Regularity structures
The reconstruction theorem

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Non-euclidean scalings of $\mathbb{R}^d$

Let $\mathfrak{s} = (\mathfrak{s}_1, \ldots, \mathfrak{s}_d)$, $\mathfrak{s}_i$ relatively prime, and let $k \in \mathbb{R}^d$. Then

1. $|\mathfrak{s}| := \sum_{i=1}^d \mathfrak{s}_i$
2. $|k|_\mathfrak{s} := \sum_{i=1}^d \mathfrak{s}_i k_i$
3. $\deg_\mathfrak{s}(X^k) := |k|_\mathfrak{s}$
4. For $x, y \in \mathbb{R}^d$

$$d_\mathfrak{s}(x, y) := \sum_{i=1}^{d} |x_i - y_i|^{1/\mathfrak{s}_i}$$

is a metric but **no** norm. We still use the notation

$$\|x - y\|_\mathfrak{s} = d_\mathfrak{s}(x, y)$$

Scaling applied to functions

Let $\eta : \mathbb{R}^d \to \mathbb{R}$, $y \in \mathbb{R}^d, \delta > 0, \mathfrak{s} \in \mathbb{N}^d$. Then

$$\mathcal{S}_{\mathfrak{s}}^{\delta}(x_1, \dots, x_d) := (\delta^{-\mathfrak{s}_1} x_1, \dots, \delta^{-\mathfrak{s}_d} x_d)$$

$$(\mathcal{S}_{\mathfrak{s},y}^{\delta} \eta)(x) := \delta^{-|\mathfrak{s}|} \eta \left(\mathcal{S}_{\mathfrak{s}}^{\delta}(y - x) \right)$$

Note that we have

$$\left\| \mathcal{S}_{\mathfrak{s}}^{\delta} x \right\|_{\mathfrak{s}} = \delta^{-1} \|x\|_{\mathfrak{s}}$$

Modelled distributions $\mathcal{D}^\gamma$

$\mathcal{D}^\gamma$ is the space of all $f : \mathbb{R}^d \to T_\gamma^-$ s.t.

$$\vert\!\vert\!\vert f \vert\!\vert\!\vert_{\gamma;\mathfrak{K}} = \sup_{x \in \mathfrak{K}} \sup_{\beta < \gamma} \|f(x)\|_\beta + \sup_{\substack{x,y \in \mathfrak{K} \\ \|x-y\|_\mathfrak{s} \leq 1}} \sup_{\beta < \gamma} \frac{\|f(x) - \Gamma_{xy} f(y)\|_\beta}{\|x-y\|_\mathfrak{s}^{\gamma-\beta}} < \infty$$

for all $\mathfrak{K} \subset\subset \mathbb{R}^d$.

Note that $\mathcal{D}^\gamma$ depends on Γ_{xy}, i.e. on the chosen model.

For models $(\Pi, \Gamma), (\overline{\Pi}, \overline{\Gamma})$, $f \in \mathcal{D}^\gamma(\Gamma), \overline{f} \in \mathcal{D}^\gamma(\overline{\Gamma})$, we define

$$\left\|\left| f; \overline{f} \right|\right\|_{\gamma,\mathfrak{K}} := \sup_{x \in \mathfrak{K}} \sup_{\beta < \gamma} \left\| f(x) - \overline{f}(x) \right\|_\beta + \sup_{\substack{x,y \in \mathfrak{K} \\ \|x-y\|_{\mathfrak{s}} \leq 1}} \sup_{\beta < \gamma} \frac{\left\| f(x) - \overline{f}(x) - \Gamma_{xy} f(y) + \overline{\Gamma}_{xy} \overline{f}(y) \right\|_\beta}{\|x - y\|_{\mathfrak{s}}^{\gamma - \beta}}$$

This yields a topology on the space $\mathcal{M} \ltimes \mathcal{D}^\gamma$.

Classical Hölder functions

η is of class $\mathcal{C}^\alpha$

1. $\alpha \in (0,1] \rightsquigarrow$ for any $\mathfrak{K} \subset\subset \mathbb{R}^d$

$$|\eta(x) - \eta(y)| \lesssim |x-y|^\alpha$$

   uniformly over $x, y \in \mathfrak{K}$

2. $\alpha > 1 \rightsquigarrow$ recursive definition: η is continuously differentiable and $\partial_{x_i}\eta \in \mathcal{C}^{\alpha-1}$, $i = 1, \dots, d$

Alternatively:

$$f(y) = f(x) + \sum_{k=1}^{\lfloor r \rfloor} \frac{f^{(k)}(x)}{k!}(y-x)^k + O(|y-x|^r)$$

$\Leftrightarrow \ \exists\ P \in \mathbb{C}[x], \deg P \leq \lfloor r \rfloor : |f(y) - P(y)| \lesssim |y-x|^r.$

$\mathcal{C}^\alpha$ and regularity structures

Let $\mathfrak{s} = (1, \dots, 1)$ and $\mathcal{T}$ contain the polynomials.

Lemma

$\eta : \mathbb{R}^d \to \mathbb{R}$ is of class $\mathcal{C}^\alpha$ iff there ex. $\widehat{\eta} \in \mathcal{D}^\alpha$ s.t.

$$\langle \mathbf{1}, \widehat{\eta}(x) \rangle = \eta(x) \ \forall \ x \in \mathbb{R}^d$$

Last lecture:

$$(\widehat{\eta})(x) = \sum_{k=0}^{\lfloor \alpha \rfloor} \frac{\eta^{(k)}(x)}{k!} X^k$$

Negative Hölder exponents

Let $\alpha < 0$, $r = -\lfloor \alpha \rfloor$

Definition

$\xi \in (\mathcal{C}_0^r)'$ is of class $\mathcal{C}_{\mathfrak{s}}^\alpha$ if for all $\mathfrak{K} \subset\subset \mathbb{R}^d$

$$|\langle \xi, \mathcal{S}_{\mathfrak{s},x}^\delta \eta \rangle| \lesssim \delta^\alpha \ \forall \ \eta \in \mathcal{B}_{\mathfrak{s},0}^r$$

where $\mathcal{B}_{\mathfrak{s},0}^r$ consists of all smooth test functions $\eta : \mathbb{R}^d \to \mathbb{R}$, s.t. $\eta \in \mathcal{C}^r, \|\eta\|_{\mathcal{C}^r} \leq 1, \operatorname{supp} \eta \subset B_{\mathfrak{s}}(0,1)$.

$$\|\xi\|_{\alpha;\mathfrak{K}} := \sup_{x \in \mathfrak{K}} \sup_{\eta \in \mathcal{B}_{\mathfrak{s},0}^r} \sup_{\delta \leq 1} \delta^{-\alpha} |\langle \xi, \mathcal{S}_{\mathfrak{s},x}^\delta \eta \rangle|$$

The Theorem

Reconstruction theorem - 1

$\mathcal{T} = (A, T, G)$ reg. str., (Π, Γ) model, $\alpha := \min A$ and $r > |\alpha|$.
$\Rightarrow$ For every $\gamma \in \mathbb{R}$, there ex. a continuous linear map $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{C}^\alpha_\mathfrak{s}$ s.t. f.a. $\mathfrak{K} \subset\subset \mathbb{R}^d$,

$$|(\mathcal{R}f - \Pi_x f(x))(\mathcal{S}^\delta_{\mathfrak{s},x}\eta)| \lesssim \delta^\gamma \, \|\Pi\|_{\gamma;\overline{\mathfrak{K}}} \, |||f|||_{\gamma;\overline{\mathfrak{K}}} \tag{R1}$$

uniformly over $\eta \in \mathcal{B}^r_{s,0}$, $\delta \in (0, 1]$, $f \in \mathcal{D}^\gamma$, $x \in \mathfrak{K}$.

If $\gamma > 0$, the bound (R1) defines $\mathcal{R}f$ uniquely.

Reconstruction theorem - 2

If $(\overline{\Pi}, \overline{\Gamma})$ is another model with reconstr. operator $\overline{\mathcal{R}}$, then

$$|(\mathcal{R}f - \overline{\mathcal{R}}\,\overline{f} - \Pi_x f(x) + \overline{\Pi}_x \overline{f}(x))(\mathcal{S}^{\delta}_{\mathfrak{s},x}\eta)| \lesssim \delta^{\gamma}\left(\|\overline{\Pi}\|_{\gamma;\overline{\mathfrak{K}}}\,|||f;\overline{f}|||_{\gamma,\overline{\mathfrak{K}}} + \|\Pi - \overline{\Pi}\|_{\gamma;\mathfrak{K}}\,|||f|||_{\gamma;\overline{\mathfrak{K}}}\right) \tag{R2}$$

uniformly over x and η.

Notes

1. For most applications, only $\gamma > 0$ is important.
2. "$(R1) \Rightarrow$ uniqueness of $\mathcal{R}$" can be proven right away.
3. $(R2) \Rightarrow \mathcal{R} : \mathcal{M}(\mathcal{T}) \ltimes \mathcal{D}^\gamma \to \mathcal{C}_{\mathfrak{s}}^\alpha$ is continouos in $(\Pi, \Gamma) \in \mathcal{M}(\mathcal{T})$.

Uniqueness of $\mathcal{R}$

$(R1)$ defines $\mathcal{R}f$ uniquely.

Proof.

Let $\gamma > 0$, $f \in \mathcal{D}^\gamma$ and $\xi_1, \xi_2 \in \mathcal{C}_\mathfrak{s}^\alpha$ that satisfy $(R1)$ if we exchange $\mathcal{R}f$ for ξ_i. Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be any smooth compactly supported test function and $\eta : B_1 \to \mathbb{R}_+$ an even function such that $\int \eta(x)\, dx = 1$. Define

$$\psi_\delta(y) := \langle \mathcal{S}_{\mathfrak{s},y}^\delta \eta, \psi \rangle = \int \psi(x)(\mathcal{S}_{\mathfrak{s},x}^\delta \eta)(y)\, dx$$

Then, for any distribution ξ, we have

$$\xi(\psi_\delta) = \int \psi(x) \langle \xi, \mathcal{S}_{\mathfrak{s},x}^\delta \eta \rangle\, dx$$

For $\xi = \xi_2 - \xi_1$, it follows from $(R1)$ that

$$|\xi(\psi_\delta)| \leq ... \\ \lesssim \delta^\gamma \int_D \psi(x)\rho(x)\, dx$$

for some function $\rho(x)$. Hence, $\xi(\psi_\delta) \to 0$ for $\delta \to 0$ and $\psi_\delta \to \psi$ in $\mathcal{C}^\infty$, i.e. $\xi(\psi_\delta) \to \xi(\psi)$. Thus, $\xi(\psi) = 0$ for every smooth compactly supported test function and it follows that $\xi = 0$.

The plan

The existence of $\mathcal{R}$ is more complicated to prove and we will need a few steps:

1. Find a *scaling function* φ with nice properties.
2. Use φ to create a wavelet analysis in $\mathbb{R}^d$. In particular, obtain an approximating set $\left\{\varphi_x^{n,\mathfrak{s}}\right\}$ of an ONB of L^2.
3. Define operators $\mathcal{R}_n : \mathcal{D}^\gamma \to \mathcal{C}^r$ using $\varphi_x^{n,\mathfrak{s}}$.
4. Show that $\mathcal{R}_n f$ converges to a limit $\mathcal{R} f$ in $\mathcal{C}_\mathfrak{s}^\alpha$.
5. Verify that $\mathcal{R}$ has the desired properties.

Wavelet analysis

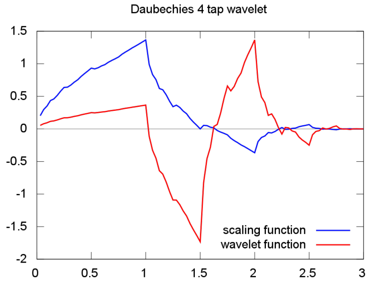
Theorem (Daubechies, 1988)

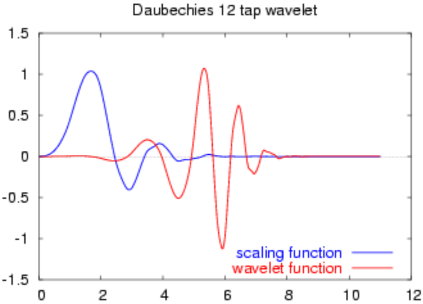
For every $r > 0$ there exists $\varphi \in \mathcal{C}^r(\mathbb{R})$ such that

1. The support of φ is compact.
2. $\int \varphi(x)\varphi(x+k)\, dx = \delta_{k,0}$ for all $k \in \mathbb{Z}$.
3. There exist *structure constants* a_k such that

$$\varphi(x) = \sum_{k \in \mathbb{Z}} a_k \varphi(2x - k)$$

If we don't need regularity, consider $\varphi = \mathbf{1}_{[0,1)}$.


Daubechies 4 tap wavelet
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5
3
scaling function
wavelet function


Daubechies 12 tap wavelet
1.5
1
0.5
0
-0.5
-1
-1.5
0
2
4
6
8
10
12
scaling function
wavelet function

Wavelet analysis in $\mathbb{R}$

For $n \in \mathbb{Z}$, define

- $\Lambda_n = 2^{-n}\mathbb{Z}$
- $\varphi_x^n(y) = 2^{n/2}\varphi(2^n(y-x)) = 2^{-n/2}(\mathcal{S}_{(1),x}^{2^{-n}}\varphi)(y)$ for $x \in \Lambda_n$.
- $V_n = \operatorname{span}\{\varphi_x^n \ : \ x \in \Lambda_n\}$ (Prop. 3 of $\varphi \Rightarrow V_n \subset V_{n+1}$)
- We can find finitely many coefficients b_k such that

$$\psi(x) := \sum_{k \in \mathbb{Z}} b_k \varphi(2x - k)$$

and defining ψ_x^n similarly to φ_x^n yields

$$V_n^{\perp} = \operatorname{span}\{\psi_x^n \ : \ x \in \Lambda_n\} \subseteq V_{n+1}$$

ψ is called *wavelet function*.

Theorem

For every $n \in \mathbb{N}_0$

$$\{\varphi_x^n \ : \ x \in \Lambda_n\} \cup \{\psi_x^m \ : \ m \geq n, x \in \Lambda_m\}$$

is an ONB of $L^2(\mathbb{R})$. For all $f \in L^2(\mathbb{R})$ we have

$$\lim_{n\to\infty} \mathcal{P}_n f := \lim_{n\to\infty} \sum_{x\in\Lambda_n} \langle f, \varphi_x^n \rangle \varphi_x^n = f$$

It is noteworthy that if $\varphi \in \mathcal{C}^r$, ψ satisfies

$$\int \psi(x) x^m \, dx = 0 \ \forall \ m \leq r$$

Wavelet analysis in $\mathbb{R}^d$

From now on, let $\mathfrak{s}$ be a scaling vector of $\mathbb{R}^d$, $\varphi_d(x) := \prod_{j=1}^d \varphi(x_j)$. We define for $n \in \mathbb{Z}$

- $\Lambda_n^{\mathfrak{s}} := 2^{-n\mathfrak{s}_1}\mathbb{Z} \times \ldots \times 2^{-n\mathfrak{s}_d}\mathbb{Z}$
- $\varphi_x^{n,\mathfrak{s}} := 2^{-\frac{n|\mathfrak{s}|}{2}}(\mathcal{S}_{\mathfrak{s},x}^{2^{-n}}\varphi_d)$ for $x \in \Lambda_n^{\mathfrak{s}}$
- $V_n = \operatorname{span}\left\{\varphi_x^{n,\mathfrak{s}} \ : \ x \in \Lambda_n^{\mathfrak{s}}\right\}$

There exists a finite collection Ψ of orthonormal compactly supported functions such that

$$V_n^{\perp} = \operatorname{span}\{\psi_x^{n,\mathfrak{s}} \ : \ \psi \in \Psi, x \in \Lambda_n^{\mathfrak{s}}\}$$

Analogously to the $\mathbb{R}^1$, we obtain an ONB for $L^2(\mathbb{R}^d)$.

Properties

- There ex. a finite set $\mathcal{K} \subset \Lambda_1^{\mathfrak{s}}$ and constants $\{a_k \ : \ k \in \mathcal{K}\}$, s.t.

$$\varphi_x^{n,\mathfrak{s}}(y) = \sum_{k \in \mathcal{K}} a_k \varphi_{x+2^{-n\mathfrak{s}}k}^{n+1,\mathfrak{s}}(y)$$

- The property of ψ to annihilate polynomials with degree $\leq r$ carries over to all functions in Ψ.

Lemma - Wavelet coefficients

For $\eta \in \mathcal{B}^r_{\mathfrak{s},0}$, $\delta \in (0,1]$ and for all $x, y \in \mathbb{R}^d$ we have

1. $|\langle \psi^{n,\mathfrak{s}}_y, \mathcal{S}^\delta_{\mathfrak{s},x}\eta\rangle| \lesssim 2^{-\frac{n|\mathfrak{s}|}{2} - nr}\delta^{-|\mathfrak{s}|-r}$
2. $|\langle \psi^{n,\mathfrak{s}}_y, \mathcal{S}^\delta_{\mathfrak{s},x}\eta\rangle| \lesssim 2^{\frac{n|\mathfrak{s}|}{2}}$ if $\delta \leq 2^{-n}$
3. $|\langle \varphi^{n,\mathfrak{s}}_y, \mathcal{S}^\delta_{\mathfrak{s},x}\eta\rangle| \lesssim 2^{-\frac{n|s|}{2}}\delta^{-|\mathfrak{s}|}$

Proof.

1. see blackboard
2. For $\delta \leq 2^{-n}$, we have

$$
\begin{aligned}
|\langle \psi_y^{n,\mathfrak{s}}, \mathcal{S}_{\mathfrak{s},x}^{\delta}\eta\rangle| &= 2^{-\frac{n|\mathfrak{s}|}{2}}\delta^{-|\mathfrak{s}|}\left|\langle \mathcal{S}_{\mathfrak{s},y}^{2^{-n}}\psi, \eta(\delta^{-\mathfrak{s}}(x-\cdot))\rangle\right| \\
&\leq 2^{-\frac{n|\mathfrak{s}|}{2}}\delta^{-|\mathfrak{s}|}\int_{\mathbb{R}^d}\left|(\mathcal{S}_{\mathfrak{s},y}^{2^{-n}}\psi)(z)\right|\, dz \\
&= 2^{-\frac{n|\mathfrak{s}|}{2}}\delta^{-|\mathfrak{s}|}\int_{\mathbb{R}^d}|\psi(\tilde{z})|\, d\tilde{z} \lesssim 2^{-\frac{n|\mathfrak{s}|}{2}}2^{n|\mathfrak{s}|} = 2^{\frac{n|\mathfrak{s}|}{2}}
\end{aligned}
$$

3. analogous to 2

□

Important tools

Proposition 3.20 ($\mathcal{C}^\alpha_\mathfrak{s}$ characterization)

Let $\alpha < 0$, $\xi \in \mathcal{S}'(\mathbb{R}^d)$, and $\varphi \in \mathcal{C}^r$ for $r > |\alpha|$.

Then $\xi \in \mathcal{C}^\alpha_\mathfrak{s}$ if and only if $\xi \in (\mathcal{C}^r_0)'$ and, for every $\mathfrak{K} \subset\subset \mathbb{R}^d$,

$$|\langle \xi, \psi^{n,\mathfrak{s}}_x \rangle| \lesssim 2^{-\frac{n|\mathfrak{s}|}{2} - n\alpha}$$

and

$$\left|\langle \xi, \varphi^0_y \rangle\right| \lesssim 1$$

uniformly over $n \geq 0$, $\psi \in \Psi$, $x \in \Lambda^\mathfrak{s}_n \cap \mathfrak{K}$ and $y \in \Lambda^\mathfrak{s}_0 \cap \mathfrak{K}$.

Lemma

Let $a, b_-, b_+ \in \mathbb{R}$ s.t. $b_+ > a > -b_-$. Then

$$\sum_{n=0}^{n_0} 2^{an} 2^{-b_-(n_0-n)} + \sum_{n=n_0}^{\infty} 2^{an} 2^{b_+(n_0-n)} \lesssim 2^{an_0}$$

Proof of Prop. 3.20. Recall that $\xi \in \mathcal{C}^\alpha_\mathfrak{s}$ iff $\xi \in (\mathcal{C}^r)'$ and for all $\mathfrak{K} \subset\subset \mathbb{R}^d$, $|\langle \xi, \mathcal{S}^\delta_{\mathfrak{s},x}\eta\rangle| \lesssim \delta^\alpha$ uniformly over $\eta \in \mathcal{B}^r_\mathfrak{s}$, $x \in \mathfrak{K}$.

If $\xi \in \mathcal{C}^\alpha_\mathfrak{s}$, then the bounds for $|\langle \xi, \psi^{n,\mathfrak{s}}_x\rangle|$ and $|\langle \xi, \varphi^0_y\rangle|$ follow by definition of $\mathcal{C}^\alpha_\mathfrak{s}$.

Let $\eta \in \mathcal{B}^r_{\mathfrak{s},0}$ and write

$$\langle \xi, \mathcal{S}^\delta_{\mathfrak{s},x}\eta\rangle = \sum_{n\geq 0}\sum_{y\in\Lambda^\mathfrak{s}_n} \langle \xi, \psi^{n,\mathfrak{s}}_y\rangle\langle \psi^{n,\mathfrak{s}}_y, \mathcal{S}^\delta_{\mathfrak{s},x}\eta\rangle + \sum_{y\in\Lambda^\mathfrak{s}_0} \langle \xi, \varphi^{0,\mathfrak{s}}_y\rangle\langle \varphi^{0,\mathfrak{s}}_y, \mathcal{S}^\delta_{\mathfrak{s},x}\eta\rangle$$

Now let $2^{-n_0} \leq \delta < 2^{-n_0+1}$. If $\operatorname{supp} \psi_y^{n,\mathfrak{s}} \cap \operatorname{supp} \mathcal{S}_{\mathfrak{s},x}^{\delta}\eta \neq \emptyset$, we have

$$\|x - y\|_{\mathfrak{s}} \leq C$$

Hence, $x, y \in \mathfrak{K} \subset\subset \mathbb{R}^d$ and we get the bounds

$$|\langle \xi, \psi_x^{n,\mathfrak{s}} \rangle| \lesssim 2^{-\frac{n|\mathfrak{s}|}{2} - n\alpha} \ , \ \left|\langle \xi, \varphi_y^0 \rangle\right| \lesssim 1$$

By our bounds on wavelet coefficients we have

$$\left|\langle \psi_y^{n,\mathfrak{s}}, \mathcal{S}_{\mathfrak{s},x}^{\delta}\eta \rangle\right| \lesssim 2^{-n\frac{|\mathfrak{s}|}{2} - rn} \cdot \delta^{-|\mathfrak{s}|-r}$$

and

$$\sum_{y \in \Lambda_n^{\mathfrak{s}}} \left|\langle \psi_y^{n,\mathfrak{s}}, \mathcal{S}_{\mathfrak{s},x}^{\delta}\eta \rangle\right| \lesssim 2^{-n\frac{|\mathfrak{s}|}{2} - rn} \cdot \delta^{-|\mathfrak{s}|-r} \cdot 2^{n|\mathfrak{s}|}\delta^{|\mathfrak{s}|} \lesssim 2^{-n\left(r - \frac{|\mathfrak{s}|}{2}\right)}\delta^{-r}$$

For $n < n_0$ we have $\delta < 2^{-n}$ and thus

$$\sum_{y \in \Lambda_n^{\mathfrak{s}}} \left| \langle \psi_y^{n,\mathfrak{s}}, \mathcal{S}_{\mathfrak{s},x}^{\delta} \eta \rangle \right| \lesssim 2^{\frac{n|\mathfrak{s}|}{2}} \cdot 2^{n|\mathfrak{s}|} \delta^{|\mathfrak{s}|} \leq 2^{\frac{n|\mathfrak{s}|}{2}}$$

Hence,

$$\begin{aligned} &\sum_{n \geq 0} \sum_{y \in \Lambda_n^{\mathfrak{s}}} \langle \xi, \psi_y^{n,\mathfrak{s}} \rangle \langle \psi_y^{n,\mathfrak{s}}, \mathcal{S}_{\mathfrak{s},x}^{\delta} \eta \rangle \\ &\lesssim \sum_{n=0}^{n_0} 2^{-\frac{n|\mathfrak{s}|}{2} - n\alpha} 2^{\frac{n|\mathfrak{s}|}{2}} + \sum_{n=n_0}^{\infty} 2^{-\frac{n|\mathfrak{s}|}{2} - n\alpha} 2^{\frac{n|\mathfrak{s}|}{2}} 2^{-(n-n_0)r} \\ &= \sum_{n=0}^{n_0} 2^{-n\alpha} + \sum_{n=n_0}^{\infty} 2^{-n\alpha} 2^{r(n_0-n)} \end{aligned}$$

Hence, we can apply the Lemma to $\underbrace{r}_{=b_+} > \underbrace{-\alpha}_{=a} > \underbrace{0}_{=-b_-}$ and get

$$\left| \sum_{n \geq 0} \sum_{y \in \Lambda_n^{\mathfrak{s}}} \langle \xi, \psi_y^{n,\mathfrak{s}} \rangle \langle \psi_y^{n,\mathfrak{s}}, \mathcal{S}_{\mathfrak{s},x}^{\delta} \eta \rangle \right| \lesssim 2^{-n_0 \alpha} \leq \delta^{\alpha}$$

For the second term we use

$$|\langle \varphi_y^{0,\mathfrak{s}}, \mathcal{S}_{\mathfrak{s},x}^{\delta} \eta \rangle| \lesssim \delta^{-|\mathfrak{s}|}$$

and thus

$$\left| \sum_{y \in \Lambda_0^{\mathfrak{s}}} \langle \xi, \varphi_y^{0,\mathfrak{s}} \rangle \langle \varphi_y^{0,\mathfrak{s}}, \mathcal{S}_{\mathfrak{s},x}^{\delta} \eta \rangle \right| \lesssim 2^{0|\mathfrak{s}|} \delta^{|\mathfrak{s}|} \cdot \delta^{-|\mathfrak{s}|} = 1$$

□

Note that the proof of Prop. 3.20 also implies that

$$\|\xi\|_\alpha \lesssim \cdot \sup \left[\left\{ |\langle \xi, \varphi_y^0 \rangle|, \frac{|\langle \xi, \psi_x^{n,\mathfrak{s}} \rangle|}{2^{-\frac{n|\mathfrak{s}|}{2} - n\alpha}} \ : \ y \in \Lambda_{\mathfrak{s}}^0, \psi \in \Psi, x \in \Lambda_n^{\mathfrak{s}}, n \geq 0 \right\} \right]$$

Theorem 3.23 (Convergence criterion in $\mathcal{C}_{\mathfrak{s}}^{\alpha}$)

Let $\alpha < 0 < \gamma$. For $n \geq 0$, let $A^n : \mathbb{R}^d \to \mathbb{R}$ such that

$$|A_x^n| \leq \|A\| \cdot 2^{-\frac{n\mathfrak{s}}{2} - \alpha n}\ ,\ |\delta A_x^n| \lesssim \|A\| \cdot 2^{-\frac{n\mathfrak{s}}{2} - \gamma n}$$

uniformly over $n \geq 0$ and $x \in \mathbb{R}^d$ for some constant $\|A\|$ and

$$\delta A_x^n = A_x^n - \sum_{k \in \mathcal{K}} a_k A_{x+2^{-n\mathfrak{s}}k}^{n+1}$$

Then

$$f_n = \sum_{x \in \Lambda_n^{\mathfrak{s}}} A_x^n \varphi_x^{n,\mathfrak{s}}$$

converges in $\mathcal{C}_{\mathfrak{s}}^{\overline{\alpha}}$ for all $\overline{\alpha} < \alpha$ and $f = \lim_{n\to\infty} f_n \in \mathcal{C}_{\mathfrak{s}}^{\alpha}$.

Proposition 3.25

Let $\{\zeta_x\}_{x\in\mathbb{R}^d} \subseteq \mathcal{S}'(\mathbb{R}^d)$ such that, for some K_1, K_2 and $\alpha < 0 < \gamma$, the bounds

$$|\langle \varphi_x^{n,\mathfrak{s}}, \zeta_x - \zeta_y\rangle| \leq K_1 \|x - y\|_{\mathfrak{s}}^{\gamma-\alpha} 2^{-\frac{n|\mathfrak{s}|}{2} - \alpha n}$$

$$|\langle \varphi_x^{n,\mathfrak{s}}, \zeta_x\rangle| \leq K_2 2^{-\alpha n - \frac{n|\mathfrak{s}|}{2}}$$

hold uniformly over all x, y such that $2^{-n} \leq \|x - y\|_{\mathfrak{s}} \leq 1$.

For $A_x^n = \langle \varphi_x^{n,\mathfrak{s}}, \zeta_x\rangle$, the assumptions of Theorem 3.23 are then satisfied and the limit distribution $f \in \mathcal{C}_{\mathfrak{s}}^{\alpha}$ satisfies

$$|(f - \zeta_x)(\mathcal{S}_{\mathfrak{s},x}^{\delta}\eta)| \lesssim K_1 \delta^{\gamma}$$

uniformly over $\eta \in \mathcal{B}_{\mathfrak{s},0}^{r}$.

The existence of $\mathcal{R}$, $\gamma > 0$

Fix scaling function with $r > |\min A|$. For $f \in \mathcal{D}^\gamma$, let

$$\mathcal{R}_n f = \sum_{x \in \Lambda_n^{\mathfrak{s}}} (\Pi_x f(x))(\varphi_x^{n,\mathfrak{s}}) \varphi_x^{n,\mathfrak{s}}$$

Let $\mathfrak{K}$ be some compact domain and $x \in \Lambda_n^{\mathfrak{s}}$ such that ψ_x^n is supported in $\overline{\mathfrak{K}}$. Then

$$|\langle \Pi_x f(x), \varphi_x^{n,\mathfrak{s}} \rangle| \lesssim |||f|||_{\gamma, \overline{\mathfrak{K}}} \|\Pi\|_{\gamma, \overline{\mathfrak{K}}} \cdot 2^{-\frac{n|\mathfrak{s}|}{2} - \alpha n}$$

by definition of $|||f|||$ and $\|\Pi\|$.

Furthermore, we have

$$\begin{aligned} |\langle \Pi_x f(x) - \Pi_y f(y), \varphi_x^{n,\mathfrak{s}} \rangle| &= |\langle \Pi_x \left(f(x) - \Gamma_{xy} f(y) \right), \varphi_x^{n,\mathfrak{s}} \rangle| \\ &\lesssim \sum_{k<\gamma} |||f|||_{\gamma;\overline{\mathfrak{K}}} \|\Pi\|_{\gamma;\overline{\mathfrak{K}}} \|x-y\|_{\mathfrak{s}}^{\gamma-k} 2^{-\frac{n|\mathfrak{s}|}{2} - kn} \end{aligned}$$

With $1 \geq \|x-y\|_{\mathfrak{s}} \geq 2^{-n}$, it follows that

$$|\langle \Pi_x f(x) - \Pi_y f(y), \varphi_x^{n,\mathfrak{s}} \rangle| \lesssim \|\Pi\|_{\gamma;\overline{\mathfrak{K}}} \cdot |||f|||_{\gamma;\overline{\mathfrak{K}}} \|x-y\|_{\mathfrak{s}}^{\gamma-\alpha} 2^{-\frac{n|\mathfrak{s}|}{2} - \alpha n}$$

$\Rightarrow$ We can apply Prop. 3.25 and get $(R1)$:

$$\left| (\mathcal{R}f - \Pi_x f(x))(\mathcal{S}_{\mathfrak{s},x}^{\delta} \eta) \right| \lesssim \delta^{\gamma} \|\Pi\|_{\gamma;\overline{\mathfrak{K}}} |||f|||_{\gamma;\overline{\mathfrak{K}}}$$

To obtain $(R2)$, we apply Prop. 3.25 to

$$\zeta_x = \Pi_x f(x) - \overline{\Pi}_x \overline{f}(x)$$

As before we have

$$|\langle \zeta_x, \varphi_x^{n,\mathfrak{s}} \rangle| \leq |\langle \Pi_x f(x), \varphi_x^{n,\mathfrak{s}} \rangle| + |\langle \overline{\Pi}_x \overline{f}(x), \varphi_x^{n,\mathfrak{s}} \rangle| \lesssim 2^{-\alpha n - \frac{n|\mathfrak{s}|}{2}}$$

Furthermore, we can write

$$\begin{aligned} \zeta_x - \zeta_y &= \Pi_x f(x) - \overline{\Pi}_x \overline{f}(x) - \Pi_y f(y) + \overline{\Pi}_y \overline{f}(y) \\ &= \Pi_x (f(x) - \Gamma_{xy} f(y) - \overline{f}(x) + \overline{\Gamma}_{xy} \overline{f}(y)) \\ &\quad + (\Pi_x - \overline{\Pi}_x)(\overline{f}(x) - \overline{\Gamma}_{xy} \overline{f}(y)) \end{aligned}$$

Recall that

$$\left|\!\left|\!\left| f;\bar{f} \right|\!\right|\!\right|_{\gamma,\mathfrak{K}} := \sup_{x\in\mathfrak{K}} \sup_{\beta<\gamma} \left\| f(x) - \bar{f}(x) \right\|_\beta + \sup_{\substack{x,y\in\mathfrak{K} \\ \|x-y\|_\mathfrak{s}\le 1}} \sup_{\beta<\gamma} \frac{\left\| f(x) - \bar{f}(x) - \Gamma_{xy} f(y) + \bar{\Gamma}_{xy}\bar{f}(y) \right\|_\beta}{\|x-y\|_\mathfrak{s}^{\gamma-\beta}}$$

It then follows that

$$|\langle \zeta_x - \zeta_y, \varphi_x^{n,\mathfrak{s}} \rangle| \lesssim \left(\|\Pi\|_{\gamma;\mathfrak{K}} \left|\!\left|\!\left| f;\bar{f} \right|\!\right|\!\right|_{\gamma;\mathfrak{K}} + \left\|\Pi - \bar{\Pi}\right\|_{\gamma;\mathfrak{K}} \left|\!\left|\!\left| \bar{f} \right|\!\right|\!\right|_{\gamma;\mathfrak{K}} \right) \cdot \|x-y\|_\mathfrak{s}^{\gamma-\alpha} \, 2^{-\frac{n|\mathfrak{s}|}{2} - \alpha n}$$

Hence, we may apply Prop. 3.25 to $\zeta_x = \Pi_x f(x) - \bar{\Pi}_x \bar{f}(x)$ and get the desired bound.

Examples - functions

Consider the following situation:

- $\mathcal{T}$ contains a copy of the reg. str. $\mathcal{T}_{d,\mathfrak{s}}$. We write $\overline{T} \subset T$ for the space associated to the abstract polynomials in T.
- (Π, Γ) describes the polynomial model when restricted to $\overline{T}$.
- Let $\gamma > 0$ and $V \subset \overline{T} + T_{\alpha}^{+}$ be a sector of regularity 0 for some $\alpha \in (0, \gamma)$.
- Apply the reconstruction theorem to elements $f \in \mathcal{D}^{\gamma}(V)$.

One could say that restricting Π to V means that we only consider polynomials and functions of Hölder regularity α or higher.

Proposition

For $f \in \mathcal{D}^\gamma(V)$, $\mathcal{R}f$ *is given by* $\mathcal{R}f(x) = \langle 1, f(x) \rangle$.

Thank you for your attention!