 1} \langle \xi, x^*x\xi \rangle \leq \|x^*x\|.$$

By (a), we have $\|x^*x\| \leq \|x^*\|\|x\|$. Together with the previous inequality, this implies $\|x\| \leq \|x^*\|$. The reverse inequality follows by exchanging the roles of x and x^*.

(c) In (b) we have already seen that $\|x\|^2 \leq \|x^*x\|$. If we combine this with (a) and apply (b) again, we obtain $\|x\|^2 \leq \|x^*x\| \leq \|x^*\|\|x\| = \|x\|^2$. Thus $\|x\|^2 = \|x^*x\|$. □

Definition 1.1.4. Let H be a Hilbert space. An operator $x \in \mathbb{B}(H)$ is called

- *normal* if $x^*x = xx^*$,
- *self-adjoint* or *symmetric* if $x = x^*$,
- *positive* if there exists $y \in \mathbb{B}(H)$ such that $x = y^*y$,
- a *projection* if $x^* = x^2 = x$,
- *unitary* if $x^*x = xx^* = 1$,
- an *isometry* if $x^*x = 1$,
- a *partial isometry* if x^*x is a projection.

Lemma 1.1.5. *If H is a Hilbert space and $x \in \mathbb{B}(H)$, then $\ker(x) = \operatorname{ran}(x^*)^\perp$ and $\overline{\operatorname{ran} x} = \ker(x^*)^\perp$.*

Proof. If $\xi \in \ker x$ and $\eta \in H$, then $\langle \xi, x^*\eta \rangle = 0$, hence $\xi \in \operatorname{ran}(x^*)^\perp$. If $\xi \in \operatorname{ran}(x^*)^\perp$, then

$$\|x\xi\|^2 = \langle \xi, x^*x\xi \rangle = 0,$$

hence $\xi \in \ker x$. The second identity follows by taking the orthogonal complement on both sides of the first identity. □

Proposition 1.1.6. *Let H be a Hilbert space. An operator $x \in \mathbb{B}(H)$ is*

(a) normal if and only if $\|x\xi\| = \|x^*\xi\|$ for all $\xi \in H$,

(b) self-adjoint if and only if $\langle \xi, x\xi \rangle \in \mathbb{R}$ for all $\xi \in H$,

(c) positive if and only if $\langle \xi, x\xi \rangle \geq 0$ for all $\xi \in H$,

(d) a projection if and only if x is the orthogonal projection onto $\operatorname{ran}(x)$*,*

(e) an isometry if and only if $\|x\xi\| = \|\xi\|$ for all $\xi \in H$,

(f) unitary if and only if it is a surjective isometry,

(g) a partial isometry if and only if it restricts to an isometry from $\ker(x)^\perp$ *to* $\operatorname{ran}(x)$*.*

Proof. (a) If x is normal, then $\|x\xi\|^2 = \langle \xi, x^*x\xi\rangle = \langle \xi, xx^*\xi\rangle = \|x^*\xi\|^2$ for all $\xi \in H$. Conversely, $\langle \xi, (x^*x - xx^*)\xi\rangle = \|x\xi\|^2 - \|x^*\xi\|^2 = 0$ for all $\xi \in H$, which implies $x^*x - xx^* = 0$.

(b) If x is self-adjoint, then $\overline{\langle \xi, x\xi\rangle} = \langle \xi, x\xi\rangle = \langle \xi, x\xi\rangle$ for all $\xi \in H$. Conversely, $\langle \xi, (x - x^*)\xi\rangle = \langle \xi, x\xi\rangle - \overline{\langle \xi, x\xi\rangle} = 0$ for all $\xi \in H$, hence $x = x^*$.

(c) If $x = y^*y$, then $\langle \xi, x\xi\rangle = \|y\xi\|^2 \geq 0$ for all $\xi \in H$. Conversely, x is self-adjoint by (b) and $\sigma(x) \subset [0, \infty)$ by the spectral theorem. Thus $x = (x^{1/2})^2$.

(d) We showed that in Mathematical Physics II.

(e) If x is an isometry, then $\|x\xi\|^2 = \langle \xi, x^*x\xi\rangle = \|\xi\|^2$ for all $\xi \in H$. Conversely, $\langle \xi, (x^*x - 1)\xi\rangle = \|x\xi\|^2 - \|\xi\|^2 = 0$ for all $\xi \in H$, hence $x^*x = 1$.

(f) If x is unitary, then x is an invertible isometry with $x^{-1} = x^*$. Thus x is surjective. Conversely, if x is a surjective isometry, then it is invertible and thus $x^{-1} = x^*xx^{-1} = x^*$, which implies $xx^* = 1$.

(g) Let x be a partial isometry. First note that $\ker x \subset \ker(x^*x)$. Conversely, if $\xi \in \ker(x^*x)$, then $\|x\xi\|^2 = \langle \xi, x^*\xi\rangle = 0$, hence $\xi \in \ker(x)$. Thus x^*x is the orthogonal projection onto $\ker(x^*x)^\perp = (\ker x)^\perp$. If $\xi \in (\ker x)^\perp$, then $\|x\xi\|^2 = \langle \xi, x^*x\xi\rangle = \|\xi\|^2$. Hence x is an isometry from $(\ker x)^\perp$ onto $\operatorname{ran} x$.

Conversely, if x is an isometry from $(\ker x)^\perp$ onto $\operatorname{ran} x$ and p the orthogonal projection onto $(\ker x)^\perp$, then

$$\langle \xi, (x^*x)^2\xi\rangle = \langle p\xi, (x^*x)^2 p\xi\rangle = \langle xp\xi, xx^*xp\xi\rangle = \langle p\xi, x^*xp\xi\rangle = \langle \xi, x^*x\xi\rangle$$

for all $\xi \in H$. Thus $(x^*x)^2 = x^*x$. □

Lemma 1.1.7 (Positive square root)**.** *Let H be Hilbert space. For every positive operator $x \in \mathbb{B}(H)$ there exists a unique positive operator $y \in \mathbb{B}(H)$ such that $x = y^2$.*

Proof. Existence: Let $f(\lambda) = \sqrt{\lambda}$ for $\lambda \geq 0$. By the spectral theorem, $f(x)$ is positive and $f(x)^2 = x$.

Uniqueness: Let $y \in \mathbb{B}(H)$ be a positive operator such that $x = y^2$. By the spectral theorem in multiplication operator form, there exists a localizable measure space $(X, \mathcal{A}, \mu)$, a measurable function $\varphi\colon \mathbb{R} \to \mathbb{R}$ and a unitary operator $u\colon L^2(X,\mu) \to H$ such that $y = uM_\varphi u^*$ and $f(y) = uM_{f\circ\varphi}u^*$ for every bounded Borel function $f\colon \sigma(y) \to \mathbb{C}$. Since $y \geq 0$, we have $\varphi \geq 0$ μ-a.e. Thus $y^2 = uM_{\varphi^2}u^*$ and

$$x^{1/2} = (y^2)^{1/2} = uM_{(\varphi^2)^{1/2}}u^* = uM_\varphi u^* = y. \qquad \square$$

Proposition 1.1.8 (Polar decomposition). *Let H be a Hilbert space. If $x \in \mathbb{B}(H)$, then there exists a unique pair (v, y) consisting of a partial isometry $v \in \mathbb{B}(H)$ and a positive operator $y \in \mathbb{B}(H)$ such that $x = vy$ and $\ker v = \ker x$.*

Proof. Existence: Let $y = (x^*x)^{1/2}$. By the spectral theorem, $\ker((x^*x)^{1/2}) = \ker(x^*x)$ and clearly $\ker x \subset \ker(x^*x)$. On the other hand, if $\xi \in \ker(x^*x)$, then $\|x\xi\|^2 = \langle \xi, x^*x\xi\rangle = 0$, hence $\xi \in \ker(x^*x)$. Thus $\ker y = \ker x$.

We define
$$v\colon\ \operatorname{ran} y \to H,\ v(y\xi) = x\xi.$$
Since
$$\|x\xi\|^2 = \langle \xi, x^*x\xi\rangle = \|(x^*x)^{1/2}\xi\|^2 = \|y\xi\|^2,$$
the operator v is well-defined and extends to an isometry from $\overline{\operatorname{ran} y}$ to H. We can extend v to a partial isometry on H by setting $v = 0$ on $(\operatorname{ran} y)^\perp = \ker x$. With this definition, $\ker v = \ker x$ and $x = vy$.

Uniqueness: Let (v', y') be a pair of bounded operators that satisfies the conditions of the proposition. We have $y^2 = x^*x = (y')^2$, hence $y = y'$ by the uniqueness of the square root. Therefore $v = v'$ on $\operatorname{ran} y$, which is dense in $\ker(v)^\perp = \ker(v')^\perp$. Since both operators are continuous, we conclude $v = v'$. $\square$

Remark 1.1.9. The decomposition $x = vy$ from the previous proposition is called the *polar decomposition* of x. As the proof shows, the positive operator y is given by $(x^*x)^{1/2}$. This operator is denoted by $|x|$. This is consistent with functional calculus for self-adjoint operators (see the exercise).

Exercises

1. Let $x \in \mathbb{B}(H)$ such that $\langle \xi, x\xi\rangle = 0$ for all $\xi \in H$. Show that $x = 0$.

2. Let $x \in \mathbb{B}(H)$ be self-adjoint and

$$f\colon \mathbb{R} \to \mathbb{R},\ \lambda \mapsto \begin{cases} 1 & \text{if } \lambda > 0 \\ 0 & \text{if } \lambda = 0 \\ -1 & \text{if } \lambda < 0 \end{cases}$$
$$g\colon \mathbb{R} \to \mathbb{R},\ \lambda \mapsto |\lambda|.$$

Show that $x = f(x)g(x)$ is the polar decomposition of x. In particular, $(x^*x)^{1/2} = |x|$ in the sense of functional calculus.

1.2 Banach algebras and C^*-algebras

In the next sections, we will take a more abstract look at spectral theory. Recall that if $x \in \mathbb{B}(H)$, the resolvent set $\rho(x)$ is defined as

$$\rho(x) = \{\lambda \in \mathbb{C} \mid x - \lambda \text{ invertible with bounded inverse}\}.$$

In operator theory, one usually checks this condition by showing that $x - \lambda$ is injective (i.e. λ is not an eigenvalue of x) and that $x - \lambda$ is surjective. Boundedness of the inverse is then a consequence of the closed graph theorem.

However, one can also approach the resolvent set more algebraically. A number $\lambda \in \mathbb{C}$ belongs to the resolvent set if and only if there exists $y \in \mathbb{B}(H)$ such that $(x - \lambda)y = y(x - \lambda) = 1$. This formulation only uses the basic algebraic operations in $\mathbb{B}(H)$ (addition, composition of operators and multiplication with scalars) and not the fact that elements of $\mathbb{B}(H)$ are linear maps on a Hilbert space. Thus spectral theory can be studied in the more general context when only these algebraic properties are given. This motivates the following definition.

Definition 1.2.1 (Algebra, invertible elements, spectrum)**.** An *algebra* is a complex vector space A together with a bilinear map $A \times A \to A$, $(a, b) \mapsto ab$, called the multiplication. The algebra A is called *unital* if there exists an element $1 \in A$ such that $1a = a1 = a$ for all $a \in A$.

If A is a unital algebra, An element $a \in A$ is called *invertible* if there exists $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$.

The *spectrum* of an element $a \in A$ is defined as

$$\sigma_A(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ not invertible}\}.$$

Remark 1.2.2. If the algebra A is unital, then the unit 1 is unique. Likewise, if an element a of a unital algebra is invertible, the inverse a^{-1} is unique.

Example 1.2.3. If H is a Hilbert space, then $\mathbb{B}(H)$ with the usual vector space structure and the multiplication given by operator composition is a unital algebra. The unit is the idenity operator. An operator $x \in \mathbb{B}(H)$ is invertible if and only if it is bijective and the spectrum $\sigma_{\mathbb{B}(H)}(x)$ coincides with the usual spectrum of an operator on a Hilbert space.

Example 1.2.4. If X is a compact Hausdorff space, then $C(X)$ with the usual vector space structure and the multiplication given by pointwise multiplication of functions is a unital algebra. The unit is the constant function 1. A function $f \in C(X)$ is invertible if and only if it has non zeros and the spectrum $\sigma_{C(X)}(f)$ equals $\operatorname{im} f$.

Proposition 1.2.5. *If A is a unital algebra and $a, b \in A$, then $\sigma_A(ab) \cup \{0\} = \sigma_A(ba) \cup \{0\}$.*

Proof. If $\lambda \in \mathbb{C} \setminus (\sigma_A(ab) \cup \{0\})$, let $c = \lambda^{-1}(1 + b(\lambda - ab)^{-1}a)$. We have

$$\begin{aligned}
(\lambda - ba)c &= (1 - \lambda^{-1}ba)(1 + b(\lambda - ab)^{-1}a) \\
&= 1 - \lambda^{-1}ba + \lambda^{-1}b(\lambda - ab)(\lambda - ab)^{-1}a \\
&= 1 - \lambda^{-1}ba + \lambda^{-1}ba \\
&= 1.
\end{aligned}$$

A similar calculation shows $(\lambda - ba)c = 1$. Thus $\lambda \in \mathbb{C} \setminus (\sigma_A(ba) \cup \{0\})$. $\square$

Remark 1.2.6. The proof of the previous lemma shows $(\lambda - ba)^{-1} = \lambda^{-1}(1 + b(\lambda - ab)^{-1}a)$. It seems like we pulled this formula out of blue air. Formally, it can be justified as follows:

$$\begin{aligned}
(\lambda - ba)^{-1} &= \lambda^{-1}(1 - \lambda^{-1}ba)^{-1} \\
&= \lambda^{-1}\sum_{n=0}^{\infty} \lambda^{-k}(ba)^k \\
&= \lambda^{-1}(1 + \lambda^{-1}b\sum_{k=0}^{\infty} \lambda^{-k}(ab)^k a) \\
&= \lambda^{-1}(1 + \lambda^{-1}b(1 - \lambda^{-1}(ab))^{-1}a) \\
&= \lambda^{-1}(1 + b(\lambda - ab)^{-1}a).
\end{aligned}$$

Note however that in this abstract algebraic setting, we do not even have a notion of convergence so that these manipulations of infinite series are not rigorous.

To be able to speak of convergence etc., that is, to actually do analysis, we need additional structure. A rich class of algebras with a topology is provided by the following definition.

Definition 1.2.7 (Banach algebra)**.** A *Banach algebra* is an algebra A with a norm $\|\cdot\|$ that satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in A$ and such that A is complete in this norm.

Remark 1.2.8. The submultiplicativity $\|ab\| \leq \|a\|\|b\|$ guarantees that the multiplications is a continuous bilinear map from $A \times A$ to A. Moreover, any norm on an algebra that makes the multiplication continuous can be replaced by an equivalent submultiplicative norm.

Definition 1.2.9 ($*$-algebra, C^*-norm, C^*-algebra)**.** A *$*$-algebra* is an algebra together with a map $A \to A$, $a \mapsto a^*$ with the following properties:

- $(\lambda a + \mu b)^* = \overline{\lambda} a^* + \overline{\mu} b^*$ for all $\lambda, \mu \in \mathbb{C}$, $a, b \in A$,
- $(ab)^* = b^* a^*$ for all $a, b \in A$,
- $(a^*)^* = a$ for all $a \in A$.

A norm $\|\cdot\|$ on a $*$-algebra A is called a *C^*-norm* if $\|ab\| \leq \|a\|\|b\|$ and $\|a^* a\| = \|a\|^2$ for all $a, b \in A$. An algebra with a complete C^* norm is called a *C^*-algebra.*

Remark 1.2.10. Clearly, every C^*-algebra is a Banach algebra. The converse is far from being true. Not only do we need an additional structure (the involution $a \mapsto a^*$) in the definition of a C^*-algebra, there are also many examples of Banach algebras with a natural involution that do not satisfy the C^* identity for the norm.

Example 1.2.11. The complex-valued polynomials in one variable form a unital algebra $\mathbb{C}[X]$ with involution given by $(\sum_k \alpha_k X^k)^* = \sum_k \overline{\alpha_k} X^k$. There are many C^*-norms on $\mathbb{C}[X]$, but no norm (whether a C^*-norm or not) that makes $\mathbb{C}[X]$ a Banach space.

Example 1.2.12. Let G be a group. Let $\mathbb{C}[G]$ be the vector space with basis G, that is, $\mathbb{C}[G]$ consists of all formal linear combinations $\sum_{g \in G} \alpha_g g$ with finitely many non-zero coefficients α_g. One can define a multiplication on $\mathbb{C}[G]$ as a bilinear extension of the multiplication of G, that is, $(\sum_g \alpha_g g)(\sum_h \beta_h h) = \sum_{\mathfrak{g},h} \alpha_g \beta_h gh$. The algebra $\mathbb{C}[G]$ is called the (complex) *group algebra.*

Moreover, there is an involution on $\mathbb{C}[G]$ defined by

$$\left(\sum_g \alpha_g g\right)^* = \sum_g \overline{\alpha_g} g^{-1}$$

With this involution, the group algebra becomes a $*$-algebra. The expression

$$\left\| \sum_g \alpha_g g \right\|_{\mathrm{u}} = \sup \left\{ \left\| \sum_g \alpha_g \pi(g) \right\| : \pi \colon G \to \mathbb{U}(H) \text{ group hom.} \right\}$$

defines a C^*-norm on $\mathbb{C}[G]$. Here, $\mathbb{U}(H)$ denotes the group of unitary operators on H. However, this norm is not complete unless G is finite.

Example 1.2.13. If H is a Hilbert space, then $\mathbb{B}(H)$ with the operation of taking adjoints and the operator norm is a C^*-algebra.

Example 1.2.14. If X is a compact Hausdorff space, then $C(X)$ with the complex conjugation as $*$-operation and the supremum norm is a C^*-algebra.

One crucial difference between the last two examples is that while multiplication in $C(X)$ is commutative, operator multiplication in $\mathbb{B}(H)$ is not (unless $\dim H \leq 1$). The goal of this chapter is to show that every commutative unital C^*-algebra is of the form $C(X)$ for some compact Hausdorff space X.

Lemma 1.2.15. *If A is a unital Banach algebra and $a \in A$ with $\|1-a\| < 1$, then a is invertible with $\|a^{-1}\| \leq (1 - \|1-a\|)^{-1}$.*

Proof. Since $\|1-a\| < 1$ and A is complete, the series $\sum_{k=0}^{\infty}(1-a)^k$ converges and the limit has norm bounded above by $(1 - \|1-a\|)^{-1}$ (see the exercises). A telescope sum trick shows $a\sum_{k=0}^{\infty}(1-a)^k = \sum_{k=0}^{\infty}(1-a)^k a = 1$. □

Proposition 1.2.16. *If A is a unital Banach algebra and* $\mathrm{Inv}(A)$ *denotes the set of invertible elements of A, then* $\mathrm{Inv}(A)$ *is open in A and $a \mapsto a^{-1}$ is continuous on* $\mathrm{Inv}(A)$.

Proof. If $a \in \mathrm{Inv}(A)$ and $b \in A$ with $\|a-b\| < \|a^{-1}\|^{-1}$, then $\|1 - a^{-1}b\| \leq \|a^{-1}\|\|a-b\| < 1$. By the previous lemma, $a^{-1}b$ is invertible and

$$\|(a^{-1}b)^{-1}\| \leq (1 - \|a^{-1}\|\|a-b\|)^{-1}.$$

In particular, b is invertible with inverse $b^{-1} = (a^{-1}b)^{-1}a^{-1}$. Therefore, $\mathrm{Inv}(A)$ is open.

Moreover,

$$\begin{aligned} \|a^{-1} - b^{-1}\| &= \|a^{-1}(b-a)b^{-1}\| \\ &\leq \|a^{-1}\|\|b-a\| \underbrace{\|b^{-1}\|}_{\|(a^{-1}b)^{-1}a^{-1}\|} \\ &\leq \|a^{-1}\|^2\|b-a\|\|a^{-1}b\| \\ &\leq \frac{\|a^{-1}\|^2\|b-a\|}{1 - \|a^{-1}\|\|b-a\|}. \end{aligned}$$

Thus $b^{-1} \to a^{-1}$ as $b \to a$. □

Proposition 1.2.17. *If A is a non-zero unital Banach algebra and $a \in A$, then $\sigma_A(a)$ is compact, non-empty and contained in $\bar{B}_{\|a\|}(0)$, and*

$$R\colon \mathbb{C} \setminus \sigma_A(a) \to A,\ z \mapsto (z-a)^{-1}$$

is (complex) differentiable.

Proof. If $\lambda \in \mathbb{C}$ with $|\lambda| > \|a\|$, then $\lambda - a = \lambda(1 - \lambda^{-1}a)$ is invertible by a previous lemma. Thus $\sigma_A(a) \subset \bar{B}_{\|a\|}(0)$. As the map $\Psi\colon \mathbb{C} \to A$, $\lambda \mapsto a - \lambda$ is continuous and $\mathbb{C} \setminus \sigma_A(a) = \Psi^{-1}(\mathrm{Inv}(A))$, we see that $\sigma_A(a)$ is closed. Therefore $\sigma_A(a)$ is compact.

If $z, w \in \mathbb{C} \setminus \sigma_A(a)$, then

$$\begin{aligned}(a-z)^{-1} - (a-w)^{-1} &= (a-z)^{-1}((a-w) - (a-z))(a-w)^{-1} \\ &= (a-z)^{-1}(z-w)(a-w)^{-1}.\end{aligned}$$

As inversion is continuous, we conclude

$$\lim_{w \to z} \left\| \frac{R(z) - R(w) - (a-z)^{-2}(z-w)}{z-w} \right\| = 0.$$

Hence R is differentiable with $R'(z) = (a-z)^{-2}$.

Suppose that $\sigma_A(a) = \emptyset$ and let $\varphi \in A^*$. If $|z| > \|a\|$, then

$$|\varphi(R(z))| \le \|\varphi\| \|R(z)\| \le \|\varphi\| |z|^{-1}(1 - |z|^{-1}\|a\|)^{-1}.$$

Hence $\varphi \circ R$ is a bounded complex differentiable function on $\mathbb{C}$ such that $\lim_{|z|\to\infty} |\varphi(R(z))| = 0$. By Liouville's theorem, $\varphi \circ R = 0$. Since $\varphi \in A^*$ is arbitrary, the Hahn–Banach theorem implies $R = 0$, which is impossible. Thus $\sigma_A(a)$ must be non-empty. □

Remark 1.2.18. Liouville's theorem is one of the results that show that complex differentiable functions behave very differently from real differentiable functions. It states the following: If $f\colon \mathbb{C} \to \mathbb{C}$ is bounded and complex differentiable, then f is constant.

Theorem 1.2.19 (Gelfand–Mazur)**.** *If A is a non-zero unital Banach algebra in which every non-zero element is invertible, then $A = \mathbb{C}1$.*

Proof. If $a \in A$, then $\sigma_A(a) \neq \emptyset$. Take $z \in \sigma_A(a)$. Since every non-zero element of A is invertible, we conclude $a - z1 = 0$, hence $a = z1$. □

Example 1.2.20. There are unital algebras in which every non-zero element is invertible and which are not isomorphic to $\mathbb{C}$. For example, let $\mathbb{C}(X) = \{P/Q \mid P, Q \in \mathbb{C}[X],\ Q \neq 0\}$. If $P \neq 0$, then P/Q is invertible with inverse $(P/Q)^{-1} = Q/P$. In particular, there is no norm on $\mathbb{C}(X)$ that makes it into a Banach algebra.

Proposition 1.2.21 (Spectral mapping theorem for polynomials)**.** *If A is a unital Banach algebra, $a \in A$ and p a complex polynomial, then $\sigma_A(p(a)) = p(\sigma_A(a))$.*

Proof. The case of a constant polynomial is easy, hence we assume that p is non-constant. For $\lambda \in \mathbb{C}$ there exist $\alpha \neq 0$ and $\mu_1, \ldots, \mu_n \in \mathbb{C}$ such that $p(X) - \lambda = \alpha \prod_{k=1}^n (X - \mu_k)$. Moreover, $p^{-1}(\lambda) = \{\mu_1, \ldots, \mu_k\}$.

We have $\lambda \in \sigma_A(p(a))$ if and only if $p(a) - \lambda$ is not invertible if and only if $a - \mu_k$ is not invertible for some $k \in \{1, \ldots, n\}$ (see the exercises). This in turn is equivalent to $p^{-1}(\lambda) \cap \sigma_A(a) \neq 0$, that is, $\lambda \in p(\sigma_A(a))$. □

Definition 1.2.22 (Spectral radius)**.** If A is a unital algebra and $a \in A$, then the *spectral radius* of a is defined as $r(a) = \sup\{|\lambda| : \lambda \in \sigma_A(a)\}$.

Example 1.2.23. If X is a compact Hausdorff space and $f \in C(X)$, then $\sigma_{C(X)}(f) = \operatorname{im} f$ and thus $r(f) = \sup\{|f(x)| : x \in X\} = \|f\|_\infty$.

If A is a non-zero unital Banach algebra, then by the previous results, $r(a) \leq \|a\|$ for every $a \in A$ and the supremum in the definition of $r(a)$ is attained. Moreover, $\sigma_A(ab) \cup \{0\} = \sigma_A(ba) \cup \{0\}$ implies that $r(ab) = r(ba)$. Note that the definition of the spectral radius only uses the algebraic structure of A. For Banach algebras, there is an equivalent characterization in terms of the norms, as we will see next.

Proposition 1.2.24 (Spectral radius formula)**.** *If A is a unital Banach algebra and $a \in A$, then $\|a^n\|^{1/n}$ converges to $r(a)$ as $n \to \infty$.*

Proof. By the spectral mapping theorem, $r(a)^n = r(a^n) \leq \|a^n\|$. Thus $r(a) \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$.

To show $\limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a)$, let $\Omega = \{z \in \mathbb{C} : |z| > r(a)\}$ and fix $\varphi \in A^*$. As seen previously, the function

$$f\colon \Omega \to \mathbb{C},\ z \mapsto \varphi((a - z)^{-1})$$

is complex differentiable. Thus it has a Laurent series expansion

$$f(z) = \sum_{k=0}^{\infty} \frac{\alpha_k}{z^k}, \qquad z \in \Omega.$$

On the other hand, we know that if $|z| > \|a\|$, then

$$f(z) = z^{-1}\varphi((z^{-1}a - 1)^{-1}) = \sum_{k=0}^{\infty} (-1)^k \frac{\varphi(a^k)}{z^{k+1}}.$$

By the uniqueness of Laurent series expansions, we conclude that $\alpha_0 = 0$ and $\alpha_k = (-1)^{k-1}\varphi(a^{k-1})$ for $k \geq 1$.

Since the Laurent series expansion converges for $|z| > r(a)$, we have $\lim_{k\to\infty} \frac{\varphi(a^k)}{|z|^{k+1}} = 0$ for all $\varphi \in A^*$ and $|z| > r(a)$. Let $T_k \colon A^* \to \mathbb{C}$, $\varphi \mapsto \frac{\varphi(a^k)}{|z|^{k+1}}$. By the Hahn–Banach theorem, $\|T_k\| = \frac{\|a^k\|}{|z|^{k+1}}$. Moreover, by the uniform boundedness principle, there exists $C > 0$ such that $\frac{\|a^k\|}{|z|^{k+1}} \leq C$. Thus

$$\limsup_{k\to\infty} \|a^k\|^{1/k} \leq \limsup_{k\to\infty} C^{1/k}|z|^{1+1/k} = |z|.$$

Taking the infimum over $|z| > r(a)$, we conclude $r(a) \geq \limsup_{k\to\infty}\|a^k\|^{1/k}$. □

Remark 1.2.25. The Laurent series expansion is another result from complex analysis. One version sufficient for our purposes states that for every bounded complex differentiable function $f \colon \mathbb{C} \setminus \bar{B}_R(0) \to \mathbb{C}$ there exists a unique sequence (α_k) in $\mathbb{C}$ such that

$$f(z) = \sum_{k=0}^{\infty} \frac{\alpha_k}{z^k}$$

for all $z \in \mathbb{C} \setminus \bar{B}_R(0)$, where the series on the right side converges uniformly.

Exercises

1. Show that there exists no norm that makes $\mathbb{C}[X]$ into a Banach space.
2. Let A be a unital $*$-algebra.
   (a) Show that $1^* = 1$.
   (b) Show that if $a \in A$ is invertible, then a^* is invertible and $(a^*)^{-1} = (a^{-1})^*$.
3. (a) Let E be a Banach space and (x_n) a sequence in E. Show that if $\sum_{n=1}^{\infty}\|x_n\| < \infty$, then $\lim_{N\to\infty}\sum_{k=0}^{N} x_k$ exists.
   (b) Let A be a Banach algebra and $a \in A$ with $\|a\| < 1$. Show that $\lim_{N\to\infty}\sum_{n=0}^{N} a^n$ exists and has norm bounded above by $(1 - \|a\|)^{-1}$.
4. Let A be a unital $*$-algebra and let $a_1, \ldots, a_n$ be commuting elements of A. Show that $a_1 \ldots a_n$ is invertible if and only if $a_1, \ldots, a_n$ are all invertible.

5. Let A be a unital C^*-algebra. Show that if $a, b \in A$ commute, then $r(ab) \leq r(a)r(b)$.

1.3 The Gelfand transform

Definition 1.3.1 ($*$-homomorphism, character, spectrum)**.** Let A, B be algebras. An *algebra homomorphism* from A to B is a linear map $\varphi\colon A \to B$ such that $\varphi(ab) = \varphi(a)\varphi(b)$for all $a, b \in A$. If A and B are unital, then φ is called *unital* if $\varphi(1) = 1$. If A and B are $*$-algebra, then $\varphi\colon A \to B$ is called a $*$*-homomorphism* if it is an algebra homomorphism and satisfies $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$.

A non-zero algebra homomorphism from A to $\mathbb{C}$ is called a *character*. If A is a commutative Banach algebra, the set of all characters of A is called the *spectrum* of A and denoted by $\Gamma(A)$.

Remark 1.3.2. If A is a unital commutative Banach algebra and $\varphi\colon A \to \mathbb{C}$ is a character, then $\varphi(1)^2 = \varphi(1^2) = \varphi(1)$, which implies $\varphi(1) \in \{0, 1\}$. If $\varphi(1) = 0$, then $\varphi(a) = \varphi(a)\varphi(1) = 0$ for all $a \in A$, which contradicts the assumption that φ is non-zero. Thus every character on a unital commutative Banach algebra is necessarily unital.

Example 1.3.3. If X and Y are compact Hausdorff spaces and $\varphi\colon X \to Y$ is continuous, then

$$\varphi^*\colon C(Y) \to C(X),\ f \mapsto f \circ \varphi$$

is a unital $*$-homomorphism. In particular, for every $x \in X$ we get a character $\delta_x\colon C(X) \to \mathbb{C}$, $f \mapsto f(x)$ from the continuous map $\varphi\colon \{*\} \to X$, $* \mapsto x$.

Example 1.3.4. Let H be a Hilbert space and $x \in \mathbb{B}(H)$ self-adjoint. Functional calculus

$$C(\sigma(x)) \to \mathbb{B}(H),\ f \mapsto f(x)$$

is a unital $*$-homomorphism. We will see soon that functional calculus exists in the context of abstract C^*-algebras, not only for $\mathbb{B}(H)$.

Definition 1.3.5 (Ideal, maximal ideal)**.** Let A be an algebra. A linear subspace I of A is called a (two-sided) *ideal* of A, denoted by $I \trianglelefteq A$, if $abc \in I$ whenever $a, c \in A$ and $b \in I$. An ideal I of A is called *proper ideal* if $I \neq A$ and *maximal ideal* if it is a proper ideal and for every proper ideal $J \trianglelefteq A$ such that $I \subset J$ one has $I = J$.

Example 1.3.6. If X is a compact Hausdorff space and $Y \subset X$ a closed subset, then $I = \{f \in C(X) \mid f|_Y = 0\}$ is an ideal of $C(X)$. This ideal is maximal if and only if Y is a singleton.

Example 1.3.7. If H is a finite-dimensional Hilbert space, then the only ideals of $\mathbb{B}(H)$ are the trivial ones: $\{0\}$ and $\mathbb{B}(H)$. If H is an infinite-dimensional separable Hilbert space, then $\mathbb{B}(H)$ has a unique non-trivial closed ideal, called the ideal of compact operators. There are many non-trivial ideals of $\mathbb{B}(H)$ that are not closed. We will study such objects in more detail in the second part of this course.

Remark 1.3.8. If A is unital and $I \trianglelefteq A$ contains an invertible element a, then $1 = aa^{-1} \in I$ and thus $b = b1 \in I$ for every $b \in A$. Hence $I = A$.

If A is an algebra and $I \trianglelefteq A$, the quotient space A/I has not only the structure of a vector space, but is again an algebra with the multiplication $(a + I)(b + I) = ab + I$. If A is unital, then A/I is again unital with unit $1 + I$

Lemma 1.3.9. *Let A be a unital Banach algebra. If $I \trianglelefteq A$ is closed, then*

$$A/I \to [0, \infty),\ a + I \mapsto \inf_{b\in I} \|a - b\|$$

is a norm that makes A/I into a Banach algebra and the quotient map

$$q\colon A \to A/I,\ a \mapsto a + I$$

is a contractive unital algebra homomorphism.

Proof. It is easy to see that the map is a norm and $\|a + I\| \le \|a\|$ for $a \in A$. To see that the norm is submultiplicative, let $a_1, a_2 \in A$. We have

$$\begin{aligned}
\inf_{b\in I} \|a_1a_2 - b\| &\le \inf_{b_1,b_2\in I} \|a_1a_2 - \underbrace{(a_1b_2 + a_2b_1 - b_1b_2)}_{I}\| \\
&= \inf_{b_1,b_2\in I} \|(a_1 - b_1)(a_2 - b_2)\| \\
&\le \inf_{b_1,b_2\in I} \|a_1 - b_1\|\|a_2 - b_2\| \\
&= (\inf_{b_1\in I} \|a_1 - b_1\|)(\inf_{b_2\in I} \|a_2 - b_2\|).
\end{aligned}$$

It remains to show that A/I with this norm is complete. If $(a_n + I)$ is a sequence in A/I such that $\sum_{n=0}^{\infty} \|a_n + I\| < \infty$, there exist $b_n \in A$ such that $a_n - b_n \in I$ and $\sum_{n=0}^{\infty} \|b_n\| < \infty$. Since A is complete, there exists $b \in A$ such that $\lim_{N\to\infty} \sum_{n=0}^{N} b_n = b$. Therefore,

$$\left\| b + I - \sum_{n=0}^{N} a_n + I \right\| = \left\| b + I - \sum_{n=0}^{N} b_n + I \right\| \le \left\| b - \sum_{n=0}^{N} b_n \right\| \to 0. \qquad \square$$

We will use the following two lemmas from commutative algebra.

Lemma 1.3.10. *Let A be a unital commutative algebra. An ideal $I \trianglelefteq A$ is maximal if and only if A/I is a field.*

Lemma 1.3.11. *Let A be a unital commutative algebra. If $\varphi\colon A \to \mathbb{C}$ is a non-zero algebra homomorphism, then* $\ker\varphi \trianglelefteq A$ *and*

$$\psi\colon A/\ker\varphi \to \mathbb{C},\ a + \ker\varphi \mapsto \varphi(a)$$

is a bijective algebra homomorphism.

Lemma 1.3.12. *If A is a unital commutative Banach algebra, then every maximal ideal of A is closed and every character of A is contractive.*

Proof. If $I \trianglelefteq A$ is a maximal ideal, then I does not contain any invertible element. In particular, $\|a - 1\| \geq 1$ for all $a \in I$. Thus $\bar{I}$ is again a proper ideal. Maximality of I implies $I = \bar{I}$.

By the previous two lemmas, if $\varphi \in \Gamma(A)$, then $\ker\varphi$ is a maximal ideal of A, hence closed by the first paragraph. Thus $\varphi = \psi \circ q$ with the maps ψ and q from previous lemmas. Moreover, $A/\ker\varphi = \{\lambda + \ker\varphi \mid \lambda \in \mathbb{C}\}$ and

$$\|\lambda + \ker\varphi\| = |\lambda| \inf_{a\in I}\|a - 1\| \geq |\lambda| = |\psi(\lambda)|.$$

Hence ψ is contractive. As q is also contractive, we conclude that φ is contractive. □

We want to give the spectrum $\Gamma(A)$ a topology. By definition, $\Gamma(A)$ is a subset of the dual space A^*. In fact, we have seen that $\Gamma(A)$ is contained in the unit ball of A^*. Since A is a Banach space, the dual space A^* comes equipped with a norm. However, this topology is not suitable for our purposes (it has too few compact sets). Instead, we use the following topology.

Definition 1.3.13 (Weak* topology)**.** Let E be a Banach space. The *weak* topology* on E^* is the finest topology that makes the maps $\varphi \mapsto \varphi(\xi)$ continuous for all $\xi \in E$.

Remark 1.3.14. There is also a more explicit description of the open sets in weak* topology: A subset U of the dual space E^* is *weak* open* if for every $\varphi \in U$ there exist $\varepsilon > 0$ and $\xi_1, \dots, \xi_n \in E$ such that

$$\{\psi \in E^* : |\varphi(x_k) - \psi(x_k)| < \varepsilon \text{ for } 1 \leq k \leq n\} \subset U.$$

However, it is often more convenient to work with the abstract characterization.

Theorem 1.3.15 (Banach–Alaoglu). *If E is a Banach space, then the unit ball of E^* is weak* compact.*

We will not prove this result in this course. However, there is an outline of the proof in the case when E is separable in the exercises.

Proposition 1.3.16. *If A is a unital commutative Banach algebra, the spectrum $\Gamma(A)$ is weak* compact.*

Proof. We have already seen that $\Gamma(A)$ is contained in the unit ball of A^*. By the Banach–Alaoglu theorem, it remains to show that $\Gamma(A)$ is weak* closed in A^*. We have $\Gamma(A) = \{\varphi \in A^* \mid \varphi(1) = 1\} \cap \bigcap_{a,b\in A}\{\varphi \in A^* \mid \varphi(ab) = \varphi(a)\varphi(b)\}$. By the definition of the weak* topology, the maps $\varphi \mapsto \varphi(1)$ and $\varphi \mapsto \varphi(ab) - \varphi(a)\varphi(b)$ for $a, b \in A$ are continuous. Thus $\Gamma(A)$ is weak* closed as intersection of weak* closed sets. $\square$

The next result motivates the terminology spectrum for the character space of a commutative C^*-algebra.

Lemma 1.3.17. *Let A be a unital commutative Banach algebra generated by $a \in A$, that is, a is not contained in a proper Banach closed unital subalgebra of A. The map*

$$\Gamma(A) \to \sigma_A(a),\ \varphi \mapsto \varphi(a)$$

is a homeomorphism.

Proof. First we have to show that $\varphi(a) \in \sigma_A(a)$ for all $\varphi \in \Gamma(A)$. Indeed, if $a - \varphi(a)$ were invertible, then

$$1 = \varphi(1) = \varphi((a - \varphi(a))(a - \varphi(a))^{-1}) = (\varphi(a) - \varphi(a))\varphi((a - \varphi(a))^{-1}) = 0,$$

a contradiction.

To see that the map is injective, let $\varphi, \psi \in \Gamma(A)$ with $\varphi(a) = \psi(a)$ and let $B = \{b \in A \mid \varphi(b) = \psi(b)\}$. Since φ and ψ are continuous, B is closed in A. Moreover, since φ and ψ are unital algebra homomorphisms, B is a unital subalgebra of A. As $a \in B$ and a generates A as Banach algebra, we conclude $B = A$. Hence $\varphi = \psi$.

To see that the map is surjective, let $\lambda \in \sigma_A(a)$ and let I be smallest closed ideal containing $a - \lambda$. Let $q\colon A \to A/I$ be the quotient map and $B = q^{-1}(\mathbb{C} + I)$. Note that $a + I = \lambda + I$, hence $a \in B$. Moreover, since the quotient map is a contractive unital algebra homomorphism, B is a closed unital subalgebra of A. As a generates A, we conclude $B = A$. Thus for every $b \in A$ there exists a unique $\varphi(b) \in \mathbb{C}$ such that $b + I = \varphi(b) + I$. It is not hard to see that $\varphi \in \Gamma(A)$ and $\varphi(a) = \lambda$.

By definition of the weak* topology, the map $\varphi \mapsto \varphi(a)$ is continuous. Since $\Gamma(A)$ is weak* compact, it is a homeomorphism. $\square$

Definition 1.3.18 (Gelfand transform)**.** Let A be a unital commutative Banach algebra. For $a \in A$ let

$$\hat{a}\colon \Gamma(A) \to \mathbb{C},\, \varphi \mapsto \varphi(a).$$

The *Gelfand transform* is the map

$$\Gamma\colon A \to C(\Gamma(A)),\, a \mapsto \hat{a}.$$

Lemma 1.3.19. *Let A be a unital commutative Banach algebra. An element $a \in A$ is contained in a maximal ideal if and only if it is not invertible.*

Proof. If a is invertible and I an ideal containing a, then $1 = a^{-1}a \in I$, hence I cannot be proper and in particular not maximal.

Conversely, assume that a is not invertible and let $I = \{ba \mid b \in A\}$. Since A is commutative, I is an ideal, and since A is unital, $a \in I$. If $1 \in I$, there would exists $b \in A$ such that $ba = 1$, in contradiction to our assumption that a is not invertible. Hence I is a proper ideal. Let $\mathcal{I}$ be the set of all proper ideals containing I, ordered by inclusion. By Zorn's lemma, $\mathcal{I}$ has a maximal element J (exercise). By definition, J is a maximal ideal containing a. □

Proposition 1.3.20. *Let A be a unital commutative Banach algebra. The Gelfand transform is a contractive unital algebra homomorphism and for every $a \in A$, the Gelfand transform $\Gamma(a)$ is invertible in $C(\Gamma(A))$ if and only if a is invertible in A.*

Proof. We have already seen that $\|\varphi\| \leq 1$ for every $\varphi \in \Gamma(A)$. Thus

$$\|\hat{a}\| = \sup_{\varphi \in \Gamma(A)} |\hat{a}(\varphi)| = \sup_{\varphi \in \Gamma(A)} |\varphi(a)| \leq \|a\|.$$

It is easy to see that the Gelfand transform is an algebra homomorphism. If $a \in A$ is invertible, then $\Gamma(a)\Gamma(a^{-1}) = \Gamma(1) = 1$, hence $\Gamma(a)$ is invertible. Conversely, if $a \in A$ is not invertible, then a is contained in a maximal ideal by the previous lemma. Thus A/I is a Banach algebra in which every element is invertible, hence $A/I \cong \mathbb{C}$ by the Banach–Mazur theorem. The quotient map $q\colon A \to A/I$ is a character and $q(a) = 0$. Thus $\hat{a}(q) = 0$, which means that $\hat{a}$ is not invertible. □

Corollary 1.3.21. *If A is a unital Banach algebra, then $\sigma(\Gamma(a)) = \sigma(a)$ and $\|\Gamma(a)\| = r(\Gamma(a)) = r(a)$ for all $a \in A$.*

Exercises

1. Recall that a topological space is called *separable* if it has a countable dense subset. Every subset of a separable metric space is again separable (this fails for general topological spaces). Let E be a separable Banach space.

   (a) Show that if $\{\xi_k \mid k \in \mathbb{N}\}$ is a dense subset of the unit ball of E, then
   $$d(\varphi, \psi) = \sum_{k=0}^{\infty} 2^{-k} |\varphi(\xi_k) - \psi(\xi_k)|$$
   is a metric on the unit ball of E^*.

   (b) Show that the metric from (a) induces the weak* topology on the unit ball of E^*.

   (c) Show that every sequence in the unit ball of E^* has a subsequence that converges with respect to d.

2. Let A be a unital commutative algebra and $I \trianglelefteq A$ a proper ideal. Show that the set of proper ideals containing I is partially ordered by inclusion and every chain has a maximal element.

3. In this exercise we construct the *Stone–Čech compactification* of the natural numbers. Let ℓ^∞ denote the space of all bounded complex sequences with the supremum norm. Clearly, ℓ^∞ is a unital commutative C^*-algebra. We denote its spectrum by $\beta\mathbb{N}$.

   (a) For $n \in \mathbb{N}$ let $\delta_n \colon \ell^\infty \to \mathbb{C}$, $x \mapsto x_n$. Show that $\{\delta_n \mid n \in \mathbb{N}\}$ is dense in $\beta\mathbb{N}$.

   (b) Show that $\{\delta_n\}$ is open in $\beta\mathbb{N}$ for every $n \in \mathbb{N}$.

   (c) Show that $\beta\mathbb{N}$ has the following universal property: For every compact Hausdorff space K and every map $f \colon \mathbb{N} \to K$ there exists a unique continuous map $\tilde{f} \colon \beta\mathbb{N} \to K$ such that $\tilde{f}(\delta_n) = f(n)$ for all $n \in \mathbb{N}$.

4. In this exercise we revisit the Banach limits from the appendix using so-called *ultralimits*. Let x be a bounded sequence in $\mathbb{C}$. By the previous exercise, there exists a unique continuous map $g \colon \beta\mathbb{N} \to \mathbb{C}$ such that $g(\delta_n) = x_n$ for $n \in \mathbb{N}$. For $\omega \in \beta\mathbb{N} \setminus \{\delta_n \mid n \in \mathbb{N}\}$ we define $\lim_{n\to\omega} x_n = g(\omega)$.

   (a) Show that if x is convergent, then $\lim_{n\to\omega} x_n = \lim_{n\to\infty} x_n$.

(b) Show that

$$\mathrm{LIM}\colon \ell^\infty \to \mathbb{C},\, x \mapsto \lim_{n\to\omega} \frac{1}{n}\sum_{k=1}^n x_k$$

is a Banach limit.

1.4 Continuous functional calculus

Let A be a (unital) C^*-algebra. The definition of normal, self-adjoint, positive etc. elements of A is the same as the algebraic definition for bounded operators. We write A_{sa} for the set of self-adjoint elements and A_+ for the set of positive elements of A. If $a, b \in A_{\mathrm{sa}}$, we define $a \leq b$ if $b - a \in A_+$.

Lemma 1.4.1. *Let A be a C^*-algebra. If $a, b \in A_{\mathrm{sa}}$ with $a \leq b$ and $c \in A$, then $c^*ac \leq c^*bc$.*

Proof. If $d \in A$ such that $b - a = d^*d$, then $c^*(b-a)c = (dc)^*(dc) \geq 0$. □

Proposition 1.4.2. *If A is a C^*-algebra and $a \in A$ is normal, then $\|a\| = r(a)$.*

Proof. If $a \in A$ is self-adjoint, then $\|a\|^2 = \|a^*a\| = \|a^2\|$. By induction one sees that $\|a\|^{2^n} = \|a^{2^n}\|$ for all $n \in \mathbb{N}$ and therefore $\|a\| = \lim_{n\to\infty}\|a^{2^n}\|^{2^{-n}} = r(a)$.

If x is normal, then $r(x^*x) \leq r(x^*)r(x) = r(x)^2$ as shown in an exercise before. Thus

$$\|x\|^2 = \|x^*x\| = r(x^*x) \leq r(x)^2 \leq \|x\|^2.$$

□

Corollary 1.4.3. *Every unital $*$-homomorphism between unital C^*-algebras is contractive and every unital $*$-isomorphism is isometric.*

Proof. If A, B are unital C^*-algebras and $\Phi\colon A \to B$ is a unital $*$-homomorphism, then $\sigma_B(\Phi(a)) \subset \sigma_A(a)$ for all $a \in A$. Thus

$$\|\Phi(a)\|^2 = \|\Phi(a^*a)\| = r(\Phi(a^*a)) \leq r(a^*a) = \|a\|^2.$$

If Φ is a unital $*$-isomorphism, then $\sigma_B(\Phi(a)) = \sigma_A(a)$ for all $a \in A$ and the inequality in the previous equation becomes an equality. □

Corollary 1.4.4. *On a given unital $*$-algebra there is at most one norm that makes it a C^*-algebra.*

Proof. The identity map is a unital $*$-isomorphism, hence isometric by the previous corollary. □

Lemma 1.4.5. *Let A be a unital C^*-algebra. If $a \in A$ is self-adjoint, then $\sigma_A(a) \subset \mathbb{R}$.*

Proof. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + i\beta \in \sigma_A(a)$. For $t \in \mathbb{R}$ let $b = a - \alpha + it$. We have $i(\beta + t) \in \sigma_A(b)$ and y is normal. Thus

$$(\beta + t)^2 \leq r(b)^2 = \|b^* b\| = \|(b - \alpha)^2 + t^2\| \leq \|b - \alpha\|^2 + t^2,$$

which implies $\beta^2 + 2t\beta \leq \|b - \alpha\|^2$. If $\beta \neq 0$, the supremum of the left side over $t \in \mathbb{R}$ is ∞. Therefore $\beta = 0$. $\square$

Proposition 1.4.6 (Invariance of the spectrum)**.** *Let A be a unital C^*-algebra and $B \subset A$ a unital C^*-subalgebra. If $b \in B$, then $\sigma_B(b) = \sigma_A(b)$.*

Proof. Note that $\mathrm{Inv}(B) \subset \mathrm{Inv}(A)$. Let $b \in B$ be self-adjoint and not invertible in B. By the previous lemma, $b - i/n \in \mathrm{Inv}(B)$ for $n \in \mathbb{N}$. If $(\|(b - i/n)^{-1}\|)$ were bounded, then ... As inversion is continuous on $\mathrm{Inv}(A)$, we see that b is not invertible in A.

For general $b \in B$ we have $b \in \mathrm{Inv}(B)$ if and only if $b^* b \in \mathrm{Inv}(B)$ if and only if $b^* b \in \mathrm{Inv}(A)$ if and only if $b \in \mathrm{Inv}(A)$. In particular, $\sigma_A(b) = \sigma_B(b)$ for all $b \in B$. $\square$

Theorem 1.4.7 (Gelfand representation theorem)**.** *If A is a unital commutative C^*-algebra, then the Gelfand transform $\Gamma \colon A \to C(\Gamma(A))$ is an isometric unital $*$-isomorphism.*

Proof. If $a \in A$ is self-adjoint, then $\operatorname{im} \Gamma(a) = \sigma(\Gamma(a)) = \sigma(a) \subset \mathbb{R}$. Thus $\Gamma(a)$ is self-adjoint. In general, we can write $a \in A$ as $a = b + ic$ with $b = \frac{1}{2}(a + a^*)$ and $c = \frac{1}{2i}(a - a^*)$ self-adjoint and $\Gamma(a) = \Gamma(b) + i\Gamma(c)$ with $\Gamma(b)$, $\Gamma(c)$ self-adjoint. Thus $\Gamma(a^*) = \overline{\Gamma(a)}$ and so Γ is a $*$-homomorphism.

As discussed before, $\varphi(1) = 1$ for all $\varphi \in \Gamma(A)$, hence Γ is unital.

We have seen before that $\|\Gamma(a)\| = r(\Gamma(a)) = r(a)$ for all $a \in A$. Thus

$$\|\Gamma(a)\|^2 = \|\Gamma(a^* a)\| = r(a^* a) = \|a^* a\| = \|a\|^2$$

for all $a \in A$, which means that Γ is isometric.

As a consequence, the image of Γ is a closed unital $*$-subalgebra of $C(\Gamma(A))$. By definition, the range of Γ separates the points of $\Gamma(A)$. It follows from the Stone–Weierstraß theorem that Γ is surjective. $\square$

Corollary 1.4.8. *Every unital commutative C^*-algebra is $*$-isomorphic to $C(X)$ for some compact Hausdorff space X.*

If A is a unital C^*-algebra and $a \in A$, then the unital C^*-algebra B generated by a is commutative if and only if a is normal. In this case, we have already seen that $\hat{a}$ is a homeomorphism from $\Gamma(B)$ onto $\sigma(a)$.

Definition 1.4.9 (Continuous functional calculus)**.** Let A be a unital C^*-algebra and $a \in A$ normal. Let B denote the unital C^*-algebra generated by a and $\Gamma\colon C(\Gamma(B)) \to B$ the Gelfand transform. For $f \in C(\sigma(a))$ we define $f(a) = \Gamma^{-1}(f \circ \hat{a}) \in B \subset A$. The map

$$C(\sigma(a)) \to A,\ f \mapsto f(a)$$

is called the *continuous functional calculus.*

Theorem 1.4.10 (Continuous funtional calculus)**.** *Let A be a unital C^*-algebra and $a \in A$ normal. The functional calculus satisfies the following properties:*

(a) If $f(z) = \sum_{k=0}^{n} \alpha_k z^k$ for $z \in \sigma(a)$, then $f(a) = \sum_{k=0}^{n} \alpha_k a^k$.

(b) If $\lambda \in \mathbb{C} \setminus \sigma(a)$ and $f(z) = (z-\lambda)^{-1}$, then $f(a) = (a-\lambda)^{-1}$.

(c) If $f \in C(\sigma(a))$, then $\sigma(f(a)) = f(\sigma(a))$ and $\|f(a)\| = \|f\|_\infty$.

(d) If B is unital C^*-algebra and $\Phi\colon A \to B$ is a unital $*$-homomorphism, then $\Phi(f(a)) = f(\Phi(a))$ for all $f \in C(\sigma(a))$

(e) If (a_n) is a sequence of normal elements in A that converges to a and Ω is a compact neighborhood of $\sigma(a)$, then $\sigma(a_n) \subset \Omega$ eventually and $f(a_n) \to f(a)$ for every $f \in C(\Omega)$.

Proof. (a) If $f(z) = 1$, then $f(a) = \Gamma^{-1}(1 \circ \hat{a}) = \Gamma^{-1}(1) = 1$, and if $f(z) = z$, then $f(a) = \Gamma^{-1}(\hat{a}) = a$. For general polynomials, the claim follows from the fact that continuous functional calculus is an algebra homomorphism.

(b) Let $g(z) = z - \lambda$. Since continuous functional calculus is a unital algebra homomorphism, we have $1 = (fg)(a) = f(a)g(a) = g(a)f(a)$. By (a), $g(a) = a - \lambda$. Thus $f(a) = (a-\lambda)^{-1}$.

(c) follows directly from the fact that Γ is an isometric $*$-isomorphism.

(d) is clear if f is a polynomial. Arbitrary $f \in C(\sigma(a))$ can be approximated by polynomials in supremum norm by the Stone–Weierstraß theorem, and then the result follows from the continuity of the Gelfand transform.

(e) Since $\sigma(a)$ is compact and Ω is a neighborhood of $\sigma(a)$, there exists $\varepsilon > 0$ such that $d(\lambda, \sigma(a)) \geq \varepsilon$ for all $\lambda \in \Omega \setminus \mathbb{C}$. By (b) and (c),

$$\|(a-\lambda)^{-1}\| = \sup_{z \in \sigma(a)} |(z-\lambda)^{-1}| = \frac{1}{d(\lambda, \sigma(a))} \leq \varepsilon^{-1}.$$

As we have shown when we proved that the invertible elements form an open subset, if $\|b - a\| = \|(b - \lambda) - (a - \lambda)\| < \|(a - \lambda)^{-1}\|^{-1} \leq \varepsilon$, then $b - \lambda$ is invertible. In particular, $\mathbb{C} \setminus \Omega \subset \rho(a_n)$ for n sufficiently large.

To see that $f(a_n) \to f(a)$, let $\varepsilon > 0$. By the Stone–Weierstraß theorem, there exists a polynomial $g \in C(\Omega)$ such that $\|f - g\|_\infty \leq \frac{\varepsilon}{2}$. Since multiplication in A is continuous, $g(a_n) \to g(a)$. Thus

$$\begin{aligned} \|f(a_n) - f(a)\| &\leq \|f(a_n) - g(a_n)\| + \|g(a_n) - g(a)\| + \|g(a) - f(a)\| \\ &\leq 2\|f - g\|_\infty + \|g(a_n) - g(a)\| \\ &\leq \varepsilon + \|g(a_n) - g(a)\|. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude $\|f(a_n) - f(a)\| \to 0$. □

1.5 Applications of Functional Calculus

Definition 1.5.1 (Real, imaginary, positive, negative part)**.** Let A be a C^*-algebra and $a \in A$. The *real* and *imaginary part* of a are defined as $\operatorname{Re} a = \frac{1}{2}(a + a^*)$ and $\operatorname{Im} a = \frac{1}{2i}(a - a^*)$. If a is self-adjoint, then its *positive* and *negative part* are defined as $a_+ = \frac{1}{2}(a + |a|)$ and $a_- = \frac{1}{2}(a - |a|)$.

Remark 1.5.2. Note that $a = \operatorname{Re} a + i \operatorname{Im} a$. In particular, every element of a C^*-algebra is a linear combination of two self-adjoint elements. Moreover, $\|\operatorname{Re} a\|, \|\operatorname{Im} a\| \leq \frac{1}{2}(\|a\| + \|a^*\|) = \|a\|$.

The positive and negative part of a self-adjoint element can equivalently be defined in terms of functional calculus (i. e. applying the function $\lambda \mapsto \lambda_\pm$ to x). It follows immediately that $\sigma(x_\pm) \subset [0, \infty)$, $x_+x_- = x_-x_+ = 0$ and $x = x_+ - x_-$.

Lemma 1.5.3. *If A is a unital C^*-algebra and $a, b \in A$ are self-adjoint with $\sigma(a), \sigma(b) \subset [0, \infty)$, then $\sigma(a + b) \subset [0, \infty)$.*

Proof. First note that $\sigma(\|a\| - a) \subset [0, \|a\|]$ and similarly for b. By the spectral radius formula, $\|\|a\| - a\| = r(\|a\| - a) \leq \|a\|$ and $\|\|b\| - b\| \leq \|b\|$. Thus

$$\begin{aligned} \sup_{\lambda \in \sigma(a+b)} (\|a\| + \|b\| - \lambda) &= r(\|a\| + \|b\| - (a + b)) \\ &\leq \|\|a\| - a\| + \|\|b\| - b\| \\ &\leq \|a\| + \|b\|. \end{aligned}$$

Hence $\sigma(a + b) \subset [0, \infty)$. □

Proposition 1.5.4. *Let A be a unital C^*-algebra. A normal element $a \in A$ is*

(a) self-adjoint if and only if $\sigma(a) \subset \mathbb{R}$,

(b) positive if and only if $\sigma(a) \subset [0, \infty)$,

(c) unitary if and only if $\sigma(a) \subset \{z \in \mathbb{C} : |z| = 1\}$,

(d) a projection if and only if $\sigma(a) \subset \{0, 1\}$.

Proof. (a), (c) and (d) follow immediately from functional calculus. We only show (a) here as a demonstration. Let $f\colon \sigma(a) \to \mathbb{C}$, $\lambda \mapsto \lambda$. By functional calculus, $\sigma(a) = \sigma(f) = \sigma(a)$ and $a^* = \bar{f}(a)$. Thus $a = a^*$ if and only if $f = \bar{f}$ if and only if $\sigma(a) = \sigma(f) \subset \mathbb{R}$.

(b) If $\sigma(a) \subset [0, \infty)$, then $a = (a^{1/2})^* a^{1/2}$ by functional calculus. Assume conversely that $a = b^*b$ for some $b \in A$. We need to show that $\sigma(a) \subset [0, \infty)$ or, equivalently, $a_- = 0$.

Let $c = ba_-$ and note that $c^*c = a_-b^*ba_- = a_-aa_- = -a_-^3$. Hence $\sigma(cc^*) \subset \sigma(c^*c) \cup \{0\} \subset (-\infty, 0]$. We have $c^*c + cc^* = 2(\operatorname{Re} c)^2 + 2(\operatorname{Im} c)^2$ and $\sigma(2(\operatorname{Re} c)^2 + 2(\operatorname{Im} c)^2) \subset [0, \infty)$ by the previous lemma. Moreover, since $\sigma(-cc^*) \subset [0, \infty)$, we also have $\sigma(c^*c) = \sigma(2(\operatorname{Re} c)^2 + 2(\operatorname{Im} c)^2 - cc^*) \subset [0, \infty]$ by the previous lemma. Therefore $\sigma(-a_-^3) = \sigma(c^*c) = 0$. As a_- is self-adjoint, this implies $\|a_-\|^3 = \|a_-^3\| = r(a_-^3) = 0$. □

Corollary 1.5.5. *An element v of a unital C^*-algebra is a partial isometry if and only if v^* is a partial isometry.*

Proof. By definition, v is a partial isometry if and only if v^*v is a projection. Since v^*v is self-adjoint, this is equivalent to $\sigma(v^*v) \subset \{0, 1\}$ by the previous proposition. As $\sigma(vv^*) \subset \sigma(v^*v) \cup \{0\}$, the conclusion follows. □

Corollary 1.5.6. *Let A be a unital C^*-algebra. The set A_+ of positive elements of A is closed and if $a, b \in A_+$ and $\lambda, \mu \geq 0$, then $\lambda a + \mu b \in A_+$. Moreover, if $a \in A$ is self-adjoint, then $-\|a\| \leq a \leq \|a\|$.*

Proof. By the previous proposition, $A_+ = \{a \in A_{\mathrm{sa}} \mid \sigma(a) \subset [0, \infty)\}$. If $a_n \in A_+$ and $a_n \to a$, then $0 = (a_n)_- \to a_-$ by continuity of functional calculus. Thus $a_- = 0$, which implies $a \in A_+$. Thus A_+ is closed. The other statements are easy consequences of the previous results. □

Proposition 1.5.7. *Every element of a unital C^*-algebra is a linear combination of four unitaries.*

Proof. Every element a of a C^*-algebra is a linear combination of two self-adjoint elements (its real and imaginary part). By rescaling, we may further assume that these self-adjoint elements have norm at most 1. If $a \in A$ is self-adjoint with $\|a\| \leq 1$, then $u = a + i(1 - a^2)^{1/2}$ is unitary:

$$uu^* = u^*u = (a + i(1 - a^2)^{1/2})(a - i(1 - a^2)^{1/2}) = a^2 + 1 - a^2 = 1.$$

Moreover, $u + u^* = x$. □

Theorem 1.5.8 (Operator monotonicity of the square root)**.** *Let A be a unital C^*-algebra. If $a, b \in A$ are positive and $a \leq b$, then $a^{1/2} \leq b^{1/2}$. Moreover, if a and b are additionally invertible, then $b^{-1} \leq a^{-1}$.*

Proof. We first assume that $a, b \in A$ are positive and invertible. Recall that $c^* A_+ c \subset A_+$ for $c \in A$. As $a \leq b$, we have $b^{-1/2} a b^{-1/2} \leq 1$, thus $r(a^{1/2} b^{-1} a^{1/2}) = r(b^{-1/2} a b^{-1/2}) \leq 1$, which implies $a^{1/2} b^{-1} a^{1/2} \leq 1$. Hence $b^{-1} \leq a^{-1}$.

Moreover, $\|a^{1/2} b^{-1/2}\|^2 = \|b^{-1/2} a b^{-1/2}\|^2 \leq 1$, which implies

$$\begin{aligned} b^{-1/4} a^{1/2} b^{-1/4} &\leq r(b^{-1/4} a^{1/2} b^{-1/4}) \\ &= r(a^{1/4} b^{-1/2} a^{1/4}) \\ &= r(a^{1/2} b^{-1/2}) \\ &\leq \|a^{1/2} b^{-1/2}\| \\ &\leq 1. \end{aligned}$$

Therefore $a^{1/2} \leq b^{1/2}$.

In general, if $a, b \in A_+$ with $a \leq b$ are not necessarily invertible and $\varepsilon > 0$, we have $a + \varepsilon \leq b + \varepsilon$ and $a + \varepsilon$, $b + \varepsilon$ are positive and invertible (since $\sigma(a+\varepsilon), \sigma(b+\varepsilon) \subset [\varepsilon, \infty)$). By the previous paragraph, $(a+\varepsilon)^{1/2} \leq (b+\varepsilon)^{1/2}$. By continuity of functional calculus, $(a + \varepsilon)^{1/2} \to a^{1/2}$ and $(b + \varepsilon)^{1/2} \to b^{1/2}$ as $\varepsilon \to 0$. □

Remark 1.5.9. If $I \subset \mathbb{R}$ is an interval, a function $f \colon I \to \mathbb{R}$ is called *operator monotone* if $f(a) \subset f(b)$ whenever a, b are self-adjoint elements of a unital C^*-algebra with $a \leq b$. Since $\mathbb{C}$ is a unital C^*-algebra with self-adjoint part $\mathbb{R}$, every operator monotone function is monotone. The converse is not true: For example, $\lambda \mapsto \lambda^2$ is not operator monotone on $[0, \infty)$. The previous result shows that $\lambda \mapsto \sqrt{\lambda}$ and $\lambda \mapsto -1/\lambda$ are operator monotone on $[0, \infty)$.

Definition 1.5.10 (Absolute value)**.** If A is a unital C^*-algebra and $a \in A$, then the *absolute value* $|a|$ of a is defined as $|a| = (a^* a)^{1/2}$.

Corollary 1.5.11. *If A is a unital C^*-algebra and $a, b \in A$, then $|ab| \leq \|a\||b|$.*

Proof. Since $a^*a \leq \|a^*a\| = \|a\|^2$, we have $b^*a^*ab \leq \|a\|^2 b^*b$. The claim now follows from the operator monotonicity of the square root function. $\square$

1.6 Bonus: Group C^*-algebras

Let G be a group. As we have seen at the very beginning of this course, the formal finite linear combinations of elements of G (more rigorously, the free complex vector space over G) form a unital $*$-algebra $\mathbb{C}[G]$ with the operations

$$\left(\sum_g \alpha_g g\right)\left(\sum_h \beta_h h\right) = \sum_{g,h} \alpha_g \beta_h gh,$$

$$\left(\sum_g \alpha_g g\right)^* = \sum_g \overline{\alpha_g} g^{-1}.$$

We also claimed that

$$\left\|\sum_g \alpha_g g\right\|_{\mathrm{u}} = \sup\left\{\left\|\sum_g \alpha_g \pi(g)\right\| : \pi\colon G \to \mathbb{U}(H) \text{ group hom.}\right\}$$

defines a C^*-norm on $\mathbb{C}[G]$, where $\mathbb{U}(H)$ denotes the group of unitary operators on H.

The C^* norm property is indeed not hard to verify given that the operator norm on $\mathbb{B}(H)$ is a C^* norm. What that takes some more considerations is the fact that the supremum on the right side is always finite and the resulting semi-norm is positive definite.

Lemma 1.6.1. *If G is a group, then $\|\cdot\|_{\mathrm{u}}$ is a norm on $\mathbb{C}[G]$.*

Proof. First note that if $\pi\colon G \to \mathbb{U}(H)$ is a group homomorphism, then

$$\|\pi(g)\|^2 = \|\pi(g)^{-1}\pi(g)\| = \|\pi(e)\| = \|1\| \leq 1.$$

Thus

$$\left\|\sum_g \alpha_g \pi(g)\right\| \leq \sum_g |\alpha_g|.$$

In particular, $\left\|\sum_g \alpha_g g\right\|_{\mathrm{u}} < \infty$.

To show that $\|\cdot\|_{\mathrm{u}}$ is positive definite, define

$$\lambda_g\colon \ell^2(G) \to \ell^2(G),\ (\lambda_g f)(h) = f(g^{-1}h)$$

for $g \in G$. Clearly, λ_g is a bijective isometry (with inverse $\lambda_{g^{-1}}$), hence a unitary. Moreover, $\lambda_e = 1$ and $\lambda_g\lambda_h = \lambda_{gh}$ for $g, h \in G$ are easy to see. Thus $\lambda\colon G \to \mathbb{U}(\ell^2(G))$ is a group homomorphism.

We have

$$\sum_g \alpha_g \lambda_g \mathbb{1}_e = \sum_g \alpha_g \mathbb{1}_g.$$

In particular, $\|\sum_g \alpha_g \lambda_g\| = 0$ if and only if $\alpha_g = 0$ for all $g \in G$. Therefore $\|\cdot\|_{\mathrm{u}}$ is positive definite. □

Definition 1.6.2 (Reduced and full group C^*-algebra)**.** Let G be a group. The representation $\lambda\colon G \to \mathbb{U}(\ell^2(G))$ from the proof of the previous lemma is called the *left regular representation* of G. The closure of $\{\sum_g \alpha_g \lambda_g \mid \alpha \in c_c(G)\}$ is called the *reduced group C^*-algebra* and denoted by $C_r^*(G)$.

The completion of $\mathbb{C}[G]$ with respect to $\|\cdot\|_{\mathrm{u}}$ is called the *full group C^*-algebra* and denoted by $C^*(G)$.

By definition, whenever $\pi\colon G \to \mathbb{U}(H)$ is a group homomorphism, then π can be extended to a contractive linear map from $C^*(G)$ to $\mathbb{B}(H)$, still denoted by the same letter π. It is not hard to see that this extension is a unital $*$-homomorphism. In particular, there is a surjective unital $*$-homomorphism $\lambda\colon C^*(G) \to C_r^*(G)$ for every group G. In general, λ is not injective.

Definition 1.6.3 (Amenable group)**.** A group G is called amenable if the left regular representation $\lambda\colon C^*(G) \to C_r^*(G)$ is injective.

Example 1.6.4. Every finite group is amenable. In this case, $C^*(G) = \mathbb{C}[G]$ and $\lambda|_{\mathbb{C}[G]}$ is injective, as we have seen before.

The name "amenable" is a word play on the word "mean" in the sense of the following definition. Invariant means are the more common way to define amenable groups, but this definition is equivalent to ours, as we will see soon.

Definition 1.6.5 (Invariant mean)**.** Let G be a group. An *left-invariant mean* is a map $\mu\colon \ell^\infty(G) \to \mathbb{C}$ such that

- $\mu(1) = 1$,
- $\mu(f) \geq 0$ if $f \geq 0$,

- $\mu(f(g^{-1}\cdot)) = \mu(f)$ for all $f \in \ell^\infty(G)$ and $g \in G$.

Theorem 1.6.6. *For a group G, the following properties are equivalent:*

(i) G is amenable.

(ii) $C_r^*(G)$ has a character.

(iii) $\ell^\infty(G)$ has a left-invariant mean.

Proof. The proof requires some techniques we have not covered in this course. Some of them can be found in Chaper **??**. We only sketch (i) $\implies$ (iii) and (ii) $\implies$ (iii) here. For a complete proof, see Brown, Ozawa. *C^*-Algebras and Finite-Dimensional Approximations*, Theorem 2.6.8.

(i) $\implies$ (ii): Let $\pi\colon G \to S^1$, $g \mapsto 1$ be the trivial representation. By the definition of $C^*(G)$, the map π can be extended to a unital $*$-homomorphism $\varphi\colon C^*(G) \to \mathbb{C}$. If G is amenable, then λ is a $*$-isomomorphism and $\varphi \circ \lambda^{-1}\colon C_r^*(G) \to \mathbb{C}$ is a character.

(ii) $\implies$ (iii): Let $\varphi\colon C_r^*(G) \to \mathbb{C}$ be a character. We can extend φ to a bounded linear functional $\psi\colon \mathbb{B}(\ell^2(G)) \to \mathbb{C}$ with $\psi(1) = 1$ and $\psi(x) \geq 0$ for all $x \geq 0$ (see Lemma **??**). Since $\psi|_{C_r^*(G)}$ is a $*$-homomorphism, we have

$$\psi(\lambda_g x \lambda_g^*) = \varphi(\lambda_g)\psi(x)\overline{\varphi(\lambda_g)} = \psi(x)$$

for all $x \in \mathbb{B}(\ell^2(G))$ and $g \in G$ (see Lemma **??**).

For $f \in \ell^\infty(G)$ consider the multiplication operator M_f on $\ell^2(G)$. We have $\lambda_g M_f \lambda_g^* = M_{f(g^{-1}\cdot)}$. Thus

$$\mu\colon \ell^\infty(G) \to \mathbb{C},\ f \mapsto \psi(M_f)$$

is a left-invariant mean. □

Corollary 1.6.7. *Every abelian group is amenable.*

Proof. If G is abelian, then $C_r^*(G)$ is commutative and thus has a character by Gelfand theory. □

Example 1.6.8. Let $\mathbb{F}_2$ be the free group on two generators. As a set, $\mathbb{F}_2$ consists of all finite words with letters a, a^{-1}, b, b^{-1} such that no two adjacent letters are inverse to each other. The identity element in this group is the empty word and the group multiplication is given by concatenation (with cancellation of adjacent inverse letters). For example, $(ab^{-1}ab)(b^{-1}aa) = ab^{-1}aaa$.

The group $\mathbb{F}_2$ is not amenable (exercise).

If G is an abelian group, then $C^*(G)$ is a unital commutative C^*-algebra. By the Gelfand representation theorem, $C^*(G) = C(\Gamma(C^*(G)))$. Let us determine the spectrum of $C^*(G)$.

Definition 1.6.9 (Pontryagin dual)**.** Let G be an abelian group. The *Pontryagin dual* $\hat{G}$ of G is the set of all group homomorphisms from G to S^1.

Note that $\hat{G}$ becomes itself a group when endowed with the multiplication $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$ for $\chi_1, \chi_1 \in \hat{G}$ and $g \in G$.

Lemma 1.6.10. *If G is an abelian group, then the map*

$$\Gamma(C^*(G)) \to \hat{G},\ \varphi \mapsto \varphi|_G$$

is bijective.

Proof. First recall that $\varphi \in \Gamma(C^*(G))$ is a unital $*$-homomorphism. In particular, if $g \in G$, then $|\varphi(g)|^2 = \varphi(g^{-1}g) = \varphi(e) = 1$. Thus $\varphi(G) \subset S^1$.

If $\chi \in \hat{G}$, then χ is a group homomorphism from G to $\mathbb{U}(\mathbb{C})$. Hence it can be extended to a unital $*$-homomorphism from $C^*(G)$ to $\mathbb{C}$. This settles surjectivity.

To see that the map is injective, note that if $\operatorname{span} G$ is dense in $C^*(G)$. Hence if two bounded linear maps on $C^*(G)$ coincide on G, they coincide on $C^*(G)$. $\square$

Remark 1.6.11. We already know that the spectrum of a unital C^*-algebra is a compact Hausdorff space. The previous lemma allows us to transport the topology on $\Gamma(C^*(G))$ to $\hat{G}$. This makes $\hat{G}$ into what is called a compact topological group (see the exercises).

Corollary 1.6.12. *If G is an abelian group, then $C^*(G)$ is $*$-isomorphic to $C(\hat{G})$.*

Example 1.6.13. Let $G = \mathbb{Z}$. Since G is a free group, the map $\hat{G} \to S^1$, $\chi \mapsto \chi(1)$ is a bijection. Moreover, if $\chi \in \hat{G}$ and $n \in \mathbb{Z}$, then $\chi(n) = \chi(1)^n$. Hence $\chi \mapsto \chi(n)$ is continuous for all $n \in \mathbb{Z}$ if and only if $\chi \mapsto \chi(1)$ is continuous. Thus $\hat{G} \to S^1$, $\chi \mapsto \chi(1)$ is a homeomorphism. It follows that $C^*(\mathbb{Z}) \cong C(S^1)$.

More generally, one can show that for $d \in \mathbb{N}$, the group C^*-algebra $C^*(\mathbb{Z}^d)$ is $*$-isomorphic to $C(\mathbb{T}^d)$, where $\mathbb{T}^d = (S^1)^d$ is the d-dimensional torus.

Example 1.6.14. Let $n \in \mathbb{N}$ and let $C_n = \{z \in S^1 \mid z^n = 1\}$ the set of n-th roots of unity. If $G = \mathbb{Z}/n\mathbb{Z}$, the map $\hat{G} \to C_n$, $\chi \mapsto \chi(1)$ is a bijection and both sets carry the discrete topology. Thus $C^*(\mathbb{Z}/n\mathbb{Z}) \cong C(C_n) \cong \mathbb{C}^n$.

Exercises

1. Let G be an abelian group.

   (a) Endow $\hat{G}$ with the finest topology that makes the maps $\hat{G} \to S^1$, $\chi \mapsto \chi(g)$ continuous for all $g \in G$. Show that the map

   $$\Gamma(C^*(G)) \to \hat{G},\ \varphi \mapsto \varphi|_G$$

   is a homeomorphism.

   (b) Show that the maps

   $$\begin{aligned} \hat{G} \times \hat{G} \to \hat{G}, (\chi_1, \chi_2) \mapsto \chi_1 \chi_2 \\ \hat{G} \to \hat{G}, \chi \mapsto \chi^{-1} \end{aligned}$$

   are continuous. Here $\hat{G} \times \hat{G}$ is endowed with the product topology.

2. Inside the free group $\mathbb{F}_2$, let A^+, A^- be the set of words beginning with a, a^{-1}, respectively, and likewise define B^+, B^-. Further, let $C = \{1, b, b^2, \dots\} \subset \mathbb{F}_2$.

   (a) Show that

   $$\begin{aligned} \mathbb{F}_2 &= A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \cup C) \\ &= A^+ \sqcup aA^- \\ &= b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C). \end{aligned}$$

   (b) Show that $\mathbb{F}_2$ is not amenable (Hint: Use the characterization with left-invariant means).

Appendix A

The Hahn–Banach theorem

The Hahn–Banach theorem is one of the most important results in functional analysis. If provides one with “sufficiently many” bounded linear functionals on a Banach space in order to distinguish its elements. This is an almost universally useful fact.

In contrast to the rest of this course, we consider both real and complex vector spaces in this section. We denote by $\mathbb{K}$ a base field that is either $\mathbb{C}$ or $\mathbb{R}$.

To start with, recall the following result from set theory.

Lemma A.0.1 (Zorn). *If $\mathcal{P}$ is a non-empty partially ordered set such that every chain in $\mathcal{P}$ has an upper bound, then $\mathcal{P}$ has a maximal element.*

Remark A.0.2. The terminology used in Zorn's lemma is defined as follows. A relation $\prec$ on a set $\mathcal{P}$ is called a *partial order* if it is

- transitive, that is, $x \prec y$ and $y \prec z$ implies $x \prec z$ for all $x, y, z \in \mathcal{P}$,
- reflexive, that is, $x \prec x$ for all $x \in \mathcal{P}$,
- anti-symmetric, that is, $x \prec y$ and $y \prec x$ implies $x = y$ for all $x, y \in \mathcal{P}$.

A subset $\mathcal{C}$ of $\mathcal{P}$ is called a *chain* if for every pair $(x, y) \in \mathcal{P}^2$ the relation $x \prec y$ or $y \prec x$ holds. An *upper bound* for $\mathcal{C}$ is an element $z \in \mathcal{P}$ such that $x \prec z$ for all $x \in \mathcal{C}$. An element $z \in \mathcal{P}$ is a *maximal element* of $\mathcal{P}$ if $x \prec z$ for all $x \in \mathcal{P}$.

We do not prove this result here. In fact, in the usual set-theoretic foundations of mathematics, Zorn's lemma is equivalent to the axiom of choice, so we may as well consider it as one of our axioms for the purpose of this course.

Definition A.0.3 (Sublinear functional, semi-norm)**.** Let E be a vector space over $\mathbb{K}$. A *sublinear functional* on E is a map $p\colon E \to \mathbb{R}$ that satisfies

- $p(\lambda\xi) = \lambda p(\xi)$ for $\lambda \geq 0$, $\xi \in E$,
- $p(\xi + \eta) \leq p(\xi) + p(\eta)$ for $\xi, \eta \in E$.

A *semi-norm* on E is a map $p\colon E \to [0, \infty)$ that satisfies

- $p(\lambda\xi) = |\lambda| p(\xi)$ for $\lambda \in \mathbb{K}$, $\xi \in E$,
- $p(\xi + \eta) \leq p(\xi) + p(\eta)$ for $\xi, \eta \in E$.

Theorem A.0.4 (Hahn–Banach, sublinear functional version)**.** *Let E be a real vector space, F a linear subspace of E and $p\colon E \to \mathbb{R}$ a sublinear functional. If $f\colon F \to \mathbb{R}$ is a linear functional such that $f(\xi) \leq p(\xi)$ for all $\xi \in F$, then there exists a linear extension $\tilde{f}$ of f to E that satisfies $\tilde{f}(\xi) \leq p(\xi)$ for all $\xi \in E$.*

Proof. Let $\mathcal{P}$ the set of all pairs (G, g) consisting of a linear subspace G of E that contains F and a linear functional $g\colon G \to \mathbb{R}$ that extends f and satisfies $f(\xi) \leq p(\xi)$ for all $\xi \in G$. Since $(F, f) \in \mathcal{P}$, the set $\mathcal{P}$ is non-empty. We define a partial order on $\mathcal{P}$ by setting $(G_1, g_1) \prec (G_2, g_2)$ if $G_1 \subset G_2$ and $g_2|_{G_1} = g_1$.

If $\mathcal{C} \subset \mathcal{P}$ is a chain, let $\hat{G} = \bigcup_{G:(G,g)\in\mathcal{C}} G$ and define $\hat{g}\colon \hat{G} \to \mathbb{R}$ by $\hat{g}(\xi) = g(\xi)$ if $x \in G$ and $(G, g) \in \mathcal{C}$. The chain property of $\mathcal{C}$ ensures that $\hat{G}$ is a subspace and $\hat{g}$ is well-defined. Moreover, $(\hat{G}, \hat{g}) \in \mathcal{P}$ and $(G, g) \prec (\hat{G}, \hat{g})$ for every $(G, g) \in \mathcal{C}$ follow directly from the construction. Thus $\mathcal{C}$ has an upper bound.

By Zorn's lemma, $\mathcal{P}$ has a maximal element $(\tilde{F}, \tilde{f})$. To finish the proof, we need to show that $\tilde{F} = E$. Suppose that this is not the case. Let $\zeta \in E \setminus \tilde{F}$ and. We want to define $h\colon \operatorname{span}(\tilde{F} \cup \{\zeta\}) \to \mathbb{R}$ such that $(\operatorname{span}(\tilde{F} \cup \{\zeta\}), h) \in \mathcal{P}$ and $(\tilde{F}, \tilde{f}) \prec (\operatorname{span}(\tilde{F} \cup \{\zeta\}), h)$.

Since p is sublinear, we have

$$\tilde{f}(\xi) + \tilde{f}(\eta) \leq p(\xi + \eta) \leq p(\xi + \zeta) + p(\eta - \zeta)$$

for all $\xi, \eta \in \tilde{F}$, hence

$$m = \sup_{\eta \in \tilde{F}} (\tilde{f}(\eta) - p(\eta - \zeta)) \leq \inf_{\xi \in \tilde{F}} (p(\xi + \zeta) - \tilde{f}(\xi)) = M.$$

Let $\alpha \in [m, M]$ and define

$$h\colon \operatorname{span}\{\tilde{F} \cup \{\zeta\}) \to \mathbb{R},\ h(\xi + \lambda\zeta) = \tilde{f}(\xi) + \lambda\alpha$$

for $\xi \in \tilde{F}$ and $\lambda \in \mathbb{R}$. Since $\tilde{F}$ and ζ are linearly independent, h is well-defined, and it is obviously an extension of $\tilde{f}$. It remains to show that h is dominated by p.

If $\lambda > 0$, then since $\alpha \leq M$, we have

$$\begin{aligned} h(\xi + \lambda\zeta) &= \tilde{f}(\xi) + \lambda\alpha \\ &\leq \tilde{f}(\xi) + \lambda \inf_{\eta \in \tilde{F}} (p(\eta + \zeta) - \tilde{f}(\eta)) \\ &\leq \tilde{f}(\xi) + \lambda \left(p\left(\frac{\xi}{\lambda} + \zeta\right) - \tilde{f}\left(\frac{\xi}{\lambda}\right)\right) \\ &= p(\xi + \lambda\zeta). \end{aligned}$$

If $\lambda < 0$, we reach the same conclusion using $\alpha \geq m$ instead. $\square$

Corollary A.0.5 (Hahn–Banach, semi-norm version)**.** *Let E be a vector space over $\mathbb{K}$, F a linear subspace of E and $p\colon E \to [0, \infty)$ a semi-norm. If $f\colon F \to \mathbb{K}$ is a linear functional such that $|f(\xi)| \leq p(\xi)$ for all $\xi \in F$, then there exists a linear extension $\tilde{f}$ of f to E that satisfies $|\tilde{f}(\xi)| \leq p(\xi)$ for all $\xi \in E$.*

Proof. Case $\mathbb{K} = \mathbb{R}$: Every seminorm is a sublinear functional. By the sublinear functional version of the Hahn–Banach theorem, the functional f can be extended to a linear functional $\tilde{f}$ on E such that $\tilde{f}(\xi) \leq p(\xi)$ for all $\xi \in E$. At the same time, $-\tilde{f}(\xi) = \tilde{f}(-\xi) \leq p(-\xi) = p(\xi)$. Thus $|\tilde{f}(\xi)| \leq p(\xi)$.

Case $\mathbb{K} = \mathbb{C}$: Let $g = \operatorname{Re} f$. This functional is real-linear, hence it can be extended to a real-linear functional $\tilde{g}$ on E such that $|\tilde{g}(\xi)| \leq p(\xi)$ for all $\xi \in E$ by the first part. Let $\tilde{f}(\xi) = \tilde{g}(\xi) - i\tilde{g}(i\xi)$ for $\xi \in E$. If $\xi \in F$, then $-ig(i\xi) = -i \operatorname{Re}(ig(\xi)) = i \operatorname{Im} \xi$, hence $\tilde{f}(\xi) = f(\xi)$ for $\xi \in F$.

By definition, $\tilde{f}$ is real-linear. However,

$$\tilde{f}(i\xi) = \tilde{g}(i\xi) + i\tilde{g}(\xi) = i\tilde{f}(\xi).$$

Together with real linearity, this implies that $\tilde{f}$ is in fact complex-linear. To show that $\tilde{f}$ is dominated by p, let $\xi \in E$ and $z \in \mathbb{C}$ with $|z| = 1$ such that $|\tilde{f}(\xi)| = z\tilde{f}(\xi)$. We have

$$|\tilde{f}(\xi)| = z\tilde{f}(\xi) = \operatorname{Re} \tilde{f}(z\xi) = \tilde{g}(z\xi) \leq p(z\xi) = p(\xi). \qquad \square$$

Corollary A.0.6 (Hahn-Banach, bounded functional version)**.** *Let E be a normed space and F a linear subspace of E. If $f\colon F \to \mathbb{K}$ is a bounded linear functional, then there exists a bounded linear extension $\tilde{f}$ of f to E with $\|\tilde{f}\| = \|f\|$.*

Proof. Apply the previous result with the seminorm $p(\xi) = \|f\|\|\xi\|$. $\square$

Corollary A.0.7. *Let E be a non-zero normed space. For every $\xi \in E$ there exists $f \in E^*$ with $\|f\| = 1$ such that $f(\xi) = \|\xi\|$.*

Proof. Let $F = \operatorname{span}\{\xi\}$ and define $g\colon F \to \mathbb{K}$, $g(\lambda\xi) = \lambda\|\xi\|$. Clearly, $\|g\| = 1$ and $g(\xi) = \|\xi\|$. By the previous corollary, g can be extended to a linear functional on E with the same norm. $\square$

Exercises

1. In this exercise we construct so-called Banach limits.

   (a) Let ℓ^∞ denote the space of bounded sequences in $\mathbb{K}$. Show that

   $$p\colon \ell^\infty \to \mathbb{K},\ x \mapsto \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k$$

   is a sublinear functional on ℓ^∞.

   (b) Show that there exists a bounded linear functional $\mathrm{LIM}\colon \ell^\infty \to \mathbb{K}$ with the following properties:

   - If $x_n \geq 0$ for all $n \in \mathbb{N}$, then $\mathrm{LIM}(x) \geq 0$.
   - If S denotes the shift operator on ℓ^∞, i.e., $(Sx)_n = x_{n+1}$, then $\mathrm{LIM}(Sx) = \mathrm{LIM}(x)$ for all $x \in \ell^\infty$.
   - If x is a convergent sequence, then $\mathrm{LIM}(x) = \lim_{n\to\infty} x_n$.

   Any such functional is called a *Banach limit*.

Appendix B

The Stone–Weierstraß theorem

Let X be a set and $\mathcal{F}$ a family of functions on X. We say that $\mathcal{F}$ separates the points of X if whenever $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{F}$ with $f(x) \neq f(y)$.

Theorem B.0.1 (Stone–Weierstraß)**.** *Let X be a compact Hausdorff space. If $A \subset C(X)$ is a unital $*$-subalgebra that separates the points of X, then A is dense in $C(X)$ (with respect to the supremum norm).*

Proof. We first note that continuity of the operations implies that $\overline{A}$ is again a unital $*$-subalgebra of $C(X)$.

Step 1: If $f \in A$ and $f \geq 0$, then $\sqrt{f} \in \overline{A}$.

By rescaling, we can assume that $0 \leq f \leq 1$. Let $g = 1 - f$. The binomial series convergence uniformly on the unit disk, hence

$$\sqrt{f(x)} = \sqrt{1 - g(x)} = \lim_{n \to \infty} \sum_{k=0}^{n} \binom{\frac{1}{2}}{k} (-1)^k g(x)^k$$

uniformly in $x \in X$. Therefore, $\sqrt{f} \in \overline{A}$.

Step 2: If $f, g \in A$ are real-valued, then $\min\{f, g\}, \max\{f, g\} \in \overline{A}$.

This follows immediately from the following identities:

$$\min\{f, g\} = \frac{f + g + \sqrt{(f-g)^2}}{2}, \quad \max\{f, g\} = \frac{f + g + \sqrt{(f-g)^2}}{2}.$$

Step 3: If $f \in C(X)$ is real-valued and $\varepsilon > 0$, then there exists $g \in A$ such that $\|f - g\|_\infty < \varepsilon$.

For $x, y \in X$ with $x \neq y$ choose $h \in A$ such that $h(x) \neq h(y)$, which exists since A separates the points of X. Otherwise replacing h by $\operatorname{Re} h$ or $\operatorname{Im} h$, we can assume that h is real-valued.

Let

$$f_{x,y}\colon X \to \mathbb{R},\ z \mapsto f(y) + (f(x) - f(y))\frac{h(z) - h(y)}{h(x) - h(y)},$$

which belongs to A since it is a linear combination of h and constant functions.

Further let

$$U_{x,y} = \{z \in X \mid f_{x,y}(z) < f(z) + \varepsilon/2\}.$$

Since f and $f_{x,y}$ are continuous, the sets $U_{x,y}$, $x, y \in X$, are open. Moreover, since $f_{x,y}(y) = f(y)$, we have $y \in U_{x,y}$. Therefore, $(U_{x,y})_{y \in X}$ is an open cover of X for every $x \in X$.

As X is compact, there exists $n \in \mathbb{N}$ and $y_1, \dots, y_n \in X$ such that $X = \bigcup_{k=1}^n U_{x,y_k}$. Let $f_x = \min_{1 \le k \le n} f_{x,y_k}$, which belongs to $\bar{A}$ by the Step 2. Furthermore, let

$$V_x = \{z \in X \mid h_x(z) > f(z) - \varepsilon/2\}.$$

Since $f_{x,y}(x) = f(x)$ for all $y \in X$, we also have $f_x(x) = f(x)$ and thus $x \in V_x$. Using once again compactness of X, we get $m \in \mathbb{N}$ and $x_1, \dots, x_m \in X$ such that $X = \bigcup_{j=1}^m V_{x_j}$. Let $\tilde{g} = \max_{1 \le j \le m} f_{x_j}$, which belongs to $\overline{A}$ by Step 2.

By construction, $f - \varepsilon/2 < \tilde{g} < f + \varepsilon/2$, hence $\|f - \tilde{g}\|_\infty < \varepsilon/2$. Take $g \in A$ with $\|g - \tilde{g}\|_\infty < \varepsilon/2$. By the triangle inequality, $\|f - g\|_\infty < \varepsilon$.

Step 4: A is dense in $C(X)$.

For arbitrary $f \in C(X)$ and $\varepsilon > 0$, we find $g, h \in A$ such that $\|\mathrm{Re}\, f - g\|_\infty < \varepsilon/2$ and $\|\mathrm{Im}\, f - h\|_\infty < \varepsilon/2$. By the triangle inequality, $\|f - (g + ih)\|_\infty < \varepsilon/2$. □

The great generality of the Stone–Weierstraß theorem makes it applicable in a variety of situations.

Corollary B.0.2 (Weierstraß). *Polynomial functions are dense in* $C([0,1])$.

Proof. Clearly, polynomial functions on a subset of $\mathbb{R}$ form a unital $*$-algebra. Moreover, linear functions already separate points. □

Corollary B.0.3. *Let* $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. *Functions of the form*

$$S^1 \to \mathbb{C},\ z \mapsto \sum_{k=-N}^{N} a_k z^k$$

with $N \in \mathbb{N}$ *and* $a_k \in \mathbb{C}$ *are dense in* $C(S^1)$.