# Self-Avoiding Walks
# Lace expansion

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# Definitions

- $c_n^\beta(x) = \sum_{\omega \in \mathcal{W}_n(0,x)} \exp\left(-\beta \cdot \sum_{0 \leq s < t \leq n} \mathbb{1}_{\omega(s)=\omega(t)}\right)$
- $c_n(x) = \lim_{\beta \to \infty} c_n^\beta(x),\ c_n = \sum_{x \in \mathbb{Z}^d} c_n(x)$
- $G_z(x) = \sum_{n=0}^\infty c_n(x) z^n$
- $\chi(z) = \sum_{x \in \mathbb{Z}^d} G_z(x) = \sum_{n=0}^\infty c_n z^n$
- $z_c^{-1} = \mu = \lim_{n \to \infty} (c_n)^{1/n}$

## Remark

- $c_n$ ... number of SAW of length $n$
- $c_n^{(0)}$ ... number of simple random walks (SRW)

# Predictions

- $c_n \sim A\mu^n n^{\gamma-1}$ as $n \to \infty$
- $\mathbb{E}_n|\omega(n)|^2 \sim Dn^{2\nu}$ as $n \to \infty$
- $G_{z_c}(x) \sim C|x|^{-d+2-\eta}$ as $|x| \to \infty$

# Results

For $d \geq 5$, Hara and Slade (1992) proved via lace expansion that

- $c_n = A\mu^n[1 + O(n^{-\varepsilon})]$
- $\mathbb{E}_n|\omega(n)|^2 = Dn[1 + O(n^{-\varepsilon})]$
- $\left(\frac{\omega(\lfloor nt \rfloor)}{\sqrt{Dn}}\right)_{t \geq 0} \to (B_t)_{t \geq 0}$

as $n \to \infty$. Hara (2008) also proved

$$G_{z_c}(x) = \frac{C}{|x|^{d-2}} \left[1 + O\left(|x|^{-\varepsilon}\right)\right]$$

as $x \to \infty$.

$d \geq 5$

$$\Rightarrow \gamma = 1, \nu = 1/2, \eta = 0$$

# Our goal

We want to prove

$$\chi(z) \asymp \left(1 - \frac{z}{z_c}\right)^{-1}$$

as $z \nearrow z_c$ for $d \geq d_0 \gg 4$. This implies $\gamma = 1$.

# The lower bound

Last time:

$$(c_m)^{1/m} \geq \lim (c_n)^{1/n} = \mu = z_c^{-1} \ \forall \ m$$

$\Rightarrow c_m \geq z_c^{-m}$

$$\Rightarrow \chi(z) = \sum_{n=0}^{\infty} c_n z^n \geq \sum_{n=0}^{\infty} (z/z_c)^n = \left(1 - \frac{z}{z_c}\right)^{-1}$$

# The upper bound

Let

$$B(z) = \sum_{x \in \mathbb{Z}^d} G_z(x)^2.$$

For the upper bound it suffices to show that we have

1. bubble condition: $B(z_c) < \infty$
2. differential inequality: $\frac{\mathrm{d}}{\mathrm{d}z}(z\chi(z)) \geq \frac{\chi(z)^2}{B(z)}$

# The differential inequality

$$Q(z) := \frac{d}{dz}(z\chi(z)) = \sum_{n=0}^{\infty}(n+1)c_n z^n$$

Understand $\chi(z)^2$, $B(z)$ as generating functions and draw diagrams!

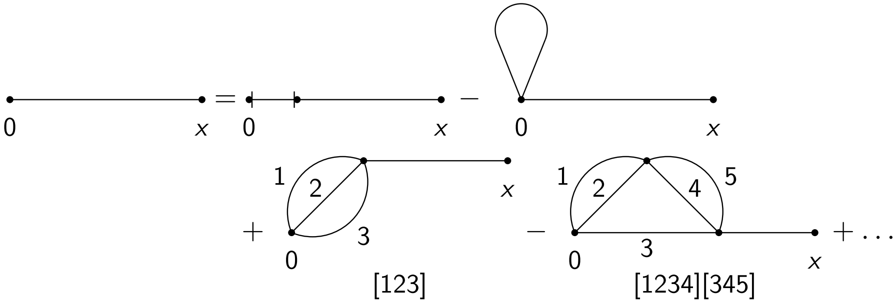
# Lace expansion

Actually, just the result, not the method. Uses convolution

$$(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(y) g(x - y)$$

## Result of lace expansion

$$c_n(x) = (c_1 * c_{n-1})(x) + \sum_{m=2}^{n} (\pi_m * c_{n-m})(x)$$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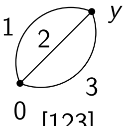
$$\pi_m(y) = -$$


$$\delta_{0y} +$$

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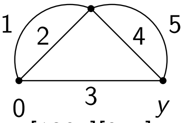


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$$+ \dots$$

$$= \sum_{N=1}^{\infty} (-1)^N \pi_m^{(N)}(y)$$

## Lace expansion

$$c_n(x) = (c_1 * c_{n-1})(x) + \sum_{m=2}^{n} (\pi_m * c_{n-m})(x)$$

# Coefficients $\pi_m$

Define

$$\Pi_z(x) = \sum_{m=2}^{\infty} \pi_m(x) z^m$$

Then,

$$G_z(x) = \delta_{0x} + z(c_1 * G_z)(x) + (\Pi_z * G_z)(x).$$

# Fourier transform on $\mathbb{Z}^d$

For an absolutely summable function $f : \mathbb{Z}^d \to \mathbb{C}$ let

$$\widehat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{ik \cdot x}$$

for $k \in [-\pi, \pi]^d$. Then,

$$\widehat{G}_z(k) = \frac{1}{1 - z\widehat{c}_1(k) - \widehat{\Pi}_z(k)} =: \frac{1}{\widehat{F}_z(k)}.$$

# Bubble condition

Parseval's relation:

$$B(z) = \sum_{x \in \mathbb{Z}^d} G_z(x)^2 = \int_{[-\pi,\pi]^d} |\widehat{G}_z(k)|^2 \frac{d^d k}{(2\pi)^d}$$

$\rightsquigarrow$ need to understand singularities of $\widehat{G}_z(k)$

Using $\widehat{G}_z(0) = \chi(z)$ write

$$\begin{aligned}\widehat{F}_z(k) &= \widehat{F}_z(0) + (\widehat{F}_z(k) - \widehat{F}_z(0)) \\ &= \chi(z)^{-1} + z(2d - \widehat{c}_1(k)) + (\widehat{\Pi}_z(0) - \widehat{\Pi}_z(k))\end{aligned}$$

$\Rightarrow$ Singularity in $(k, z) = (0, z_c)$. Are there others?

$$\widehat{c}_1(k) = 2\sum_{j=1}^{d} \cos(k_j)$$

$$\Rightarrow |2d - \widehat{c}_1(k)| \sim |k|^2$$

$$\begin{aligned}|\widehat{\Pi}_z(0) - \widehat{\Pi}_z(k)| &= |\sum_{x\in\mathbb{Z}^d} (1-\cos(k\cdot x))\Pi_z(x)| \\ &\leq O((d-4)^{-1}) \cdot (2d - \widehat{c}_1(k))\end{aligned}$$

$\rightsquigarrow$ no other singularities for $d \gg 4$

# Estimates on $\Pi_z$

$$\Pi_z(x) = \sum_{m=2}^{\infty} \sum_{N=1}^{\infty} (-1)^N \pi_m^{(N)}(x) z^m = \sum_{N=1}^{\infty} (-1)^N \Pi_z^{(N)}(x)$$

Let $H_z(x) = G_z(x) - \delta_{0x}$ and $z \geq 0, k \in [-\pi, \pi]^d$. Then

$$\sum_{x \in \mathbb{Z}^d} \Pi_z^{(1)}(x) \leq 2d \cdot z \cdot \|H_z\|_\infty$$

$$\sum_{x \in \mathbb{Z}^d} (1 - \cos(k \cdot x)) \Pi_z^{(1)}(x) = 0$$

and for $N \geq 2$

$$\sum_{x \in \mathbb{Z}^d} \Pi_z^{(N)}(x) \leq \|H_z\|_\infty \|G_z * H_z\|_\infty^{N-1}$$

$$\sum_{x \in \mathbb{Z}^d} (1 - \cos(k \cdot x)) \Pi_z^{(N)}(x) \leq N^2 \|(1 - \cos(k \cdot x)) H_z\|_\infty \|G_z * H_z\|_\infty^{N-1}$$

Define analogue of $G_z(x)$ for the SRW:

$$C_z(x) = \sum_{n=0}^{\infty} c_n^{(0)}(x) z^n$$

$\Rightarrow$ critical value $z_0 = (2d)^{-1}$ and

$$\widehat{C}_z(k) = \frac{1}{1 - z\widehat{c}_1(k)}$$

For $d > 4$, we have

$$\left\|\widehat{C}_{z_0}\right\|_2^2 = \int_{[-\pi,\pi]^d} \frac{4d^2}{(2d - \widehat{c}_1(k))^2} \frac{\mathrm{d}^d k}{(2\pi)^d} \leq 1 + \frac{K}{d-4}$$

# Coupling between $\widehat{G}_z$ and $\widehat{C}_z$

For $z \in [0, z_c)$ let $p(z) \in [0, z_0)$ be such that

$$\chi(z) = \widehat{G}_z(0) = \widehat{C}_{p(z)}(0) = (1 - 2d \cdot p(z))^{-1}$$

i.e.

$$p(z) = z + \frac{1}{2d}\widehat{\Pi}_z(0).$$

We hope that $\widehat{G}_z(k) \approx \widehat{C}_{p(z)}(k)$ even for $k \neq 0$.

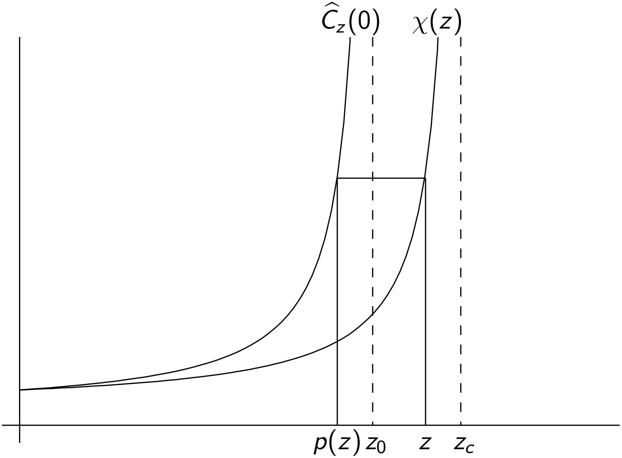


Figure: The definition of $p(z)$.

$$B(z_c) = \lim_{z \nearrow z_c} B(z) = \lim_{z \nearrow z_c} \left\| \widehat{G}_z \right\|_2^2 \le a^2 \lim_{z \nearrow z_c} \left\| \widehat{C}_{p(z)} \right\|_2^2$$

$$= a^2 \int \frac{4d^2}{2d - \widehat{c}_1(k)} \frac{\mathrm{d}^d k}{(2\pi)^d} \le 2a^2 < \infty$$

## Lemma

*Let* $a < b$ *and* $f : [z_1, z_2) \to \mathbb{R}$ *be continuous with* $f(z_1) \leq a$. *Suppose for every* $z \in (z_1, z_2)$ *we have*

$$f(z) \leq b \Rightarrow f(z) \leq a.$$

*Then* $f(z) \leq a$ *for all* $z \in [z_1, z_2)$.

Apply lemma to

$$f(z) = \max\{f_1(z), f_2(z), f_3(z)\}$$

on $[0, z_c)$ and $a = 1 + c \cdot (d-4)^{-1}, b = 4$. Here

$$f_1(z) = 2d \cdot z \ , \ \ f_2(z) = \sup_{k \in [-\pi,\pi]^d} \frac{|\widehat{G}_z(k)|}{|\widehat{C}_{p(z)}(k)|}$$

$$f_3(z) \approx \sup_{k,l \in [-\pi,\pi]^d} \frac{|\Delta_k \widehat{G}_z(l)|}{|\Delta_k \widehat{C}_{p(z)}(l)|}$$

Thank you for your attention!