# A Characterization of Effective Resistance Metrics

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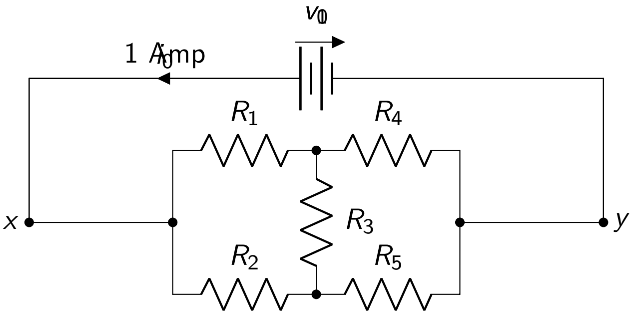
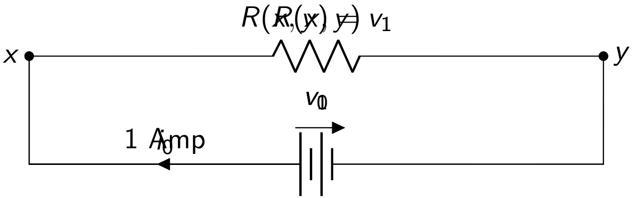
# Effective Resistance




**Ohm's Law**

$V = R \cdot I$



# Mathematical model

## Weighted simple graph

$G = (V, c)$ consists of

- countable **vertex set** $V \neq \emptyset$
- **edge weights** $c : V \times V \to \mathbb{R}_{\geq 0}$ s.t.
  - $c(x, y) = c(y, x) \; \forall \; x, y \in V$
  - $c(x, x) = 0 \; \forall \; x \in V$
  - $c_x := \sum_y c(x, y) < \infty \; \forall \; x \in V$ (locally finite)

$\rightsquigarrow$ set of edges $E = \operatorname{supp} c \subseteq V^2$

- (basically) undirected
- no self-edges
- no multiple edges

**Assumption:** Every $G$ is connected

# Mathematical model (2)

## Interpretation

Resistor between $x, y \in V$ with

- **conductance** $c(x, y)$
- **resistance** $r(x, y) := c(x, y)^{-1}$

$f : V \to \mathbb{R}$

## Laplacian (normalized)

$$(\Delta f)(x) = f(x) - \sum_{y} \frac{c(x, y)}{c_x} f(y)$$

## Energy form

$$\mathcal{E}(f) := \frac{1}{2} \sum_{x,y \in V} c(x, y)(f(x) - f(y))^2$$

# Finite graphs

# What is **electrical** current?

1 Amp from $x$ to $y$

- **Flow** $I : V^2 \to \mathbb{R}$, $I(x, y) = -I(y, x)$
- Kirchhoff's Voltage Law $\Rightarrow \ \exists$ potential $\phi : V \to \mathbb{R}$ s.t

$$I(x, y) = (\nabla\phi)(x, y) = c(x, y)(\phi(x) - \phi(y))$$

- Kirchhoff's Current Law

$$\Rightarrow \Delta\phi = \frac{1}{c_x}\mathbb{1}_x - \frac{1}{c_y}\mathbb{1}_y \tag{D}$$

# Effective Resistance - finite graphs

$$\Delta\phi = \frac{1}{c_x}\mathbb{1}_x - \frac{1}{c_y}\mathbb{1}_y \qquad \text{(D)}$$

## Effective Resistance

$\phi^{xy}$ unique solution of (D) with $\phi^{xy}(y) = 0$

$$R(x, y) := \phi^{xy}(x)$$

## Variational Problem

$$R(x, y) = \left(\min\left\{\mathcal{E}(u) \mid u : V \to \mathbb{R}, u(x) = 1, u(y) = 0\right\}\right)^{-1}$$

# Probabilistic representation

## Random walk on $G$

$(X_k)$ Markov chain on $V$

- transition prob. $p(x, y) = c(x, y)/c_x$
- distributions $(\mathbb{P}_x)_{x\in V}$ (walk starting in $x$)

$\Rightarrow \Delta = \mathrm{Id} - p$

## Tetali, 1991

$\tau_y := \inf \{k \geq 0 \mid X_k = y\}$

$$\phi^{xy}(z) = \frac{1}{c_z}\mathbb{E}_x\left[\sum_{k=0}^{\tau_y - 1} \mathbb{1}_z(X_k)\right]$$

Note: $(c_x)_{x\in V}$ is invariant measure of $(X_k)$

# Probabilistic representation (2)

$$\phi^{xy}(z) = \frac{1}{c_z}\mathbb{E}_x\left[\sum_{k=0}^{\tau_y - 1} \mathbb{1}_z(X_k)\right]$$

## Effective Resistance

$$R(x,y) = \frac{1}{c_x}\mathbb{E}_x\left[\sum_{k=0}^{\tau_y - 1} \mathbb{1}_x(X_k)\right] = \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y \leq \tau_x^+]}$$

$$= \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y < \tau_x^+]}$$

# $R$ as a metric

## Theorem (Tetali, 1991)

Effective Resistance $R : V \times V \to \mathbb{R}$ is a metric on $V$

## Potential as metric data

$$\phi^{xy}(z) = \frac{1}{2}(R(x,y) + R(y,z) - R(x,z)) \geq 0 \ \forall\, x, y, z \in V$$

# Geodesic metric

## Path length

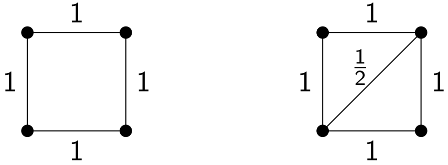
$p := (x_0, \ldots, x_n) \in V^{n+1}$, s.t. $(x_k, x_{k+1}) \in E$

$$L(p) := \sum_{k=0}^{n-1} r(x_k, x_{k+1})$$

## Geodesic metric

$$d_G(x, y) := \inf \{L(p) \mid p \text{ path } x \to y \text{ in } G\}$$

# Geodesic metric (2)




$\rightsquigarrow$ same geodesic metric, different effective resistance

$R$ captures more information about $G$ than $d_G$

# Reconstructing $G$ from $R$

- $G$ is completely defined by $c$
- $\phi^{xy}$ is metric data
- LES for $x, y \in V$

$$\Delta\phi^{xy} = \frac{1}{c_x}\mathbb{1}_x - \frac{1}{c_y}\mathbb{1}_y \tag{D}$$

- **Main idea:** Convert (D) to LES in variables $c(\cdot, y)$ for fixed $y$
- For each $x \neq y$, take $y$-equation of (D)

$$\sum_{z \in V} c(y, z)\phi^{xy}(z) = 1$$

- With $c(y, y) = 0$, we have $|V|$ equations

# Reconstructing $G$ from $R$ (2)

## Proposition

$R$ eff. res. of $G = (V, c)$, $A_y \in \mathbb{R}^{V \times V}$, $b_y \in \mathbb{R}^V$ be

$$A_y(x,z) := \begin{cases} 1 & , \ x = y = z \\ \frac{1}{2}(R(x,y) + R(y,z) - R(x,z)) & , \ \text{otherwise} \end{cases},$$

$$b_y(x) := 1 - \delta_{xy}.$$

$\Rightarrow \det A_y > 0$ and

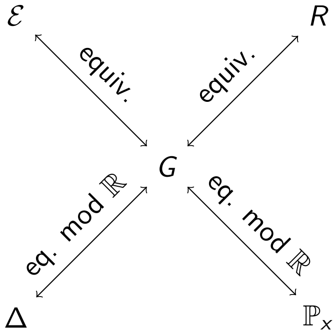
$$A_y \cdot c(\cdot, y) = b_y$$

## Corollary

$R_1, R_2$ be effective resistances of $(V, c_1), (V, c_2)$, resp. Then,

$$R_1 = R_2 \Leftrightarrow c_1 = c_2.$$

# Equivalence



# Characterization

For $(V, d)$ finite metric space, $y \in V$, let

$$A_y(x,z) := \begin{cases} 1 & , \ x = y = z \\ \frac{1}{2}(d(x,y) + d(y,z) - d(x,z)) & , \ \text{otherwise} \end{cases}$$

matrix of **triangle inequality defects**.

**Theorem**

$(V, d)$ finite metric space. Ex. $G = (V, c)$ with eff. res. $d$ if and only if

1. $\exists\ y \in V : \det A_y > 0$
2. $\forall\ y \in V$ : solutions of $A_y \cdot c(\cdot, y) = b_y$ are non-negative

# Characterization - Sketch of Proof

Need to show: Solution $c$ exists and defines connected graph

1. $\det A_y$ is independent of $y \in V$
   - $A_y$ contains all information about $(V, d)$
   - $\exists\ T^{xy} : A_y = T_{xy} \cdot A_x \cdot T_{xy}^t$
   - $\det(T_{xy}) = -1$

   $\Rightarrow$ solution matrix $c$ exists
2. $c$ is symmetric
   - Cramer's rule: $c(x, y) = \frac{\det A_{yx}}{\det A_y}$
   - $\det(T_{xy} A_{xy} T_{xy}^t) = \det(T_{yx} A_{yx} T_{yx}^t)$

$$\Rightarrow c(x, y) = \frac{\det A_{yx}}{\det A_y} = \frac{\det A_{xy}}{\det A_x} = c(y, x)$$

3. $G = (V, c)$ is connected
   - $G$ has no isolated vertices $\Rightarrow \Delta$ well-defined
   - $\phi^{xy}(z) := A_y(x, z)$ solves Dirichlet problem of $G \Rightarrow G$ connected $\square$

# Infinite graphs

# Harmonic Dirichlet functions

Let

$$\mathcal{HD}(G) := \{f : V \to \mathbb{R} \mid \Delta f \equiv 0, \mathcal{E}(f) < \infty\}$$

There exist graphs $G$ such that

$$\mathbb{1}_V \cdot \mathbb{R} \subsetneq \mathcal{HD}(G)$$

$\Rightarrow$ Non-uniqueness of solutions for

$$\Delta \phi^{xy} = \frac{1}{c_x} \mathbb{1}_x - \frac{1}{c_y} \mathbb{1}_y$$
$$\phi^{xy}(y) = 0$$

$G$ recurrent $\Rightarrow \mathcal{HD}(G) = \mathbb{1}_V \cdot \mathbb{R}$

# Resistance metric

**Definition (Kigami)**

$R : X \times X \to \mathbb{R}$ is **resistance metric** iff for all $Y \subseteq X$, $|Y| < \infty$, there ex. $G_Y = (Y, c_Y)$ with effective resistance $R\restriction_{Y \times Y}$.

$\rightsquigarrow$ Apply finite characterization to every $Y$

**Proposition**

$V$ countably infinite, $(V_n)_n$ finite exhaustion of $V$. Then, $R : V \times V \to \mathbb{R}$ is a resistance metric on $V$ if and only if $R_n := R\restriction_{V_n}$ satisfies the criterion from before for all $n \in \mathbb{N}$.

Is $R$ associated with a graph/Markov chain?

# Limit graph

$V$ countably infinite, $R$ resistance metric on $V$, $(V_n)_n$ exhaustion of $V$

- $\exists$ graphs $G_n = (V_n, c_n)$ s.t. $R_{G_n} = R\restriction_{V_n}$
- $R_n = R_{n+1}\restriction_{V_n} \Rightarrow c_n(x,y) \geq c_{n+1}(x,y)$
- $c(x,y) := \lim_{n\to\infty} c_n(x,y)$ exists for all $x, y \in V$

**Limit graph**

$$\Gamma_R := (V, c)$$

**Proposition**

If $R$ "comes from" graph $G$, then $\Gamma_R = G$.

$\rightsquigarrow$ Use $\Gamma_R$ to get probabilistic representation?

# Convergence of random walks

- $\mathbb{P}_x^n$ random walk on $G_n$
- $\mathbb{P}_x$ random walk on $\Gamma_R$

**Theorem**

1. $(\mathbb{P}_{x_0}^n)_{n\in\mathbb{N}}$ converges weakly if and only if

$$\sum_{y\in V} \lim_{n\to\infty} c_n(x,y) = \lim_{n\to\infty} \sum_{y\in V_n} c_n(x,y) < \infty \ \forall\, x \in V$$

2. In case of convergence, $\mathbb{P}_{x_0}^n \Rightarrow \mathbb{P}_{x_0}$
3. If $\Gamma_R$ is recurrent, then

$$R(x,y) = \frac{1}{c_x}\mathbb{E}_x\left[\sum_{k=0}^{\tau_y - 1} \mathbb{1}_x(\omega_k)\right]$$

# Free Effective Resistance

$(V_n)$ finite exhaustion of $V$ such that $G_n := (V_n, c\restriction_{V_n})$ connected

## Definition

$$R^F(x,y) := \lim_{n\to\infty} R_{G_n}(x,y)$$

Does

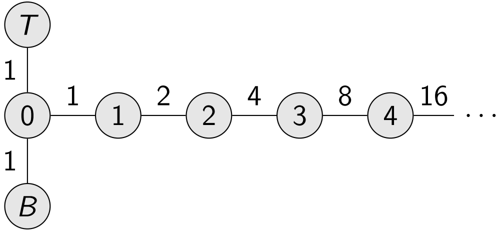
$$R^F(x,y) = \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y \le \tau_x^+]}$$

or

$$R^F(x,y) = \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y < \tau_x^+]}$$

hold?

# Transient $\mathcal{T}$



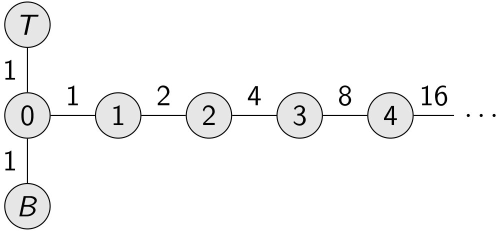

- transient, $\mathcal{HD}(\mathcal{T}) = \mathbb{R}$
- $R^F(B, T) = 2$
- $\mathbb{P}_B[\tau_T < \tau_B^+] = \mathbb{P}_0[\tau_T < \tau_B] = 1 - \mathbb{P}_0[\tau_B < \tau_T] - \mathbb{P}_0[\tau_B = \tau_T = \infty]$
- Symmetry $\Rightarrow \mathbb{P}_0[\tau_B < \tau_T] = \mathbb{P}_0[\tau_T < \tau_B]$

$$\mathbb{P}_B[\tau_T < \tau_B^+] = \frac{1 - \mathbb{P}_0[\tau_B = \tau_T = \infty]}{2} < \frac{1}{2}$$

and

$$\mathbb{P}_B[\tau_T \leq \tau_B^+] = \frac{1 + \mathbb{P}_0[\tau_B = \tau_T = \infty]}{2} > \frac{1}{2}$$

# Transient $\mathcal{T}$ (2)




Hence,

$$R^F(B,T) \neq \frac{1}{c_B \cdot \mathbb{P}_B[\tau_T < \tau_B^+]}$$

and

$$R^F(B,T) \neq \frac{1}{c_B \cdot \mathbb{P}_B[\tau_T \leq \tau_B^+]}$$

$\Rightarrow$ Probabilistic representation does **not** solely depend on $\mathcal{HD} = \mathbb{R}$ !

# $\mathcal{T}$ in other graphs

**Theorem**

$G$ transient. Then,

$$R^F(x,y) = \frac{1}{c_x \cdot \mathbb{P}_x[\tau_y < \tau_x^+]}$$

if and only if $G$ is subgraph of a line.

**Main idea:** If $G$ is not a subgraph of a line, it contains something like $\mathcal{T}$.

**Thank you for your attention!**

# Computing effective resistances

## Proposition (Star-mesh transform)

$G = (V, c)$, fix $x_0 \in V$. Let $V' := V \setminus \{x_0\}$ and

$$c'(x,y) := \begin{cases} 0 & , x = y \\ c(x,y) + \frac{c(x,x_0)c(x_0,y)}{c_{x_0}} & , x \neq y \end{cases}, x, y \in V'$$

$\Rightarrow R_{(V',c')}(x,y) = R_{(V,c)}(x,y)$ for all $x, y \in V'$

$$\text{In particular, } c'_x = c_x - \frac{c(x_0,x)^2}{c_{x_0}}$$

## Corollary

$G = (V, c)$, $G' = (V', c')$ s.t. $V' \subseteq V$ and $R_G\restriction_{V'} = R_{G'}$. Then

1. $c'(x,y) \geq c(x,y)$ for $x, y \in V'$
2. $c'_x \leq c_x$ for $x \in V'$

# Resistance forms

$X$ any set, $D$ lin. subspace of $\mathbb{R}^X$ and $\mathcal{F} : D \to \mathbb{R}$ non-negative symmetric quadratic form

## Definition - Kigami 2001

$(\mathcal{F}, D)$ is a **resistance form** if

1. $1 \in D$ and $\mathcal{F}(f) = 0 \Leftrightarrow f$ constant
2. $({}^{D}/_{\mathbb{R}}, \mathcal{F})$ is a Hilbert space
3. $\forall\ Y \subseteq X, |Y| < \infty, f \in l(Y)\ \exists\ g \in D : g\restriction_Y = f$
4. For $x, y \in X$

$$R_{(\mathcal{F},D)}(x, y) := (\min\{\mathcal{F}(f) \mid f \in D, f(x) = 1, f(y) = 0\})^{-1} < \infty$$

**Resistance metric**

5. $f \in D$, $\widetilde{f} := \max(\min(f, 1), 0) \Rightarrow \widetilde{f} \in D$ and $\mathcal{F}(\widetilde{f}) \leq \mathcal{F}(f)$

# Spaces of functions

$G = (V, c)$ graph, $\mathcal{E}$ energy form

## Domain of $\mathcal{E}$

$$\operatorname{dom}\mathcal{E} := \{f : V \to \mathbb{R} \mid \mathcal{E}(f) < \infty\}$$

## Potentials of finite support

Define $\mathcal{F}in \subseteq \operatorname{dom}\mathcal{E}$ as $\mathcal{E}$-closure of span $\{\mathbb{1}_x\}_{x\in V}$.

## Royden Decomposition

$$\operatorname{dom}\mathcal{E} = \mathcal{F}in \oplus \mathcal{HD}(G)$$

## Proposition

If $\mathcal{F}in \subseteq D \subseteq \operatorname{dom}\mathcal{E}$ s.t. $(\mathcal{E}, D)$ is res. form, then

$$\Gamma_{R_{(\mathcal{E},D)}} = G$$