Universität Leipzig
Fakultät für Mathematik und Informatik
Mathematisches Institut

Spanning tree measures, electrical networks and effective resistance

Diplomarbeit

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vorgelegt von
Tobias Weihrauch
Matrikelnummer 1812326

Betreuender Hochschullehrer: Professor Dr. Max von Renesse
Mathematisches Institut
Universität Leipzig

Abstract

We present an introduction into the theory of networks, focusing on the study of random walks on networks, spanning tree measures, and Hilbert spaces relating to electrical networks. We define and investigate the effective resistance of a finite network, and show connections to Green's functions and Γ-convergence in the spirit of Kasue [4]. Eventually, we extend some of the given statements to infinite networks, laying the foundation for further analysis.

Contents

1 Introduction

Graphs are very basic objects. They consist of a set of *vertices* and a set of *edges* connecting these vertices. If we assign a positive weight to each edge, we obtain a *network*. Due to this simplicity, graphs and networks can be interpreted in various ways and can be used to model a lot of different phenomena. In this work we want to mainly consider two interpretations: As a network of streets that one can walk around on or as an electrical network.

The main purpose of this work is to study both interpretations on a level that is intelligible to math students that have a fundamental understanding of probability theory and functional analysis. This will enable us to investigate spanning tree measures, random walks on networks, and Hilbert spaces related to electrical networks. All of the aforementioned are objects of recent mathematical research.

Throughout this work, we will mainly follow three research papers while providing our own observations and results along the way. Our notion of a graph and the study of random walks and electrical networks (Chapters 3, 4 and 6) are roughly based on [1] by Benjamini, Lyons, Peres and Schramm. The definition and first results regarding a network's effective resistance (Chapter 5) are due to Tetali [9]. Furthermore, results regarding the connection of the effective resistance to Green's functions and to Gamma convergence (Chapters 5 and 7) are based on [4] by Kasue.

Among our own contributions is the development of a theory of random walks on networks that use edges instead of vertices as their state space, see Section 3.2. This is especially useful when considering networks that allow multiple edges. Furthermore, we show that the simple structure of a network is completely determined by its effective resistance. More precisely, two networks that contain no self-edges and no multiple edges have the same effective resistance if and only if they are isomorphic.

A quick note to our method of referencing: Since a considerable part of our work was finding proofs for statements, which were merely stated in our references, or enhancing given proofs to be more rigorous, we mark statements and proofs separately. If a reference is only given in the title of a statement but not at the beginning of the associated proof, then the given proof is mainly due to our own work.

2 Graph theory

In the following, we introduce our notion of a graph and a network. We also present basic concepts and statements of graph theory with respect to our definition.

2.1 Basic definitions and statements

Definition 2.1 (Graph)**.** A *graph* $G = (V, E)$ consists of

- a countable set V of *vertices* and
- a set $E \subseteq V \times V \times \mathbb{N}$ of *directed edges* such that $(x, y, n) \in E$ implies $(y, x, n) \in E$.

If V and E are finite, we say that G is *finite*.

For $e = (x, y, n)$, we call $\widehat{e} := (y, x, n)$ the *reversed edge*, $\underline{e} := x$ the *tail* and $\overline{e} := y$ the *head* of e. Furthermore, we call $\mathfrak{e} := \{e, \widehat{e}\}$ the *(undirected) edge* associated to e and $\widehat{e}$ and denote by $\mathfrak{E} := \{\{e, \widehat{e}\} \mid e \in E\}$ the set of undirected edges.

Remarks.

1. For any directed edge $e = (x, y, n)$, the number $n \in \mathbb{N}$ can be thought of as an identification number. This way we can differentiate between multiple edges with the same endpoints x and y.
2. As defined above, we will denote directed edges using latin letters (e, f, etc.) and the corresponding (undirected) edges using fractal letters ($\mathfrak{e}, \mathfrak{f}$, etc.). The same method is used when dealing with sets of edges.
3. Although $G = (V, E)$ is technically a directed graph, we will mostly interpret it as an undirected graph $G = (V, \mathfrak{E})$ since $(x, y, n) \in E$ if and only if $(y, x, n) \in E$.

Example 2.2. Let $G = (V, E)$ such that $V = \{a, b, c, d\}$ and

$$E = \{(a, b, 1), (b, a, 1), (a, b, 2), (b, a, 2), (a, d, 1), (c, d, 1), (b, c, 1), (c, b, 1)\}.$$

The visualization of a graph is quite intuitive. We draw the vertices and use lines between vertices to represent edges. Instead of writing the identification number of each edge next to it, we will use different colors for edges with different identification numbers. One usually uses arrowheads to mark the direction of a directed edge. However, since we know that for every directed edge of G its reversed edge also belongs to G, we draw only one edge associated with (x, y, n) and (y, x, n) and omit the arrowheads. Two possible diagrams depicting the same graph G are shown in Figure 2.1.

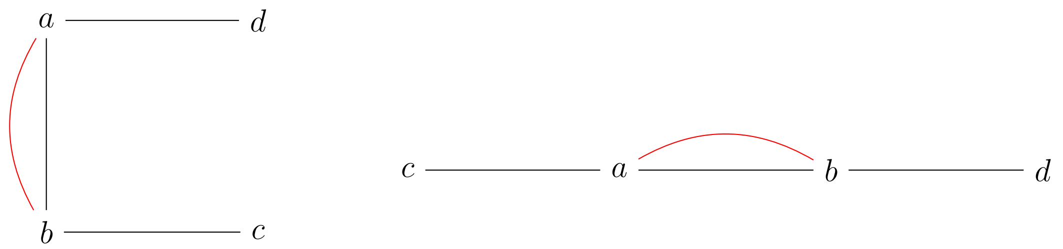

Figure 2.1: Two diagrams showing the same graph G.

Definition 2.3. Let $G = (V, E)$ be a graph. For $e \in E$, $x, y \in V$, we will use the following notation:

- $|G| := |\mathfrak{E}|$.
- $V(e) := V(\mathfrak{e}) := \{\underline{e}, \overline{e}\}$ - the set of vertices connected by e.
- $E(x) := \{e \in E \mid \underline{e} = x\}$ - the set of directed edges with tail x.
- $E(x, y) := \{e \in E \mid \underline{e} = x, \overline{e} = y\}$ - the set of directed edges with tail x and head y.
- $\mathfrak{E}(x) := \{\{e, \widehat{e}\} \mid e \in E(x)\}$ - the set of edges touching x.
- $\mathfrak{E}(x, y) := \{\{e, \widehat{e}\} \mid e \in E(x, y)\}$ - the set of edges connecting x and y.

We call G *simple* if $|E(x, x)| = 0$ and $|E(x, y)| \leq 1$ for all $x, y \in V$, i.e. G contains neither loops nor multiple edges.

Remark. Each notation can be canonically extended to sets of edges and vertices (e.g. $V(A) = \bigcup_{e \in A} V(e)$ for $A \subseteq E$).

Definition 2.4 (Paths and cycles)**.** Let $G = (V, E)$ be a graph and $x, y \in V$.

1. A *path in G from x to y* is a sequence $p = (e_1, \dots, e_n)$ of directed edges such that $\underline{e_1} = x$, $\overline{e_i} = \underline{e_{i+1}}$ for $i = 1, \dots, n-1$ and $\overline{e_n} = y$. We use the notation *p is a path $x \to y$.*
2. Let $p = (e_1, \dots, e_n)$ be a path in G. Its *reversed path* is defined by $\widehat{p} := (\widehat{e_n}, \dots, \widehat{e_1})$.
3. For two paths $p_1 = (e_1, \dots, e_n)$, $p_2 = (f_1, \dots, f_m)$ such that $\overline{e_n} = \underline{f_1}$, we define $p_1 p_2 := (e_1, \dots, e_n, f_1, \dots, f_m)$.
4. A path $(e_1, \dots, e_n)$ is called *self-avoiding* if $V(e_i) \cap V(e_j) = \emptyset$ for all $i \neq j$.
5. A *loop* is an edge $e \in E$ such that $\underline{e} = \overline{e}$.
6. A *generalized cycle* is a path $(e_1, \dots, e_n)$ such that $\underline{e_1} = \overline{e_n}$.

7. A *cycle* is a generalized cycle $(e_1, \ldots, e_n)$ such that $e_i \neq \widehat{e_j}$ for all $i \neq j$.
8. A cycle $c = (e_1, \ldots, e_n)$ is called *simple* if $(e_1, \ldots, e_{n-1})$ is self-avoiding, i.e. if c does not contain a smaller cycle.

Example 2.5. The way we have defined a cycle in Definition 2.4 might seem a

Figure 2.2: The graphs G_1 and G_2.

little bit strange, since we demand of a cycle $(e_1, \ldots, e_n)$ to satisfy

$$e_i \neq \widehat{e_j} \;\forall\; i \neq j.$$

This is due to our definition of a graph. Consider $G_i = (\{a, b, c\}, E_i)$, $i = 1, 2$, where

$$E_1 = \{(a, b, 1), (b, a, 1), (b, c, 1), (c, b, 1)\}$$

and

$$E_2 = E_1 \cup \{(c, a, 1), (a, c, 1)\},$$

as shown in Figure 2.2. By intuition, one would say that G_1 does not contain any cycles while G_2 contains cycles. With respect to our definition of a cycle this is true. However, if we had defined a cycle to only be a path starting at some vertex x and returning to it, then G_1 would contain the cycle $(e, \widehat{e})$ where $e = (a, b, 1)$. Since this is not a cycle with respect to our definition, we call it a generalized cycle.

The additional requirement of $e_i \neq \widehat{e_j}$ for all $i \neq j$ in Definition 2.4 states that a cycle cannot use both a directed edge and its reversed edge.

Lemma 2.6. *Let $(e_1, \ldots, e_n)$ be a path, $\underline{e_1} = \overline{e_n}$ and there exists an i such that $\widehat{e_i} \neq e_j$ for all $j \neq i$. Then there exists a subsequence $(e_{n_1}, \ldots, e_{n_k})$ which is a cycle.*

Proof. Without loss of generality, we can assume that $i = 1$ since

$$(e_i, e_{i+1}, \ldots, e_n, e_1, \ldots, e_{i-1})$$

satisfies all requirements if and only if $(e_1, \ldots, e_n)$ does. We can also assume that $\overline{e_j} \neq \underline{e_1}$ for all $1 \neq j \neq n$ because otherwise we can just consider $(e_1, \ldots, e_j)$. If $e_1 = \widehat{e_1}$, then (e_1) is a cycle and we are done. Now assume that $e_1 \neq \widehat{e_1}$. Remove from $(e_1, \ldots, e_n)$ all sequences $(e_i, e_{i+1}, \ldots, e_j)$ such that $e_j = \widehat{e_i}$ in the order they appear (i.e. if applicable, remove $(e_2, \ldots, \widehat{e_2})$ and move on to the next remaining e_i). The resulting sequence $(e_{n_1}, \ldots, e_{n_k})$ will satisfy $e_{n_1} = e_1$ and $e_{n_k} = e_n$, i.e. e_n will not be removed: If $e_i = \widehat{e_n}$ then $\overline{e_{i-1}} = \underline{e_i} = \overline{e_n} = \underline{e_1}$ which we already ruled out. We also know that $\overline{e_{n_i}} = \underline{e_{n_{i+1}}}$ for all $i = 1, \ldots, k-1$ since each removal of a sequence $(e_j, \ldots, \widehat{e_j})$ does not affect this property. Obviously, $\underline{e_{n_1}} = \underline{e_1} = \overline{e_n} = \overline{e_{n_k}}$ does hold as well as $e_{n_i} \neq \widehat{e_{n_j}}$ for all $i \neq j$. Hence, $(e_{n_1}, \ldots, e_{n_k})$ is a cycle. □

Remark. Lemma 2.6 will be quite useful whenever we want to prove the mere existence of a cycle, since for any sequence of edges $(e_1, \ldots, e_n)$, which might contain a cycle, we do not have to show

$$\forall\, i\; \forall\, j \neq i : e_i \neq \widehat{e_j}$$

but only

$$\exists\, i\; \forall\, j \neq i : e_i \neq \widehat{e_j}.$$

Definition 2.7 (Spanning tree)**.** Let $G = (V, E)$ be a graph.

1. G is called *connected* if for any $x, y \in V$, there exists a path $x \to y$ in G.
2. G is called a *forest* if it contains no cycles.
3. G is called a *tree* if it is a connected forest.
4. A graph $H = (V_H, E_H)$ is called a *subgraph* of G if $V_H \subseteq V$ and $E_H \subseteq E$. In this case, we write $H \subseteq G$.
5. A subgraph $H = (V_H, E_H)$ of G is called *spanning* if $V_H = V$.
6. A *spanning tree* T of G is a spanning subgraph of G that is a tree.

By $ST(G)$ we denote the set of spanning trees of G.

Remarks.

1. Note that H is a subgraph if $V_H \subseteq V$, $E_H \subseteq E \cap (V_H \times V_H \times \mathbb{N})$ and $(x, y, n) \in F$ if and only if $(y, x, n) \in F$.
2. Since each spanning tree $T = (V, E_T) \in ST(G)$ has the same vertex set as G, we may consider it a subset of edges of G, i.e. we may identify T with $\mathfrak{E}_T$ or E_T.
3. If G is not connected then $ST(G) = \emptyset$.

Important: Throughout this work, we will assume (if not stated otherwise) that every considered graph is connected.

Example 2.8. Consider the graph G depicted in Figure 2.3. G has three spanning trees, which are shown in Figure 2.4.

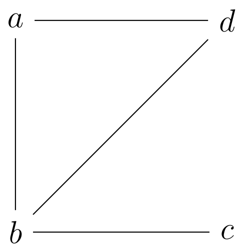

Figure 2.3: The graph G.

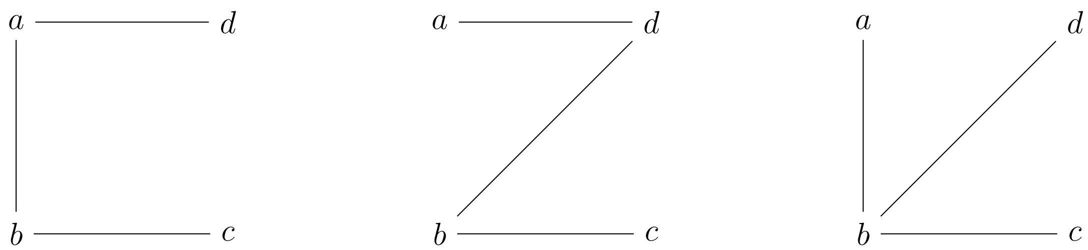

Figure 2.4: The three spanning trees of G.

Lemma 2.9 (Properties of spanning trees)**.** *Let $G = (V, E)$ be a finite graph and $T = (V, E_T) \in ST(G)$. Then*

1. $|T| = |V| - 1$.
2. T *is simple.*
3. *For every $x, y \in V$, there is a unique self-avoiding path in T from x to y.*

Proof.

1. If $|V| = 1$, then the only spanning tree is $T = (V, \emptyset)$, and hence $0 = |T| = |V| - 1$ holds. Now consider $|V| > 1$. Since T is spanning, connected and contains no cycles, we can find an enumeration $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ of $\mathfrak{E}_T$ such that for any $1 < j \leq n$, we have
$$\left| \left(V(\mathfrak{e}_j) \setminus \bigcup_{i=1}^{j-1} V(\mathfrak{e}_i) \right) \right| = 1$$
because there is always at least one edge $\mathfrak{e}_j$ connecting $V_j := \bigcup_{i=1}^{j-1} V(\mathfrak{e}_i)$ and $V \setminus V_j$. Hence, $|V_1| = 2$, $|V_2| = 2 + 1$, $|V_3| = 2 + 1 + 1$, and so on. This gives us $|V_j| = 1 + j$, and thus
$$\left| \bigcup_{\mathfrak{e} \in E_T} V(\mathfrak{e}) \right| = \left| \bigcup_{i=1}^{n} V(\mathfrak{e}_i) \right| = 1 + n = 1 + |\mathfrak{E}_T| = 1 + |T|.$$
Since T is spanning and connected, we also have
$$V = \bigcup_{\mathfrak{e} \in \mathfrak{E}_T} V(\mathfrak{e}),$$

and hence $|V| = 1 + |T|$.

2. By Definition 2.4, a loop is a cycle, i.e. T does not contain any loops. Suppose there exist $x, y \in V$ such that $x \neq y$ and $e, f \in E(x,y), e \neq f$. Then $(e, \widehat{f})$ is a cycle in T, which is a contradiction.
3. Since T is connected, there is at least one self-avoiding path p_1 in T from x to y. Assume there is another path $p_2 \neq p_1$ from x to y. Then $p_1\widehat{p_2}$ is a path in T that satisfies all requirements of Lemma 2.6. Hence, $p_1\widehat{p_2}$ contains a cycle, which is a contradiction to $T \in ST(G)$. □

Corollary 2.10. *Let $G = (V, E)$ be a finite graph. Then*

$$|ST(G)| \leq \binom{|G|}{|V|-1}$$

holds, where $\binom{|G|}{|V|-1}$ denotes the binomial coefficient indexed by n and k.

Proof. By Lemma 2.9, we have $|T| = |V| - 1$ for all $T \in ST(G)$. Since we can identify $T = (V, E_T)$ with $\mathfrak{E}_T$, we can think of $ST(G)$ as a subset of $2^{\mathfrak{E}}$. Then

$$ST(G) \subseteq \{\mathfrak{F} \subseteq \mathfrak{E} \mid |\mathfrak{F}| = |V| - 1\} =: M.$$

Hence, $|ST(G)| \leq |M| = \binom{|\mathfrak{E}|}{|V|-1} = \binom{|G|}{|V|-1}$. □

Definition 2.11 (Connected component)**.** Let $G = (V, E)$ be a finite graph that is not necessarily connected and let $v \in V$. The *connected component of G, which contains v*, is the subgraph $H = (V_H, E_H) \subseteq G$ where

$$V_H = \{x \in V \mid \exists \text{ path } v \to x \text{ in } G\}$$

and $E_H = \{e \in E \mid \underline{e}, \overline{e} \in V_H\}$.

Remarks.

1. Let $x, y \in V$. If there is a path $x \to y$ in G, then the connected components containing x and y are equal.
2. Every finite graph $G = (V, E)$ has a finite number $n \in \mathbb{N}$ of connected components $H_i = (V_i, E_i)$, $i = 1, \dots, n$ satisfying

$$V = \bigcup_{i=1}^{n} V_i \ , \ E = \bigcup_{i=1}^{n} E_i$$

and $V_i \cap V_j = E_i \cap E_j = \emptyset$ for all $i \neq j$. This means that $\{V_1, \dots, V_n\}$ and $\{E_1, \dots, E_n\}$ are partitions of V and E, respectively.

Lemma 2.12. *Let $T = (V_T, E_T)$ be a subgraph of $G = (V, E)$, where $|V| \geq 2$. Then the following statements are equivalent:*

1. *T is a spanning tree of G.*
2. *T is spanning, connected and satisfies $|T| = |V| - 1$.*
3. *T contains no cycles and satisfies $|T| = |V| - 1$.*

Proof.
1 ⇒ 2 and 1 ⇒ 3: If T is a spanning tree, then T is connected, spanning and contains no cycles. By Lemma 2.9, we also have $|T| = |V| - 1$.

2 ⇒ 1: We only need to show that T contains no cycles. Assume T is connected and spanning and contains a cycle $(f_1, \dots, f_n)$. We know that

$$V = \bigcup_{\mathfrak{e} \in \mathfrak{E}_T} V(\mathfrak{e})$$

because T is spanning and connected. We can also find an enumeration $\mathfrak{e}_1, \dots, \mathfrak{e}_m$ of $\mathfrak{E}_T$ such that

$$|V(\mathfrak{e}_j) \setminus V_j| \leq 1 \quad \forall\, j = 2, \dots, m$$

where $V_j := \bigcup_{i=1}^{j-1} V(\mathfrak{e}_i)$. Hence,

$$|V| = \left| \bigcup_{j=1}^{m} V(\mathfrak{e}_j) \right| \leq 1 + m = 1 + |\mathfrak{E}_T| = 1 + |T|.$$

Note that this can only be an equality if $|V(\mathfrak{e}_j) \setminus V_j| = 1$ for all $j = 2, \dots, m$. Now let j_0 be the index such that

$$\left\{ \underline{f_1}, \overline{f_1}, \dots, \underline{f_n}, \overline{f_n} \right\} \not\subseteq V_{j_0} \ , \ \left\{ \underline{f_1}, \overline{f_1}, \dots, \underline{f_n}, \overline{f_n} \right\} \subseteq V_{j_0+1},$$

i.e. the edge $\mathfrak{e}_{j_0}$ closes the cycle $(f_1, \dots, f_n)$. It follows that $|V(\mathfrak{e}_{j_0}) \setminus V_{j_0}| = 0$. Hence, $|V| < 1 + |T|$, which is a contradiction.

3 ⇒ 1: First, we will show that T is spanning. Consider any enumeration $\mathfrak{e}_1, \dots \mathfrak{e}_n$ of $\mathfrak{E}_T$. Let $V_j := \bigcup_{i=1}^{j-1} V(\mathfrak{e}_i)$. Then

$$|V(\mathfrak{e}_j) \setminus V_j| \geq 1 \ \forall\, j = 2, \dots, n,$$

and since T does not contain any loops, $|V(\mathfrak{e}_1)| = 2$. Hence,

$$|V_T| \geq |V(\mathfrak{E}_T)| = \left| \bigcup_{j=1}^{n} V(\mathfrak{e}_j) \right| \geq 1 + n = 1 + |\mathfrak{E}_T| = 1 + |V| - 1 = |V|.$$

Thus, $V_T = V$, and we have shown that T is spanning.

Now let $T_i := (V_i, E_i)$, $i = 1, \dots, m$, be the connected components of T. Since T contains no cycles, so does each T_i. Therefore, they are spanning trees of themselves satisfying $|T_i| = |\mathfrak{E}_i| = |V_i| - 1$. Hence,

$$|V| - 1 = |\mathfrak{E}_T| = \sum_{i=1}^{m} |\mathfrak{E}_i| = \sum_{i=1}^{m} |V_i| - 1 = |V| - m,$$

implying $m = 1$. This proves that T is connected. $\square$

Definition 2.13 (Generated (sub)graph)**.** Let $G = (V, E)$ be a graph and $W \subseteq V, F \subseteq E$. We define two graphs:

- The *subgraph* $G(W) := (W, E_W)$ *of* G *generated by* W, where

$$E_W := \{e, \widehat{e} \mid e \in E(v, w), v, w \in W\}$$

- The *graph* $G(F) := (V(F), F')$ *generated by* F, where

$$F' := \{e, \widehat{e} \mid e \in F\}$$

Remarks.

1. Note that we do not need any information about G when considering the graph $G(F)$ generated by a set of edges F, since the edges contain the information which vertices are needed.
2. Using Definition 2.13, we can extend basic properties of graphs (such as being connected) onto sets of edges or vertices. For example, let $G = (V, E)$ and $F \subseteq E$. We say that F is connected (in G) if $G(F)$ is connected.

Definition 2.14 (Network and weights)**.**

1. A *network* $\mathcal{N}$ is a triplet (V, E, C) (or a pair (G, C)) where $G = (V, E)$ is a connected graph and $C : E \to \mathbb{R}^+$ is a *weight function* such that $C(e) = C(\widehat{e})$ for all $e \in E$.
2. For $e \in E$, we call $C(e)$ (or $C(\mathfrak{e})$) the *weight* of e (or $\mathfrak{e}$).
3. We call $\mathcal{N}$ *locally finite* if $\sum_{e \in E(v)} C(e) < \infty$ for all $v \in V$.
4. For a set of directed edges $A \subseteq E$ (or $\mathfrak{A} \subseteq \mathfrak{E}$), we define

$$\text{weight}_C(A) := \text{weight}_C(\mathfrak{A}) := \prod_{\mathfrak{e} \in \mathfrak{A}} C(\mathfrak{e}).$$

Remarks.

1. In this work we will only consider locally finite networks if not stated otherwise.
2. C can also be considered as a mapping $C : \mathfrak{E} \to \mathbb{R}^+$ since $C(e) = C(\widehat{e})$ for all $e \in E$.
3. Sometimes we may refer to G being a network. In this case, we either assume that C is a previously defined, fixed weight function or that C can be an arbitrary weight function.
4. Note that the weight of a set of directed edges equals the weight of the associated set of undirected edges.

Example 2.15. When drawing a network's diagram, we will write each edge's weight right next to it: Consider $G = (\{a, b, c\}, \{e_1, \widehat{e_1}, e_2, \widehat{e_2}, e_3, \widehat{e_3}\})$ such that

$$e_1 = (a, b, 1), e_2 = (b, c, 1), e_3 = (c, a, 1)$$

and $C(e_i) = 2^i$. A possible diagram of the network (G, C) is shown in Figure 2.5.

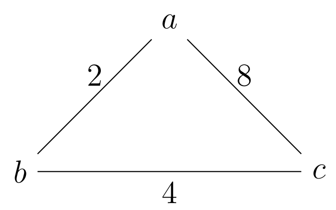

Figure 2.5: A diagram showing the network (G, C).

Lemma 2.16. *Let (V, E, C) be a network, $A, B \subseteq E$. Then*

1. $\text{weight}_C(\emptyset) = 1$.
2. $\text{weight}_C(\mathfrak{A} \cup \mathfrak{B}) = \text{weight}_C(\mathfrak{A}) \cdot \text{weight}_C(\mathfrak{B}) \cdot (\text{weight}_C(\mathfrak{A} \cap \mathfrak{B}))^{-1}$.
3. $\text{weight}_C(\mathfrak{A} \cup \mathfrak{B}) = \text{weight}_C(\mathfrak{A} \setminus \mathfrak{B}) \cdot \text{weight}_C(\mathfrak{B}) = \text{weight}_C(\mathfrak{B} \setminus \mathfrak{A}) \cdot \text{weight}_C(\mathfrak{A})$.
4. $\text{weight}_C(A \cup B) = \text{weight}_C(A) \cdot \text{weight}_C(B) \cdot (\text{weight}_C(A \cap B))^{-1}$ *if* $\widehat{A} = A, \widehat{B} = B$.
5. $\text{weight}_C(A \cup B) = \text{weight}_C(A \setminus B) \cdot \text{weight}_C(B) = \text{weight}_C(B \setminus A) \cdot \text{weight}_C(A)$ *if* $\widehat{A} = A, \widehat{B} = B$.

Proof. 1 is the definition of an empty product. 2 and 3 are easy calculations, which imply 4 and 5 if one uses the following: For $C := A \cup B$ we always have $\mathfrak{C} = \mathfrak{A} \cup \mathfrak{B}$. For $D := A \cap B$, $M := A \setminus B$ and $N := B \setminus A$ we have $\mathfrak{D} = \mathfrak{A} \cap \mathfrak{B}$, $\mathfrak{M} = \mathfrak{A} \setminus \mathfrak{B}$ and $\mathfrak{N} = \mathfrak{B} \setminus \mathfrak{A}$ if $\widehat{A} = A, \widehat{B} = B$. ∎

2.2 Graph homomorphisms

Definition 2.17. Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs. A *graph homomorphism* $\Gamma : G \to H$ between G and H is a pair $\Gamma = (\Gamma_V, \Gamma_E)$ such that

1. $\Gamma_V : V_G \to V_H$, $\Gamma_E : E_G \to E_H$.
2. $\Gamma_E(\widehat{e}) = \widehat{\Gamma_E(e)}$ for all $e \in E_G$.
3. $\underline{\Gamma_E(e)} = \Gamma_V(\underline{e})$ for all $e \in E_G$.

We denote the set of graph homomorphisms between G and H by $Hom(G, H)$.

Remarks.

1. We will omit the indices and use the notation $\Gamma(v) := \Gamma_V(v)$ for all $v \in V$ and $\Gamma(e) := \Gamma_E(e)$ for all $e \in E$ if a confusion of Γ_V and Γ_E is improbable.
2. Since $\{\Gamma(e), \Gamma(\widehat{e})\} = \left\{\Gamma(e), \widehat{\Gamma(e)}\right\}$, Γ_E can be canonically extended to a mapping on the sets of undirected edges $\mathfrak{E}_G \to \mathfrak{E}_H$.

Lemma 2.18 (First properties of graph homomorphisms)**.** *Let $\Gamma : G \to H$ be a graph homomorphism. Then*

1. $\overline{\Gamma_E(e)} = \Gamma_V(\overline{e})$ *holds for all $e \in E_G$.*
2. $\overline{\Gamma(e)} = \underline{\Gamma(f)}$ *holds for all $e, f \in E_G$ such that $\overline{e} = \underline{f}$.*

Proof.

1. Let $e \in E$. Then it follows that $\overline{\Gamma_E(e)} = \underline{\widehat{\Gamma_E(e)}} = \underline{\Gamma_E(\widehat{e})} = \Gamma_V(\underline{\widehat{e}}) = \Gamma_V(\overline{e})$.
2. Let $e, f \in E$ be as described. It follows that $\overline{\Gamma(e)} = \Gamma(\overline{e}) = \Gamma(\underline{f}) = \underline{\Gamma(f)}$. □

Remark. The second statement of the lemma above states that adjacent edges will be mapped to adjacent edges.

Lemma 2.19. *Let $\Gamma \in Hom(G, G')$, $p = (e_1, \ldots, e_n)$ be a sequence of directed edges in G and set $\Gamma(p) := (\Gamma(e_1), \ldots, \Gamma(e_n))$.*

1. *If p is a path in G from x to y, then $\Gamma(p)$ is a path in G' from $\Gamma(x)$ to $\Gamma(y)$.*
2. *If Γ_V is injective and $\Gamma(p)$ is a path in G' from $\Gamma(x)$ to $\Gamma(y)$, then p is a path in G from x to y.*
3. *If $A \subseteq E_G$ is connected (in G), then so is $\Gamma(A) := \{\Gamma(e) \mid e \in A\}$ (in G').*

Proof.

1. As a direct consequence of statement 2 of Lemma 2.18, $(\Gamma(e_1), \ldots, \Gamma(e_n))$ is a path in G'. It starts in $\underline{\Gamma(e_1)} = \Gamma(\underline{e_1}) = \Gamma(x)$ and ends in $\overline{\Gamma(e_n)} = \Gamma(\overline{e_n}) = \Gamma(y)$.

2. $\Gamma(x) = \underline{\Gamma(e_1)} = \Gamma(\underline{e_1})$ implies $x = \underline{e_1}$ because Γ_V is injective. Analogously $\Gamma(y) = \overline{\Gamma(e_n)} = \Gamma(\overline{e_n})$ implies $y = \overline{e_n}$ and $\Gamma(\overline{e_i}) = \overline{\Gamma(e_i)} = \underline{\Gamma(e_{i+1})} = \Gamma(\underline{e_{i+1}})$ implies $\overline{e_i} = \underline{e_{i+1}}$ for $i = 1, \dots, n-1$. Hence, p is a path in G from x to y.
3. Let $v_1', v_2' \in V(\Gamma(A))$. Then there are $v_1, v_2 \in V_G$ such that $\Gamma(v_i) = v_i'$. Since A is connected, there is a path $(e_1, \dots, e_n)$ in A such that $\underline{e_1} = v_1$, $\overline{e_n} = v_2$. By 1 $(\Gamma(e_1), \dots, \Gamma(e_n))$ is a path from $\underline{\Gamma(e_1)} = \Gamma(\underline{e_1}) = \Gamma(v_1) = v_1'$ to $\overline{\Gamma(e_n)} = v_2'$ in $\Gamma(A)$. This proves that $\Gamma(A)$ is connected (in G'). □

Lemma 2.20. *Let $\Gamma \in Hom(G, G')$ and $p = (e_1, \dots, e_n)$ be a sequence of directed edges in G.*
1. *If Γ_E is injective and p is a cycle in G, then $\Gamma(p)$ is a cycle in G'.*
2. *If Γ_V is injective and $\Gamma(p)$ is a cycle in G', then p is a cycle in G.*

Proof.
1. By Lemma 2.19, $\Gamma(p)$ is a path in G' such that $\underline{\Gamma(e_1)} = \overline{\Gamma(e_n)}$. Now let i, j be indices such that $\Gamma(e_i) = \widehat{\Gamma(e_j)}$ (if no such indices exist, there is nothing to show). Then $\Gamma(e_i) = \widehat{\Gamma(e_j)} = \Gamma(\widehat{e_j})$ and by the injectivity of Γ_E, $e_i = \widehat{e_j}$. Since p is a cycle in G, this implies $i = j$.
2. By Lemma 2.19, p is a path in G such that $\underline{e_1} = \overline{e_n}$. Let i, j be indices such that $e_i = \widehat{e_j}$. Then $\Gamma(e_i) = \Gamma(\widehat{e_j}) = \widehat{\Gamma(e_j)}$. Hence, $i = j$, and it follows that p is a cycle. □

Lemma 2.21. *Let $\Gamma \in Hom(G, G')$ and Γ_V, Γ_E bijective. Then*

$$\Gamma^{-1} := (\Gamma_V^{-1}, \Gamma_E^{-1}) \in Hom(G', G).$$

Proof.
1. Set $f := \Gamma^{-1}(e)$. Then $\widehat{\Gamma^{-1}(e)} = \widehat{f} = \Gamma^{-1}(\Gamma(\widehat{f})) = \Gamma^{-1}\left(\widehat{\Gamma(f)}\right) = \Gamma^{-1}(\widehat{e})$.
2. Let $e' \in E_{G'}$. Then $\underline{\Gamma_E^{-1}(e')} = \Gamma_V^{-1}\left(\Gamma_V\left(\underline{\Gamma_E^{-1}(e')}\right)\right) = \Gamma_V^{-1}\left(\underline{\Gamma_E(\Gamma_E^{-1}(e'))}\right) = \Gamma_V^{-1}(\underline{e'})$. □

Definition 2.22 (Graph isomorphism)**.** Let $\Gamma \in Hom(G, G')$.
1. We call Γ a *graph isomorphism* if Γ_V and Γ_E are bijective.
2. We say two graphs G, G' are *isomorphic* if there exists an isomorphism $\Gamma : G \to G'$. In this case, we write $G \simeq G'$.

Remark. If two graphs are isomorphic, then they are basically the same graph. The only 'real' differences that may occur are regarding the labeling of vertices, edges or identification numbers of certain edges.

2.3 Contracting edges

Definition 2.23 (Contracted graph)**.** Let $G = (V, E)$ be a graph and $H = (W, F)$ a subgraph of G. The graph $G/H = (V_{G/H}, E')$ resulting from G by *contracting all edges in* H is defined as follows:

- $V_{G/H}$ is a partition[1]of V such that for any $v' \in V_{G/H}$, we have $x, y \in v'$ if and only if there exists a path $x \to y$ in H.
- E' is a set of directed edges such that there is a bijection $\eta : E' \to E$ satisfying

$$\underline{\eta(e')} \in \underline{e'}\ ,\ \ \eta(\widehat{e'}) = \widehat{\eta(e')}\ \ \forall\ e' \in E'.$$

For any subset $F \subseteq E$ of directed edges of G, we set $G/F := G/G(F)$ (see Def. 2.13). Analogously, we define $G/\mathfrak{F}$ for a set $\mathfrak{F}$ of (undirected) edges of G.

For a network $\mathcal{N} = (G, C)$, we define the contracted network $\mathcal{N}/H := (G/H, C_H)$ where $C_H := C \circ \eta$.

Remarks.

1. Although this definition seems a bit complicated, the operation it describes is quite simple: We merge all vertices of G that are connected in H and basically keep the edges as they were.
2. The two requirements for η imply $\overline{\eta(e')} \in \overline{e'}$ as well.
3. Note that E' and η are not uniquely defined. The resulting graphs are, however, isomorphic.
4. Using η we will identify the edges in G and G/H. If we want to consider a subset of edges A as a set of edges in G, we will write A_G and analogously $A_{G/H}$ if we want to consider it as a set of edges in G/H.

Example 2.24. Consider the graph shown in Figure 2.6. Now let $\mathfrak{F} = \{\mathfrak{e}\}$ and

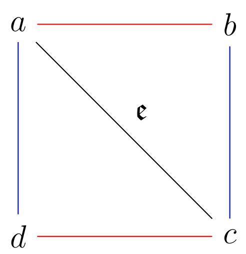

Figure 2.6: The graph G.

$H_{\mathfrak{F}} := (\{a, c\}, \mathfrak{e})$. In $G/\mathfrak{F}$ the vertices a and c are merged to one vertex $\{a, c\}$ because they are connected in $H_{\mathfrak{F}}$. Hence, $G/\mathfrak{F}$ is the graph in Figure 2.7.

[1]Let X be a set. $P \subseteq 2^X$ is called a *partition* of X if $\bigcup_{p \in P} p = X$ and $p \cap q = \emptyset\ \forall\ p, q \in P, p \neq q$.

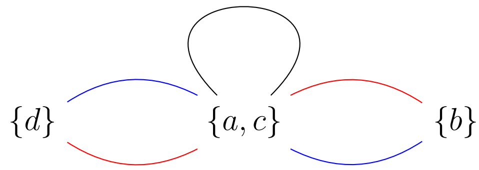

Figure 2.7: The graph $G/\mathfrak{F}$ resulting from G by contracting the edges in $\mathfrak{F}$.

Lemma 2.25. *Let $F \subseteq G \subseteq H$ be finite graphs. Then $G/F \subseteq H/F$ in the sense that G/F is isomorphic to a subgraph of H/F.*

Proof. To prove the statement, we have to show that there exists a graph homomorphism $\Gamma : (\Gamma_V, \Gamma_E) : G/F \to H/F$ such that both Γ_V and Γ_E are injective. Restricting the codomain of Γ_V and Γ_E to their respective ranges will then produce the desired isomorphism.

Let $G = (V_G, E_G), H = (V_H, E_H), G/F = (V_{G/F}, E_{G/F})$, $H/F = (V_{H/F}, E_{H/F})$, and $\eta_G : E_{G/F} \to E_G$, $\eta_H : E_{G/H} \to E_H$ be the associated bijections. Since two vertices $x, y \in V_G$ are connected by edges in F if and only if they are connected by edges in F when considered vertices in H, it follows that

$$V_{H/F} = V_{G/F} \cup \{\{x\} \mid x \in V_H \setminus V_G\}\,.$$

Hence, $\Gamma_V : V_{G/F} \to V_{H/F}$, $\Gamma_V(v) = v$ is well-defined and injective. Now let $\Gamma_E : E_{G/F} \to E_{H/F}$ be

$$\Gamma_E = \eta_H^{-1} \circ \eta_G.$$

As the composition of injective mappings, Γ_E clearly is injective. Also, $\Gamma_E(\widehat{e}) = \widehat{\Gamma_E(e)}$ holds for all $e \in E_{G/F}$ since η_G and η_H satisfy this property.

The only thing left to show is that $\underline{\Gamma_E(e)} = \Gamma_V(\underline{e})$ holds for all $e \in E_{G/F}$. By definition of η_G and η_H, we have

$$\underline{\eta_G(e)} \in \underline{e}\;\forall\; e \in E_{G/F}\;,\;\; \underline{f} \in \underline{\eta_H^{-1}(f)}\;\forall\; f \in E_H.$$

Hence, $\underline{\eta_G(e)} \in \underline{e}$ and $\underline{\eta_G(e)} \in \underline{\eta_H^{-1}(\eta_G(e))} = \underline{\Gamma_E(e)}$ for all $e \in E_{G/F}$. Since $V_{H/F}$ is a partition of V_H, $\underline{e} \cap \underline{\Gamma_E(e)} \neq \emptyset$ implies $\underline{\Gamma_E(e)} = \underline{e} = \Gamma_V(\underline{e})$. It follows that $\Gamma = (\Gamma_V, \Gamma_E)$ is graph homomorphism such that Γ_V and Γ_E are both injective. ☐

Lemma 2.26 (Successively contracting edges)**.** *Let $G = (V, E)$, $F, F' \subseteq E$. Then*

$$(G/F)/F'_{G/F} \simeq G/(F \cup F'_G)$$

or simply

$$(G/F)/F' \simeq G/(F \cup F').$$

Proof. Let $G/F := (V_1, E_1)$, $(G/F)/F'_{G/F} =: (V_2, E_2)$ and $G/(F \cup F'_G) = (V_3, E_3)$. Furthermore, let $\eta_1 : E_1 \to E$, $\eta_2 : E_2 \to E_1$ and $\eta_3 : E_3 \to E$ be the bijections as in Definition 2.23. Now we define $\Gamma_V : V_2 \to V_3$ by

$$\Gamma_V(v) := \bigcup_{x \in v} x \ , \ v \in V_2$$

and $\Gamma_E : E_2 \to E_3$ by

$$\Gamma_E := \eta_3^{-1} \circ \eta_1 \circ \eta_2.$$

We claim that $\Gamma := (\Gamma_V, \Gamma_E)$ is an isomorphism $(G/F)/F'_{G/F} \to G/(F \cup F'_G)$.

1. Γ_V is well-defined, i.e. $\Gamma_V(v) \in V_3$ for all $v \in V_2$:
   For $v \in V_2$, we have $v \subseteq V_1 \subseteq 2^V$, i.e. $\Gamma_V(v) \subseteq V$. Fix $y_0 \in \Gamma_V(v)$. We have to show that

   $$\Gamma_V(v) = \{y \in V \mid \text{There exists a path in } F \cup F'_G \text{ from } y_0 \text{ to } y\} \in V_3.$$

   Let $y \in \Gamma_V(v)$. By definition of Γ_V, there exist $x_0, x \in v$ such that $y_0 \in x_0$ and $y \in x$. By definition of $v \in V_2$, there exists a path in $F'_{G/F}$ from x_0 to x in G/F using only edges in $F'_{G/F}$. Since all vertices in G/F are sets of vertices of G that are connected by edges in F, we can find a path in G from $y_0 \in x_0$ to $y \in x$ using only edges in $F \cup F'_G$. This shows "$\subseteq$".
   Now let $y \in V$ such that there exists a path

   $$y_0 \overset{e_1}{\to} y_1 \overset{e_2}{\to} \dots \overset{e_n}{\to} y_n = y$$

   in $F \cup F'_G$ from y_0 to y. Furthermore, let $1 \leq i_1 < \dots < i_m \leq n$ be the indices of the edges in $F'_G \setminus F$ and $x_i \in V_1$ be the vertices in G/F such that $y_i \in x_i$. Then $x_0 \in v$ because $y_0 \in x_0$, $x_0 = x_{i_1 - 1}$ and $x_{i_m} = x_n$. Thus,

   $$x_0 \overset{e_{i_1}}{\to} x_{i_1} \overset{e_{i_2}}{\to} \dots \overset{e_{i_m}}{\to} x_{i_m} = x_n$$

   is a path in G/F from x_0 to x_n using only edges in $F'_{G/F}$. By definition of $v \in V_2$, this implies $x_n \in v$. Hence,

   $$y = y_n \in x_n \subseteq \bigcup_{x \in v} x = \Gamma_V(v).$$

2. Γ_V is surjective:
   Let $v_3 \in V_3$ and $y_0 \in v_3$. Then

   $$v_3 = \{y \in V \mid \text{There exists a path in } F \cup F'_G \text{ from } y_0 \text{ to } y\}.$$

By definition of V_1 and V_2, there exist $v_1 \in V_1$, $v_2 \in V_2$ such that $y_0 \in v_1$ and $v_1 \in v_2$. By 1, we have

$$\Gamma_V(v_2) = \{y \in V \mid \text{There exists a path in } F \cup F'_G \text{ from } y_0 \text{ to } y\} = v_3.$$

3. Γ_V is injective: Let $v, v' \in V_2$ such that

$$\Gamma_V(v) = \bigcup_{x \in v} x = \bigcup_{x' \in v'} x' = \Gamma_V(v').$$

Since $\Gamma_V(v) \neq \emptyset$, there exist $x \in v$, $x' \in v'$ such that $x \cap x' \neq \emptyset$. Since V_1 is a partition of V and $x, x' \in V_1$, this implies $x = x'$. Hence, $v \cap v' \neq \emptyset$. Again, V_2 is a partition of V_1, which implies $v = v'$.
4. Γ_E is a bijection since η_i are bijections. Furthermore, $\Gamma_E(\widehat{e}) = \widehat{\Gamma_E(e)}$ holds for all $e \in E_2$ since $\eta_i(\widehat{e}) = \widehat{\eta_i(e)}$ holds for all $e \in E_i$, $i = 1, 2, 3$.
5. $\underline{\Gamma_E(e)} = \Gamma_V(\underline{e})$ for all $e \in E_2$:
By Definition 2.23, we have $\underline{\eta_i(f)} \in \underline{f}$ for all $f \in E_i$, $i = 1, 2, 3$. Let $e \in E_2$ and $v := \underline{\eta_3(\Gamma_E(e))}$. Hence, $v \in \underline{\Gamma_E(e)}$. On the other hand, we have

$$v = \underline{\eta_1(\eta_2(e))} \in \underline{\eta_2(e)} \in \underline{e}$$

which implies

$$v \in \underline{\eta_2(e)} \subseteq \bigcup_{x \in \underline{e}} x = \Gamma_V(\underline{e}).$$

Thus, $\underline{\Gamma_E(e)} \cap \Gamma_V(\underline{e}) \neq \emptyset$ and $\underline{\Gamma_E(e)}, \Gamma_V(\underline{e}) \in V_3$. Since V_3 is a partition of V, this implies $\underline{\Gamma_E(e)} = \Gamma_V(\underline{e})$.

Γ is a graph homomorphism (1, 4 and 5), and both Γ_V and Γ_E are bijective (2, 3 and 4). Hence, Γ is a graph isomorphism $(G/F)/F'_{G/F} \to G/(F \cup F'_G)$. ☐

Lemma 2.27. *Let G be a finite graph and $F \subseteq E$, $\widehat{F} = F$, a set of directed edges that does not contain any cycles. Furthermore, let $ST_F(G) := \{T \in ST(G) \mid F \subseteq T\}$. Then the following statements hold.*

1. *$T_G \in ST_F(G) \Rightarrow T_{G/F} \setminus F \in ST(G/F)$.*
2. *$T_{G/F} \in ST(G/F) \Rightarrow T_G \cup F \in ST_F(G)$.*

Proof.

1. Let T_G be a spanning tree of G containing F. T_G is connected and spanning in G and every $f \in F$ is a loop in $T_{G/F}$. Hence, $T_{G/F} \setminus F$ is connected and spanning in G/F. Suppose there is a cycle in $T_{G/F} \setminus F$. Considering the associated edges in T_G and possibly adding edges of F to it will produce a cycle in T_G which is impossible. Hence, there are no cycles in $T_{G/F} \setminus F$.

2. Let $T_{G/F}$ be a spanning tree of G/F. Then $T_G \cup F$ is connected and spanning in G. Every possible cycle in $T_G \cup F$ corresponds to a cycle in $T_{G/F}$ which is impossible. Hence, there are no cycles in $T_G \cup F$. □

Corollary 2.28. *Let (G, C) be a finite network and $F \subseteq E$, $\widehat{F} = F$, a set of directed edges in G that does not contain any cycles. The mapping*

$$\varphi_F : ST_F(G) \to ST(G/F)$$

defined by $\varphi_F(T_G) = T_{G/F} \setminus F$ is a bijection satisfying

$$\mathrm{weight}_C(\varphi_F(T_G)) \cdot \mathrm{weight}_C(F) = \mathrm{weight}_C(T_G).$$

Proof. $\varphi^{-1} : ST(G/F) \to ST_F(G)$, $\varphi_F^{-1}(T_{G/F}) := T_G \cup F$ is by Lemma 2.27 the inverse mapping to φ_F. Hence, φ_F is a bijection. Furthermore by Lemma 2.16, we have for any $T \in ST_F(G)$:

$$\begin{aligned} \mathrm{weight}_C(\varphi_F(T_G)) \cdot \mathrm{weight}_C(F) &= \mathrm{weight}_C(T_{G/F} \setminus F) \cdot \mathrm{weight}_C(F) \\ &= \mathrm{weight}_C(T_G \cup F) \overset{F \subseteq T_G}{=} \mathrm{weight}_C(T_G). \end{aligned}$$ □

3 Probability theory on finite networks

At the end of Section 2.1, we have defined networks, which are connected graphs equipped with a function that assigns each of the given graph a positive weight. In this chapter, we will investigate two probabilistic objects that one can define on networks, namely spanning tree measures and random walks.

3.1 Spanning tree measures

Definition 3.1 (Spanning tree measure)**.** Let $\mathcal{N} = (G, C)$ be a finite network. Let $\mu_{\mathcal{N}} : 2^{ST(G)} \to \mathbb{R}_{\geq 0}$ be the unique probability measure on $(ST(G), 2^{ST(G)})$ such that $\mu_{\mathcal{N}}(T) := \mu_{\mathcal{N}}(\{T\})$ is proportional to $\mathrm{weight}_C(T)$ for all $T \in ST(G)$. We call $\mu_{\mathcal{N}}$ the *spanning tree measure* of $\mathcal{N}$.

Remarks.

1. For $A \subseteq E$ and $e \in E$, we will often use the notation

$$\mu_{\mathcal{N}}[A \subseteq T] := \mu_{\mathcal{N}}(\{T \in ST(G) \mid A \subseteq T\})$$

and

$$\mu_{\mathcal{N}}[e \in T] := \mu_{\mathcal{N}}(\{T \in ST(G) \mid e \in T\}).$$

2. Sometimes we will consider $\mu_{\mathcal{N}}$ to be a probability measure on 2^E that is carried by the set $\{E_1, \ldots, E_n\}$ where $ST(G) = \{(V, E_1), \ldots, (V, E_n)\}$.

Example 3.2. Let G be any finite graph and $C \equiv 1$. Then

$$\mu_{\mathcal{N}}(T) = \mu_{\mathcal{N}}(T') \quad \forall\, T, T' \in ST(G),$$

i.e.

$$\mu_{\mathcal{N}} = \frac{1}{|ST(G)|} \sum_{T \in ST(G)} \delta_{\{T\}} \;.$$

$\mu_{\mathcal{N}}$ is called the *uniform spanning tree measure (UST)*.

Lemma 3.3.
1. *If $\mathcal{N}_1 = (G, C_1)$ and $\mathcal{N}_2 = (G, C_2)$ are two networks such that $C_1 = \lambda \cdot C_2$ for some $\lambda \in \mathbb{R}_{\geq 0}$, then $\mu_{\mathcal{N}_1} = \mu_{\mathcal{N}_2}$.*
2. *For any network $\mathcal{N} = (G, C)$, there exists a network $\widetilde{\mathcal{N}}(G, \widetilde{C})$ such that*

$$\mu_{\mathcal{N}}(T) = \mu_{\widetilde{\mathcal{N}}}(T) = \text{weight}_{\widetilde{C}}(T) \quad \forall\, T \in ST(G).$$

Proof.
1. For any $T \in ST(G)$, we have $|T| = |V| - 1$ and therefore

$$\text{weight}_{C_1}(T) = \lambda^{|V|-1} \cdot \text{weight}_{C_2}(T).$$

Thus for $T_0 \in ST(G)$, we compute

$$\begin{aligned}\mu_{\mathcal{N}_1}(T_0) &= \frac{\text{weight}_{C_1}(T_0)}{\sum_{T \in ST(G)} \text{weight}_{C_1}(T_0)} = \frac{\lambda^{|V|-1} \cdot \text{weight}_{C_2}(T_0)}{\sum_{T \in ST(G)} \lambda^{|V|-1} \cdot \text{weight}_{C_2}(T_0)} \\ &= \frac{\text{weight}_{C_2}(T_0)}{\sum_{T \in ST(G)} \text{weight}_{C_2}(T_0)} = \mu_{\mathcal{N}_2}(T_0).\end{aligned}$$

2. Set $K := \sum_{T \in ST(G)} \text{weight}_C(T)$ and define

$$\widetilde{C}(e) := K^{\frac{1}{1-|V|}} \cdot C(e).$$

By statement 1, we have $\mu_{\widetilde{\mathcal{N}}} = \mu_{\mathcal{N}}$ and thus

$$\begin{aligned}\mu_{\widetilde{\mathcal{N}}}(T_0) = \mu_{\mathcal{N}}(T_0) &= \frac{\text{weight}_C(T_0)}{K} = \frac{(K^{\frac{1}{|V|-1}})^{|V|-1} \text{weight}_{\widetilde{C}}(T_0)}{K} \\ &= \text{weight}_{\widetilde{C}}(T_0)\end{aligned}$$

for all $T_0 \in ST(G)$. ∎

Proposition 3.4 (BLPS [1]). *Let (G, C) be a finite network and $F \subseteq E$, $\widehat{F} = F$, a set of directed edges in G containing no cycles. Then*

$$\mu_{\mathcal{N}}[A \subseteq T_G \mid F \subseteq T_G] = \mu_{\mathcal{N}/F}[A \setminus F \subseteq T_{G/F}]$$

holds for every set $A \subseteq E$ of directed edges.

Proof. By Corollary 2.28, we have for every $T \in ST_F(G)$

$$\begin{aligned} A \setminus F \subseteq \varphi_F(T_G) &\Leftrightarrow A \setminus F \subseteq T_{G/F} \setminus F \\ &\Leftrightarrow A \subseteq T_{G/F} \cup F \\ &\Leftrightarrow A \subseteq T_G. \end{aligned}$$

Using this, we compute

$$\begin{aligned} \mu_{\mathcal{N}/F}[A \setminus F \subseteq T_{G/F}] &= \frac{\sum_{\substack{T \in ST(G/F) \\ A \setminus F \subseteq T}} \mathrm{weight}(T)}{\sum_{T \in ST(G/F)} \mathrm{weight}(T)} \overset{2.28}{=} \frac{\sum_{\substack{T \in ST_F(G) \\ A \setminus F \subseteq \varphi_F(T)}} \mathrm{weight}(\varphi_F(T))}{\sum_{T \in ST_F(G)} \mathrm{weight}(\varphi_F(T))} \\ &= \frac{\sum_{\substack{T \in ST_F(G) \\ A \subseteq T}} \mathrm{weight}(T)}{\sum_{T \in ST_F(G)} \mathrm{weight}(T)} = \frac{\mu_{\mathcal{N}}[A \subseteq T_G \wedge F \subseteq T_G]}{\mu_{\mathcal{N}}[F \subseteq T_G]} \\ &= \mu_{\mathcal{N}}[A \subseteq T_G \mid F \subseteq T_G]. \end{aligned}$$

□

3.2 Random walks on networks

Throughout this chapter let $\mathcal{N} = (V, E, C) = (G, C)$ be a locally finite, connected network.

An intuitive way to interpret a network is to consider edges as streets and vertices as crossroads of these streets. The weight $C(e)$ can then be interpreted as the width of street e. Now imagine a person being put on this network of streets at some vertex $v \in V$. At each vertex the person randomly chooses one of the adjacent streets to use next and travels in this fashion from vertex to vertex. To further adapt the network's structure we will assume the probability with which the next street is chosen to be proportional to the street's width, i.e. the edges' weight.

This *random walk* on the given network can be modelled as a stochastic process, more precisely as a Markov chain. This is quite intuitive: simply put, the main characteristic of a Markov chain is its lack of memory. The next chosen step only depends on the current state and is independent of all past steps. In our example, the probability of choosing a certain street only depends on the place we are in and how wide the streets we can see are.

For a quick introduction into the theory of Markov chains, we recommend [7].

Definition 3.5 (Markov chain)**.** Let $(\Omega, \mathcal{F})$ be a measurable space, $\mathbb{P}$ a distribution on $(\Omega, \mathcal{F})$ and S a countable set. Furthermore, let λ be a distribution on $(S, 2^S)$ and $P \in M(S \times S)$ a stochastic[2] matrix. A family $X = (X_i)_{i \in \mathbb{N}}$ of measurable mappings $X_i : (\Omega, \mathcal{F}) \to (S, 2^S)$ is called a $\mathbb{P}$-*Markov chain* with *initial distribution* λ and *transition matrix* P if

$$\mathbb{P}[X_1 = s] = \lambda(\{s\}) \quad \forall\, s \in S$$

and

$$\mathbb{P}[X_{n+1} = s_{n+1} | X_n = s_n, \dots, X_1 = s_1] = \mathbb{P}[X_{n+1} = s_{n+1} | X_n = s_n] = P_{s_{n+1} s_n}$$

holds for all $s_1, \dots, s_{n+1} \in S$ such that $\mathbb{P}[X_n = s_n, \dots, X_1 = s_1] > 0$. We call S the *state space* of X.

Lemma 3.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a measure space and X a $\mathbb{P}$-Markov chain with state space S. Then*

$$\mathbb{P}[X_{n+1} = s_{n+1} | X_n = s_n, X_{n-1} \in A_{n-1}, \dots, X_1 \in A_1] = \mathbb{P}[X_2 = s_{n+1} | X_1 = s_n]$$

holds for all $A_{n-1}, \dots, A_1 \subseteq S, s_{n+1}, s_n \in S$.

Proof. Let $K := \mathbb{P}[X_n = s_n, X_{n-1} \in A_{n-1}, \dots, X_1 \in A_1]$. Then the following holds

$$\begin{aligned}
& \mathbb{P}[X_{n+1} = s_{n+1} | X_n = s_n, X_{n-1} \in A_{n-1}, \dots, X_1 \in A_1] \\
= \;& \mathbb{P}[X_{n+1} = s_{n+1}, X_n = s_n, X_{n-1} \in A_{n-1}, \dots, X_1 \in A_1] / K \\
= \;& \frac{1}{K} \cdot \sum_{\substack{s_i \in A_i \\ 1 \le i \le n-1}} \mathbb{P}[X_{n+1} = s_{n+1}, \dots, X_1 = s_1] \\
= \;& \frac{1}{K} \cdot \sum_{\substack{s_i \in A_i \\ 1 \le i \le n-1}} \mathbb{P}[X_{n+1} | X_n = s_n, \dots, X_1 = s_1] \cdot \mathbb{P}[X_n = s_n, \dots, X_1 = s_1] \\
= \;& \frac{1}{K} \cdot \mathbb{P}[X_2 = s_{n+1} | X_1 = s_n] \cdot \sum_{\substack{s_i \in A_i \\ 1 \le i \le n-1}} \mathbb{P}[X_n = s_n, \dots, X_1 = s_1] \\
= \;& \frac{1}{K} \cdot \mathbb{P}[X_2 = s_{n+1} | X_1 = s_n] \cdot \mathbb{P}[X_n = s_n, X_{n-1} \in A_{n-1}, \dots, X_1 \in A_1] \\
= \;& \mathbb{P}[X_2 = s_{n+1} | X_1 = s_n].
\end{aligned}$$

□

[2]A matrix $P \in M(S \times S)$ is called *stochastic* if $\sum_{t \in S} P_{st} = 1$ for all $s \in S$.

Definition 3.7. For vertices $v, w \in V$ and an edge $e \in E$ we define

1. $C(v,w) := \sum_{e \in E(v,w)} C(e)$.
2. $C_v := \sum_{e \in E(v)} C(e)$.
3. $p(e) := C(e)/C_{\underline{e}}$.
4. $p(v,w) := C(v,w)/C_v$.

Remark. Note that $p(e) \in (0,1]$ and $p(v,w) \in [0,1]$ for all $e \in E, v, w \in V$. Furthermore,

$$p(v,w) = \frac{C(v,w)}{C_v} = \sum_{e \in E(v,w)} \frac{C(e)}{C_v} = \sum_{e \in E(v,w)} p(e)$$

holds.

Proposition 3.8. *Let $\Omega_E = E^{\mathbb{N}}$ and μ be any distribution on $(V, 2^V)$. Then there exists a unique distribution $\mathbb{P}_\mu$ on $(\Omega_E, 2^{\Omega_E})$ such that*

$$\mathbb{P}_\mu(A_1 \times \ldots \times A_n \times E^{\mathbb{N}}) := \sum_{\substack{(e_1,\ldots,e_n) \\ \in A_1 \times \ldots \times A_n}} \mu(\{\underline{e_1}\}) \cdot p(e_1) \cdot \prod_{i=2}^{n} \mathbb{1}_{E(\overline{e_{i-1}})}(e_i) \cdot p(e_i)$$

for all $n \in \mathbb{N}$, $A_1, \ldots, A_n \subseteq E$.

Proof. The essential step of the proof uses a corollary of Caratheodory's extension theorem (see [8], p. 297-298, Proposition 9). It is not complicated to check that

$$\mathcal{C} := \left\{ A_1 \times \ldots \times A_n \times E^{\mathbb{N}} \mid n \in \mathbb{N},\ A_i \subseteq E \right\}$$

is a semialgebra of sets. In fact, $\emptyset \in \mathcal{C}$ and $A \cap B \in \mathcal{C}$ for all $A, B \in \mathcal{C}$ is trivial. What is left to be shown is that for $A, B \in \mathcal{C}$ there exist $n \in \mathbb{N}$ and disjoint sets $C_1, \ldots, C_n \in \mathcal{C}$ such that

$$A \setminus B = \bigcup_{i=1}^{n} C_i.$$

Let $A = A_1 \times \ldots \times A_l \times E^{\mathbb{N}}$ and $B = B_1 \times \ldots \times B_m \times E^{\mathbb{N}}$. Without loss of generality, we may assume $l = m$ since we can let $A_i = E$ or $B_i = E$ for some i. For any sets D_1, D_2, E_1, E_2, the formula

$$(D_1 \times D_2) \setminus (E_1 \times E_2) = ((D_1 \setminus E_1) \times D_2) \cup ((D_1 \cap E_1) \times (D_2 \setminus E_2))$$

holds. By induction we get

$$(A_1 \times \ldots \times A_m) \setminus (B_1 \times \ldots \times B_m) = \bigcup_{i=1}^{m} \underbrace{\left[\prod_{j=1}^{i-1} (A_j \cap B_j) \times (A_i \setminus B_i) \times \prod_{j=i+1}^{n} A_i \right]}_{=:F_i}.$$

Hence,

$$\begin{aligned} A \setminus B &= (A_1 \times \ldots \times A_m \times E^{\mathbb{N}}) \setminus (B_1 \times \ldots \times B_m \times E^{\mathbb{N}}) \\ &= (A_1 \times \ldots \times A_m \setminus (B_1 \times \ldots \times B_m)) \times E^{\mathbb{N}} \\ &= (F_1 \cup \ldots \cup F_m) \times E^{\mathbb{N}} = \bigcup_{i=1}^{m} F_i \times E^{\mathbb{N}}. \end{aligned}$$

Since $F_i \times E^{\mathbb{N}} \in \mathcal{C}$, we have proven that $\mathcal{C}$ is a semialgebra of sets. We define $\mathbb{P}_\mu$ on $\mathcal{C}$ as desired. Now let $A_i \in \mathcal{C}$ be a sequence of pairwise disjoint sets such that $\bigcup_{i=1}^{\infty} A_i \in \mathcal{C}$. By definition of $\mathbb{P}_\mu$, the equality

$$\mathbb{P}_\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}_\mu(A_i)$$

holds. Hence, $\mathbb{P}_\mu$ has a unique extension to a measure on the algebra $\mathcal{A}(\mathcal{C})$ generated by $\mathcal{C}$. By Caratheodory's extension theorem (see [8], p. 295, Theorem 8), this measure can in turn be uniquely extended to a measure on $\sigma(\mathcal{A}(\mathcal{C}))$.

Let $A = A_1 \times A_2 \times \ldots$ be any subset of Ω_E. Then $C_k := A_1 \times \ldots \times A_k \times E^{\mathbb{N}} \in \mathcal{C}$ for all $k \in \mathbb{N}$. Hence, $C_k \in \sigma(\mathcal{A}(\mathcal{C}))$ and

$$A = \bigcap_{k \in \mathbb{N}} C_k \in \sigma(\mathcal{A}(\mathcal{C})).$$

It follows that $\sigma(\mathcal{A}(\mathcal{C})) = 2^{\Omega_E}$ and $\mathbb{P}_\mu$ is a measure on $(\Omega_E, 2^{\Omega_E})$ with mass 1, i.e. $\mathbb{P}_\mu$ is a distribution. □

Definition 3.9. Let $\Omega_E := E^{\mathbb{N}}$. We define

1. $X_k : \Omega_E \to E$ such that $X_k(e_1, e_2, \ldots) = e_k$ for $k \in \mathbb{N}$.
2. $Y_k : \Omega_E \to V$ such that $Y_k(e_1, e_2, \ldots) = \overline{e_k}$ for $k \in \mathbb{N}$ and $Y_0(e_1, e_2, \ldots) = \underline{e_1}$.

Remark. Note that X_k is only defined for $k \geq 1$ while Y_k is defined for $k \geq 0$.

Lemma 3.10. *Let $w \in V$ be a fixed vertex and μ be a distribution on V. Then*

1. $\mathbb{P}_\mu[Y_0 = w] = \mu(\{w\})$.
2. *If $\mu(\{w\}) \neq 0$, then $\mathbb{P}_\mu[A \mid Y_0 = w] = \mathbb{P}_{\delta_w}(A)$ for all $A \subseteq \Omega_E$.*

Proof.

1. We have $\{Y_0 = w\} = E(w) \times E^{\mathbb{N}}$, i.e.

$$\begin{aligned} \mathbb{P}_\mu[Y_0 = w] = \mathbb{P}_\mu(E(w) \times E^{\mathbb{N}}) &= \sum_{e \in E(w)} \mu(\{\underline{e}\}) \cdot p(e) \\ &= \sum_{e \in E(w)} \mu(\{w\}) \cdot \frac{C(e)}{\sum_{f \in E(w)} C(f)} = \mu(\{w\}). \end{aligned}$$

2. It suffices to show the equality for $A = A_1 \times \ldots \times A_n \times E^{\mathbb{N}}$ for some $n \in \mathbb{N}$ and $A_1, \ldots, A_n \subseteq E$. The equality for more general events follows by Caratheodory's extension theorem analogously to Proposition 3.8. For this simple case, we compute

$$\begin{aligned}
\mathbb{P}_\mu[A \mid Y_0 = w] &= \frac{1}{\mu(\{w\})} \cdot \mathbb{P}_\mu((A_1 \cap E(w)) \times A_2 \times \ldots \times A_n \times E^{\mathbb{N}}) \\
&= \sum_{\substack{(e_1,\ldots,e_n) \\ \in A_1 \times \ldots \times A_n}} \mathbb{1}_w(\underline{e_1}) \cdot \frac{\mu(\{\underline{e_1}\})}{\mu(\{w\})} \cdot p(e_1) \cdot \prod_{i=2}^{n} \mathbb{1}_{E(\overline{e_{i-1}})}(e_i) \cdot p(e_i) \\
&= \sum_{\substack{(e_1,\ldots,e_n) \\ \in A_1 \times \ldots \times A_n}} \delta_w(\{\underline{e_1}\}) \cdot p(e_1) \cdot \prod_{i=2}^{n} \mathbb{1}_{E(\overline{e_{i-1}})}(e_i) \cdot p(e_i) \\
&= \mathbb{P}_{\delta_w}(A).
\end{aligned}$$

□

Definition 3.11. For $v \in V$, we define $\mathbb{P}_v := \mathbb{P}_{\delta_v}$ and for any measurable $X : \Omega_E \to \mathbb{R}$

$$\mathbb{E}_v[X] := \int_{\Omega_E} X \, \mathrm{d}\mathbb{P}_v.$$

Lemma 3.12. *Let μ be a distribution on V and $v_0, \ldots, v_{n+1} \in V$, $e_1, \ldots, e_{n+1} \in E$. Then the following equalities hold:*

1. $\mathbb{P}_\mu[X_1 = e_1, \ldots, X_n = e_n] = \mu(\{\underline{e_1}\}) \cdot p(e_1) \cdot \prod_{i=2}^{n} \mathbb{1}_{E(\overline{e_{i-1}})}(e_i) p(e_i)$
2. *If* $\mathbb{P}_\mu[X_1 = e_1, \ldots, X_n = e_n] > 0$, *then*
$$\mathbb{P}_\mu[X_{n+1} = e_{n+1} \mid X_n = e_n, \ldots, X_1 = e_1] = \mathbb{1}_{E(\overline{e_n})}(e_{n+1}) p(e_{n+1}).$$
3. $\mathbb{P}_\mu[Y_0 = v_0, \ldots, Y_n = v_n] = \mu(\{v_0\}) \cdot \prod_{i=1}^{n} p(v_{i-1}, v_i)$.
4. *If* $\mathbb{P}_\mu[Y_0 = v_0, \ldots, Y_n = v_n] > 0$, *then*
$$\mathbb{P}_\mu[Y_{n+1} = v_{n+1} \mid Y_n = v_n, \ldots, Y_0 = v_0] = p(v_n, v_{n+1}).$$
5. *Let* $e \in E$, $v \in V$, $n \in \mathbb{N}$. *If* $\mathbb{P}_\mu[Y_n = v] > 0$, *then*
$$\mathbb{P}_\mu[X_{n+1} = e \mid Y_n = v] = \mathbb{1}_v(\underline{e}) \cdot p(e).$$

Proof. The equations 1-4 are easy calculations using the definition of $\mathbb{P}_\mu$ (Proposition 3.8) and the fact that

$$\{X_1 = e_1, \ldots, X_n = e_n\} = \{e_1\} \times \ldots \times \{e_n\} \times E^{\mathbb{N}}$$

and

$$\{Y_0 = v_0, \dots, Y_n = v_n\} = E(v_0, v_1) \times \ldots \times E(v_{n-1}, v_n) \times E^{\mathbb{N}}.$$

Regarding 5: Let $E(\cdot, v) := \bigcup_{x\in V} E(x, v)$ be the set of edges leading towards v. Then we get

$$\begin{aligned}
\mathbb{P}_\mu[X_{n+1} = e, Y_n = v] &= \mathbb{P}_\mu(E^{n-1} \times E(\cdot, v) \times \{e\} \times E^{\mathbb{N}}) \\
&= \sum_{\substack{e_1,\dots,e_{n-1}\in E \\ e_n\in E(\cdot,v)}} \mu(\{\underline{e_1}\})p(e_1) \prod_{i=2}^{n} \mathbb{1}_{E(\overline{e_{i-1}})}(e_i)p(e_i) \cdot \mathbb{1}_{E(v)}(e)p(e) \\
&= \mathbb{1}_{E(v)}(e)p(e) \cdot \sum_{\substack{e_1,\dots,e_{n-1}\in E \\ e_n\in E(\cdot,v)}} \mu(\{\underline{e_1}\})p(e_1) \prod_{i=2}^{n} \mathbb{1}_{E(\overline{e_{i-1}})}(e_i)p(e_i) \\
&= \mathbb{1}_{E(v)}(e)p(e) \cdot \mathbb{P}_\mu(E^{n-1} \times E(\cdot, v) \times E^{\mathbb{N}}) \\
&= \mathbb{1}_{E(v)}(e)p(e) \cdot \mathbb{P}_\mu[Y_n = v].
\end{aligned}$$

Hence, $\mathbb{P}_\mu[X_{n+1} = e \mid Y_n = v] = \mathbb{1}_{E(v)}(e)p(e) = \mathbb{1}_v(\underline{e})p(e)$. □

Definition 3.13 (Random network walk)**.** Let X_k and Y_k be as in Defintion 3.9. We call the discrete-time processes $X := (X_k)_{k\in\mathbb{N}}$ and $Y := (Y_k)_{k\in\mathbb{N}_0}$ *random network walks (RNW).*

Remarks.

1. Note that X is a stochastic process with values in E while Y is the projection of X onto V.
2. For any $n \in \mathbb{N}$, $\underline{X_{n+1}} = \overline{X_n}$ holds almost surely because

$$\begin{aligned}
\mathbb{P}_v[\underline{X_{n+1}} = \overline{X_n}] &= \sum_{w\in V} \mathbb{P}_v[\underline{X_{n+1}} = w \mid \overline{X_n} = w] \cdot \mathbb{P}_v[\overline{X_n} = w] \\
&= \sum_{w\in V} \mathbb{P}_v[Y_n = w] \cdot \sum_{e\in E(w)} \mathbb{P}_v[X_{n+1} = e \mid Y_n = w] \\
&= \sum_{w\in V} \mathbb{P}_v[Y_n = w] \cdot \sum_{e\in E(w)} \mathbb{1}_w(\underline{e}) \cdot p(e) \\
&= \sum_{w\in V} \mathbb{P}_v[Y_n = w] \cdot 1 = 1.
\end{aligned}$$

Proposition 3.14. *Let μ be a distribution on V and define μ_E by*

$$\mu_E(\{e\}) := \mu(\{\underline{e}\}) \cdot p(e) \ , \ e \in E.$$

Then the following hold.

1. X is a $\mathbb{P}_\mu$-Markov chain with initial distribution μ_E and transition matrix $P^E = (\mathbb{1}_{E(\overline{e})}(\underline{f}) \cdot p(f))_{e,f\in E}$.

2. Y is a $\mathbb{P}_\mu$-Markov chain with initial distribution μ and transition matrix $P^V = (p(u,v))_{u,v\in V}$.

Proof.

1. We have to show that for any $e_1, \dots, e_{n+1} \in E$

$$\mathbb{P}_\mu[X_{n+1} = e_{n+1} \mid X_n = e_n, \dots, X_1 = e_1] = \mathbb{P}_\mu[X_{n+1} = e_{n+1} \mid X_n = e_n]$$

holds. By Lemma 3.12, the left side of the equation equals $\mathbb{1}_{E(\overline{e_n})}(e_{n+1})p(e_{n+1})$. On the other hand, we have

$$\begin{aligned}\mathbb{P}_\mu[X_{n+1} = e_{n+1} \mid X_n^v = e_n] &= \frac{\mathbb{P}_\mu[X_{n+1} = e_{n+1}, X_n^v = e_n]}{\mathbb{P}_\mu[X_n^v = e_n]} \\ &= \frac{\mathbb{P}_\mu\left(E^{n-1} \times \{e_n\} \times \{e_{n+1}\} \times E^{\mathbb{N}}\right)}{\mathbb{P}_\mu\left(E^{n-1} \times \{e_n\} \times E^{\mathbb{N}}\right)} \\ &= \frac{\mathbb{P}_\mu\left(E^{n-1} \times \{e_n\} \times E^{\mathbb{N}}\right) \cdot \mathbb{1}_{E(\overline{e_n})}(e_{n+1}) \cdot p(e_{n+1})}{\mathbb{P}_\mu\left(E^{n-1} \times \{e_n\} \times E^{\mathbb{N}}\right)} \\ &= \mathbb{1}_{E(\overline{e_n})}(e_{n+1}) \cdot p(e_{n+1}).\end{aligned}$$

This proves that X is a P_μ-Markov chain with transition matrix P^E. Its initial distribution is given by

$$\mathbb{P}_\mu[X_1 = e] = \mathbb{P}_\mu(\{e\} \times E^{\mathbb{N}}) = \mu(\{\underline{e}\}) \cdot p(e) = \mu_E(\{e\}).$$

2. Analogously to 1, we have to show that

$$\mathbb{P}_\mu[Y_{n+1} = v_{n+1} \mid Y_n = v_n, \dots, Y_0 = v_0] = \mathbb{P}_\mu[Y_{n+1} = v_{n+1} \mid Y_n = v_n].$$

By Lemma 3.12, the left side equals $p(v_n, v_{n+1})$. Computing the right side of the equation gives

$$\begin{aligned}\mathbb{P}_\mu[Y_{n+1} = v_{n+1} \mid Y_n = v_n] &= \sum_{e\in E(v_n,v_{n+1})} \mathbb{P}_\mu[X_{n+1} = e \mid Y_n = v_n] \\ &= \sum_{e\in E(v_n,v_{n+1})} \mathbb{1}_{v_n}(\underline{e}) \cdot p(e) \\ &= \sum_{e\in E(v_n,v_{n+1})} p(e) = p(v_n, v_{n+1}).\end{aligned}$$

Thus, Y is a $\mathbb{P}_\mu$-Markov Chain with transition matrix P^V. Its initial distribution is given by

$$\mathbb{P}_\mu[Y_0 = v] = \mathbb{P}_\mu(E(v) \times E^{\mathbb{N}}) = \mu(\{v\}).$$

□

Remarks.

1. Both X and Y are irreducible[3] and, if $\mathcal{N}$ is finite, recurrent[4].
2. If we speak of a *RNW starting in a vertex* $v \in V$, we want to consider X (or Y) as a $\mathbb{P}_v$-Markov chain since then $\mathbb{P}_v[Y_0 = v] = 1$.

Definition 3.15 (Hitting time)**.** We define the *first time a RNW hits* A to be the random variable

$$\tau_A := \inf \{n \in \mathbb{N}_0 \mid Y_n \in A\}.$$

For $w \in V$, we set $\tau_w := \tau_{\{w\}}$.

Remark. Note that $\tau_A = \infty$ is not impossible. This just means that Y does not hit A.

Lemma 3.16. *Let $\mathcal{N}$ be finite, $v \in V$ and $w \in V$. Then $\mathbb{P}_v[\tau_w < \infty] = 1$.*

Proof. The statement is trivial for $v = w$. Assume $v \neq w$. Y is irreducible and recurrent both as a $\mathbb{P}_v$-Markov chain and as a $\mathbb{P}_w$-Markov chain, i.e. for $T_w^+ := \inf \{n \in \mathbb{N} \mid Y_n = w\}$, we have

$$\mathbb{P}_w[T_w^+ < \infty] = 1.$$

Now let $n := \inf \{l \geq 1 \mid \mathbb{P}_w[Y_l = v] > 0\}$. We have $n < \infty$ since Y is irreducible. By definition of n, we also get

$$\mathbb{P}_w[Y_n = v] = \mathbb{P}_w[Y_n = v, Y_{n-1} \neq w, \dots, Y_1 \neq w].$$

Hence,

$$\begin{aligned}
0 &= 1 - \mathbb{P}_w[T_w^+ < \infty] = \mathbb{P}_w[T_w^+ = \infty] \\
&\geq \mathbb{P}_w[Y_n = v, Y_k \neq w \ \forall\ k \neq n, k \geq 1] \\
&= \mathbb{P}_w[Y_k \neq w \ \forall\ k > n \mid Y_n = v] \cdot \mathbb{P}_w[Y_n = v, Y_{n-1} \neq w, \dots, Y_1 \neq w] \\
&= \mathbb{P}_v[Y_k \neq w \ \forall\ k > 0] \cdot \mathbb{P}_w[Y_n = v] \\
&= \mathbb{P}_v[\tau_w = \infty] \cdot \mathbb{P}_w[Y_n = v] \geq 0.
\end{aligned}$$

This implies $\mathbb{P}_v[\tau_w = \infty] \cdot \mathbb{P}_w[Y_n = v] = 0$. Thus, $\mathbb{P}_v[\tau_w = \infty] = 0$ and

$$\mathbb{P}_v[\tau_w < \infty] = 1 - \mathbb{P}_v[\tau_w = \infty] = 1.$$

□

[3] A Markov chain X with state-space S is called *irreducible* if $\forall\, s, t \in S\ \exists\, n \in \mathbb{N} : \mathbb{P}_s[X_n = t] > 0$.

[4] A state $s \in S$ of a Markov chain X is called *recurrent* if $\mathbb{P}_s[\inf \{n \geq 2 \mid X_n = s\} < \infty] = 1$. X is called *recurrent* if all states $s \in S$ are recurrent.

Lemma 3.17. *Let $\mathcal{N}$ be finite, $c = (e_1, \dots, e_n)$ be a cycle in $\mathcal{N}$ and $v, w \in V$ such that $\underline{e_i} \neq w$ for all $i = 1, \dots, n$. Then*

$$\mathbb{E}_v \left[\sum_{k=1}^{\tau_w} \mathbb{1}_{(e_1,\dots,e_n)}(X_k, \dots, X_{k+n-1}) \right] = \mathbb{E}_v \left[\sum_{k=1}^{\tau_w} \mathbb{1}_{(\widehat{e_n},\dots,\widehat{e_1})}(X_k, \dots, X_{k+n-1}) \right],$$

i.e. in expectation a RNW from v to w uses the cycle c the same number of times it uses the cycle $\widehat{c}$.

Proof. We have $\underline{e_1} = \overline{e_n} = \underline{\widehat{e_n}}$ because c is a cycle. Now suppose $\mathbb{P}_v[Y_{k-1} = \underline{e_1}] = \mathbb{P}_v[Y_{k-1} = \underline{\widehat{e_n}}] > 0$ for some $k \in \mathbb{N}^+$. It follows that

$$\begin{aligned}
\mathbb{P}_v[(X_k, \dots, X_{k+n-1}) = c \mid Y_{k-1} = \underline{e_1}] &= \mathbb{P}_{\underline{e_1}}[X_1 = e_1, \dots, X_n = e_n] \\
&= \frac{C(e_1)}{C_{\underline{e_1}}} \cdot \ldots \cdot \frac{C(e_n)}{C_{\underline{e_n}}} \\
&= \frac{C(\widehat{e_n})}{C_{\underline{e_1}}} \cdot \frac{C(\widehat{e_{n-1}})}{C_{\underline{e_n}}} \cdot \ldots \cdot \frac{C(\widehat{e_1})}{C_{\underline{e_2}}} \\
&= \frac{C(\widehat{e_n})}{C_{\underline{\widehat{e_n}}}} \cdot \ldots \cdot \frac{C(\widehat{e_1})}{C_{\underline{\widehat{e_1}}}} \\
&= \mathbb{P}_{\underline{\widehat{e_n}}}[X_1 = \widehat{e_n}, \dots, X_n = \widehat{e_1}] \\
&= \mathbb{P}_v[(X_k, \dots, X_{k+n-1}) = \widehat{c} \mid Y_{k-1} = \underline{\widehat{e_n}}].
\end{aligned}$$

Let $Z(c,t) := \sum_{k=1}^{t} \mathbb{1}_c(X_k, \dots, X_{k+n-1})$. Since $\mathbb{P}_v[Z(c, \tau_w) = \infty] \leq \mathbb{P}_v[\tau_w = \infty] = 0$, we have

$$\mathbb{E}_v[Z(c, \tau_w)] = \sum_{m=1}^{\infty} m \cdot \mathbb{P}_v[Z(c, \tau_w) = m] = \sum_{m=1}^{\infty} m \cdot \sum_{t=1}^{\infty} \mathbb{P}_v[Z(c,t) = m \ , \ \tau_w = t].$$

Using that $\mathbb{P}_v[Z(c,t) = m, \ \tau_w = t]$ is just the finite sum of

$$\mathbb{P}_v(\{(f_1, \dots, f_t)\} \times E^{\mathbb{N}}) = \mathbb{P}_v[X_1 = f_1, \dots, X_t = f_t]$$

over all paths $(f_1, \dots, f_t)$ such that $\underline{f_1} = v$, $\overline{f_k} \neq w$ for $k = 1, \dots, t-1$, $\overline{f_t} = w$, and

$$Z(c,t)(\omega) = m \ \forall \ \omega \in \{(f_1, \dots, f_t)\} \times E^{\mathbb{N}},$$

it follows by the above calculation that

$$\mathbb{P}_v[Z(c,t) = m, \ \tau_w = t] = \mathbb{P}_v[Z(\widehat{c},t) = m \ , \ \tau_w = t]$$

for all $m, t \in \mathbb{N}$. Hence, the desired statement follows. □

Definition 3.18 (Expected visits)**.**

1. For $u, v, w \in V$, we define
$$U_w^{uv} := \mathbb{E}_u \left[\sum_{k=0}^{\tau_v - 1} \mathbb{1}_w(Y_k) \right]$$
to be the *expected number of visits of Y to w when starting in u before reaching v.*
2. For $u, v \in V$, $e \in E$, we define
$$J_e^{uv} := \mathbb{E}_u \left[\sum_{k=1}^{\tau_v} \mathbb{1}_e(X_k) \right]$$
to be the *expected number of times X uses e when starting in u and before reaching v.*

Remarks.

1. For $u = v$, we have $\tau_v = 0$ and interpret both sums in Definition 3.18 to be equal to 0.
2. Note that $U_v^{uv} = 0$ for any $v \in V$ (in particular for $u = v$) and $U_u^{uv} \geq 1$ for $u \neq v$.

Lemma 3.19. *Let $u, v \in V$, $u \neq v$ and $w \in V \setminus \{u, v\}$. It follows that*
$$U_w^{uv} = \sum_{x \in V} U_x^{uv} \cdot \frac{C(x,w)}{C_x}$$
and
$$U_u^{uv} = 1 + \sum_{x \in V} U_x^{uv} \cdot \frac{C(x,u)}{C_x} \ .$$

Proof. For any $x, y \in V$, we have
$$\mathbb{P}_u[Y_k = y \mid Y_{k-1} = x] = p(x,y) = \frac{C(x,y)}{C_x}$$
if $\mathbb{P}_u[Y_{k-1} = x] > 0$. Furthermore, $\{k \leq \tau_v\} = \bigcap_{j=0}^{k-1} \{Y_j \in V \setminus \{v\}\}$. By Lemma 3.6, this implies for $x \neq v$
$$\begin{aligned}
\mathbb{P}_u[Y_{k-1} = x, Y_k = y, k \leq \tau_v] &= \mathbb{P}_u[Y_k = y \mid Y_{k-1} = x, k \leq \tau_v] \cdot \mathbb{P}_u[Y_{k-1} = x, k \leq \tau_v] \\
&= \mathbb{P}_u[Y_k = y \mid Y_{k-1} = x] \cdot \mathbb{P}_u[Y_{k-1} = x, k \leq \tau_v] \\
&= \mathbb{P}_u[Y_{k-1} = x, k \leq \tau_v] \cdot \frac{C(x,y)}{C_x} \ .
\end{aligned}$$

Using this and the law of total probability, we calculate for $u \neq w \neq v$

$$\begin{aligned}
U_w^{uv} &= \mathbb{E}_u\left[\sum_{k=0}^{\tau_v-1} \mathbb{1}_w(Y_k)\right] = \mathbb{E}_u\left[\sum_{k=1}^{\tau_v} \mathbb{1}_{\{Y_k=w\}}\right] = \mathbb{E}_u\left[\sum_{k=1}^{\infty} \mathbb{1}_{\{Y_k=w,k\leq\tau_v\}}\right] \\
&= \sum_{k=1}^{\infty} \mathbb{P}_u[Y_k = w, k \leq \tau_v] = \sum_{x\in V}\sum_{k=1}^{\infty} \mathbb{P}_u[Y_{k-1} = x, Y_k = w, k \leq \tau_v] \\
&= \sum_{x\in V} \frac{C(x,w)}{C_x} \sum_{k=1}^{\infty} \mathbb{P}_u[Y_{k-1} = x, k \leq \tau_v] = \sum_{x\in V} \frac{C(x,w)}{C_x} \mathbb{E}_u\left[\sum_{k=1}^{\infty} \mathbb{1}_{\{Y_{k-1}=x,k\leq\tau_v\}}\right] \\
&= \sum_{x\in V} \frac{C(x,w)}{C_x} \mathbb{E}_u\left[\sum_{k=0}^{\tau_v-1} \mathbb{1}_x(Y_k)\right] = \sum_{x\in V} U_x^{uv} \cdot \frac{C(x,w)}{C_x} .
\end{aligned}$$

The second equality is proven analogously

$$\begin{aligned}
U_u^{uv} &= \mathbb{E}_u\left[\sum_{k=0}^{\tau_v-1} \mathbb{1}_u(Y_k)\right] = \sum_{k=0}^{\infty} \mathbb{P}_u[Y_k = u, k \leq \tau_v] \\
&= 1 + \sum_{k=1}^{\infty} \mathbb{P}_u[Y_k = u, k \leq \tau_v] = 1 + \sum_{x\in V} U_x^{uv} \cdot \frac{C(x,u)}{C_x} . \qquad \square
\end{aligned}$$

Lemma 3.20. *Let $u, v \in V$ and $e \in E$. Then it follows that*

$$J_e^{uv} = U_{\underline{e}}^{uv} \cdot p(e).$$

Proof. Analogously to the proof of Lemma 3.19, we have for $x, v \in V$, $e \in E$

$$\begin{aligned}
\mathbb{P}_u[Y_{k-1} = x, X_k = e, k \leq \tau_v] &= \mathbb{P}_u[X_k = e \mid Y_{k-1} = x] \cdot \mathbb{P}_u[Y_{k-1} = x, k \leq \tau_v] \\
&= \mathbb{1}_{\underline{e}}(x) p(e) \cdot \mathbb{P}_u[Y_{k-1} = x, k \leq \tau_v]
\end{aligned}$$

by Lemmas 3.6 and 3.12. Hence,

$$\begin{aligned}
J_e^{uv} &= \mathbb{E}_u\left[\sum_{k=1}^{\tau_v} \mathbb{1}_e(X_k)\right] = \mathbb{E}_u\left[\sum_{k=1}^{\infty} \mathbb{1}_{\{X_k=e,k\leq\tau_v\}}\right] \\
&= \sum_{k=1}^{\infty} \mathbb{P}_u[X_k = e, k \leq \tau_v] = \sum_{k=1}^{\infty}\sum_{x\in V} \mathbb{P}_u[Y_{k-1} = x, X_k = e, k \leq \tau_v] \\
&= \sum_{k=1}^{\infty}\sum_{x\in V} \mathbb{1}_{\underline{e}}(x) p(e) \cdot \mathbb{P}_u[Y_{k-1} = x, k \leq \tau_v] = p(e) \sum_{k=1}^{\infty} \mathbb{P}_u[Y_{k-1} = \underline{e}, k \leq \tau_v] \\
&= p(e) \mathbb{E}_u\left[\sum_{k=1}^{\tau_v} \mathbb{1}_{\{Y_{k-1}=\underline{e}\}}\right] = p(e) \mathbb{E}_u\left[\sum_{k=0}^{\tau_v-1} \mathbb{1}_{\underline{e}}(Y_k)\right] = U_{\underline{e}}^{uv} \cdot p(e). \qquad \square
\end{aligned}$$

3.3 Wilson's Algorithm

Let $\mathcal{N} = (V, E, C) = (G, C)$ be a finite connected network. Using the random network walk on $\mathcal{N}$, Wilson [11] discovered an algorithm that will produce a random spanning tree with distribution $\mu_{\mathcal{N}}$. This result does not only show the strong connection between random network walks and spanning tree measures but will also enable us to establish a connection between spanning tree measures and electrical currents in Chapter 4.

Definition 3.21 (Loop erasure)**.** Let $(e_1, \ldots, e_n)$ be a path in G. We will now define the *loop erasure* $LE(e_1, \ldots, e_n) := (f_1, \ldots, f_m)$.

If $\underline{e_1} = \overline{e_n}$, then $LE(e_1, \ldots, e_n) := ()$, i.e. $m = 0$. Otherwise, define $(f_1, \ldots, f_m)$ inductively:

1. $f_1 := e_j$ where $j = \max\left\{i \mid \underline{e_i} = \underline{e_1}\right\}$.
2. Suppose that f_j has been set. If $\overline{f_j} = \overline{e_n}$, then let $m = j$ and we are finished. Otherwise, let $k = \max\left\{i \mid \underline{e_i} = \overline{f_j}\right\}$ and $f_{j+1} := e_k$.

Remark. There is a way of generating the loop erasure of a path $(e_1, \ldots, e_n)$ 'along the way': While walking along our path, we remember all previously visited vertices (and used edges) and if we hit one of these vertices again, we delete the generalized cycle that led us here the second time from our memory. When we are done, the memorized path is exactly the loop erasure $LE(e_1, \ldots, e_n)$.

Example 3.22. Consider the path shown in Figure 3.1.

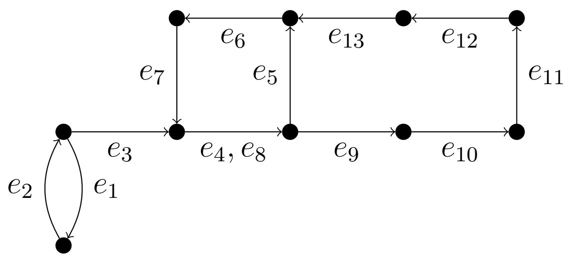

Figure 3.1: The path $(e_1, \ldots, e_{13})$.

Its loop erasure is $LE(e_1, \ldots, e_{13}) = (e_3, e_8, e_9, e_{10}, e_{11}, e_{12}, e_{13})$, which is shown in Figure 3.2.

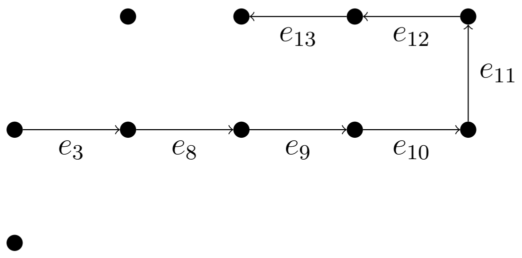

Figure 3.2: The loop erasure $LE(e_1, \ldots, e_{13})$.

Definition 3.23 (Loop erased random walk)**.** Let X be a RNW on $\mathcal{N}$ and $W \subseteq V$. We define

$$LE(X_1, \ldots X_{\tau_W})$$

to be the *loop erased RNW* of $\mathcal{N}$ *stopping as soon as it hits* W.

Remark. By Lemma 3.16, we have $\mathbb{P}[\tau_v < \infty] = \sum_{x \in V} \mathbb{P}_x[\tau_v < \infty] \cdot \mathbb{P}[\underline{X_1} = x] = 1$ for all $v \in V$. Hence, $(X_1, \ldots, X_{\tau_W})$ is almost surely a finite path, and we may obtain its loop erasure.

Definition 3.24 (Rooted spanning tree)**.** Let $G = (V, E)$ be a graph and fix $r \in V$. A *spanning tree rooted at* r is a set of directed edges $T_r \subseteq E$ such that

1. T_r contains no generalized cycles.
2. $|\{e \in T_r \mid \underline{e} = v\}| = 1$ for all $v \in V \setminus \{r\}$.

Remarks.

1. A spanning tree rooted at r is basically a spanning tree of G such that for each $v \in V \setminus \{r\}$, the edges connecting v and r are directed towards r.
2. T_r is a spanning tree rooted at r is if it satisfies the following three conditions.
   (a) $[e \in T_r \Rightarrow \widehat{e} \notin T_r]$ for all $e \in E$.
   (b) $|\{e \in T_r \mid \underline{e} = v\}| = 1$ for all $v \in V \setminus \{r\}$.
   (c) $G(T_r) = (V, \{e, \widehat{e} \mid e \in T_r\}) \in ST(G)$.

   Obviously, any set of directed edges satisfying all three conditions is a spanning tree rooted at r. Suppose T_r is a spanning tree rooted at r. It immediately follows that T_r satisfies condition (a) and (b). Now assume there exists a cycle $(e_1, \ldots, e_n)$ in $G(T_r)$, and let $f_1, \ldots, f_n \in T_r$ such that for all $i = 1, \ldots, n$, we have $f_i = e_i$ or $\widehat{f_i} = e_i$. Since T_r does not contain generalized cycles, it follows that $\underline{f_i} = \underline{f_{i+1}}$ and $\overline{f_j} = \overline{f_{j+1}}$ for some $i \neq j - 1$. Hence, there exists $v \neq r$ such that $|\{e \in T_r \mid \underline{e} = v\}| \in \{0, 2\}$, which is a contradiction. Thus, $G(T_r)$ cannot contain any cycles and satisfies $|G(T_r)| = |V| - 1$. It follows that $G(T_r) \in ST(G)$ by Lemma 2.12.

Lemma 3.25. *Let $r \in V$. There is a natural bijection between the set $RST(G, r)$ of spanning trees rooted at r and the set $ST(G)$ of spanning trees of G.*

Proof. Let $T_r \in RST(G, r)$. Then $\varphi(T_r) := G(T_r)$ is a spanning tree of $ST(G)$. Hence, we obtain $\varphi : RST(G, r) \to ST(G)$. For $T \in ST(G)$ and $v \in V$ let $e_1^v, \ldots, e_{n_v}^v$ be the unique self-avoiding path $v \to r$ in T and define

$$\psi(T) := \left\{e_1^v, \ldots, e_{n_v}^v \mid v \in V \setminus \{r\}\right\}.$$

Then $\psi(\varphi(T_r)) = T_r$ and $\varphi(\psi(T)) = T$. □

Algorithm 3.26 (Wilson's algorithm)**.**
Input: finite network $\mathcal{N} = (V, E, C)$, $|V| = n + 1$, and a root $r \in V$

We inductively define a sequence of increasing subsets $T(i) \subseteq E$:

1. Choose an arbitrary enumeration $(v_1, \ldots, v_n)$ of the vertices in $V \setminus \{r\}$.
2. Set $T(0) := \emptyset$.
3. Suppose that $T(i)$ has been defined. If v_{i+1} already is an endpoint of some edge in $T(i)$ then let $T(i + 1) := T(i)$. Otherwise $T(i)$ is generated by adding the loop erasure of a RNW starting at v_{i+1} and stopping as soon as it hits a vertex of $T(i)$ (or r for $i = 0$).

Output: spanning tree $T(n)$ rooted at r

Remarks.

1. By Lemma 3.16 Wilson's algorithm almost surely terminates.
2. It is easy to verify that Wilson's algorithm will produce a rooted spanning tree with respect to Definition 3.24. By construction of $T(n)$, every vertex $v \in V \setminus \{r\}$ has exactly one outgoing edge in $T(n)$, i.e.

$$\{e \in T(n) \mid \underline{e} = v\} = 1,$$

and it is impossible for the algorithm to produce any generalized cycles.

Theorem 3.27 (Wilson [11], 1996)**.** *Wilson's algorithm will produce a random spanning tree rooted at r with distribution proportional to the tree's weight.*

Proof (Wilson [11]). Let r be the designated root and $(v_1, \ldots, v_n)$ be the enumeration of the remaining vertices in $V \setminus \{r\}$. For every vertex $v \in V \setminus \{r\}$, let $S_v = (S_{v,i})_{i=1}^{\infty}$ be a stack of random edges such that all items in all stacks are mutually independent and such that the probability of $S_{v,i} = e$ is given by $\mathbb{1}_{E(v)}(e) \cdot C(e)/C_v$. Also, let the stack associated with r be empty, i.e. $S_r = ()$. Looking only at the top of each stack, we see a set G_{vis} of $|V| - 1$ directed edges containing exactly one outgoing edge for each $v \in V \setminus \{r\}$. We call this set G_{vis} the *visible graph*.

Suppose there is a generalized cycle in the visible graph. We then *pop* this cycle off the top of the stacks, meaning that we replace $S_x = (S_{x,1}, S_{x,2}, \ldots)$ with $(S_{x,2}, S_{x,3}, \ldots)$ for each vertex x occurring in this generalized cycle. We repeat this cycle-popping process until there are no more generalized cycles in G_{vis}. If this procedure terminates, G_{vis} contains exactly one outgoing edge for each $v \in V \setminus \{r\}$ and no generalized cycles, i.e. $G_{vis} \in RST(G, r)$. We will later see that this will happen almost surely. However, let us first consider the effect the choice of which cycle to pop next will have on the resulting rooted spanning tree.

For any stack item $S_{v,i}$, let i be the *label* of this item. This way, we can define a *labeled cycle* to be a sequence $c = ((e_1, i_1), \ldots, (e_k, i_k))$ such that $(e_1, \ldots, e_k)$ is a generalized cycle and $e_j = S_{\underline{e_j}, i_j}$ for $j = 1, \ldots, k$. During the process of cycle-popping a generalized cycle may be popped more than once, but a labeled cycle can only be popped once. If no more cycles can be popped, the result is a labeled spanning tree rooted at r.

Suppose we have a fixed configuration of stacks. We will now show that every labeled cycle C will either always or never be popped in order to 'reveal' a certain rooted spanning tree: Let C be a labeled cycle that can be popped, i.e. there are labeled cycles $C_1, C_2, \ldots, C_k = C$ that may be popped one after another until C is popped. Now suppose that the first labeled cycle to be popped is not C_1 but some C'. If C' shares no vertices with $C_1, \ldots, C_k$, popping C' will not affect the other labeled cycles, meaning C will still get popped. Otherwise let C_i be the first labeled cycle sharing a vertex w with C'. Since C' and $C_j, j < i$ share no vertices, the unique edge e in C' satisfying $\underline{e} = w$ also appears in C_i and even has the same label as it has in C'. Hence, C_i and C' also share the vertex $\overline{e}$. Iterating this argument we see that C_i equals C' as a labeled (simple) cycle. Hence, we can pop

$$C' = C_i, C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_k = C,$$

i.e. C will still get popped if we choose to pop C' before C_1. If there exist infinitely many labeled cycles which can be popped, the algorithm never terminates. This will not change by altering the order in which to pop cycles.

We have seen that each configuration of stacks uniquely defines a rooted spanning tree and a set of labeled cycles stacked on top of this tree. To pop these cycles we start at S_{v_1} and walk around on the top of the stacks in the following fashion: Whenever we are at S_x, the next stack to visit is associated with the vertex $\overline{top(S_x)}$. If we encounter a stack that we have already visited, we pop this cycle off the top of the stacks. This process will halt as soon as we hit the stack S_r because it is empty.

Note that starting at some stack S_v and repeating this process will not change the top of the stacks that we visited while walking from S_{v_1} to S_r since it already 'shows' a self-avoiding path to r. Hence, repeating the process for each starting vertex v_i, $i = 2, \ldots, n$ will pop cycles off the top of the stacks until the visible graph is a spanning tree rooted at r.

Note that this walking around on the top of the stacks is just another way of simulating a RNW on G since the distribution of each item in the stack S_v is exactly the distribution of the first step of a RNW starting at v (i.e. of X_1 with respect to $\mathbb{P}_v$). Hence, walking from S_{v_1} to S_r and erasing cycles as soon as we encounter them is equivalent to generating $LE(X_1, \ldots, X_{\tau_r})$ and will almost surely terminate (see Lemma 3.16). Since repeating this process with another starting vertex will not change the already generated path r, we may think of it as $T(1)$. It follows that repeating this process with starting vertex v_i, $i = 2, \ldots, n$, is just the execution of Wilson's algorithm.

Fix $T_r \in RST(G, r)$, and for $v \in V \setminus \{r\}$ let $e(T_r, v)$ be the unique edge in T_r with $\underline{e} = v$. The probability $\Pr(G_{vis} = T_r)$ that we see T_r on top of the stacks is

$$\Pr(G_{vis} = T_r) = \prod_{v \in V \setminus \{r\}} \Pr(S_{v,1} = e(T_r, v)) = \prod_{e \in T_r} \frac{C(e)}{C_{\underline{e}}} = C_r \prod_{v \in V} \frac{1}{C_v} \cdot \mathrm{weight}_C(T_r)$$

since all items in all stacks are i.i.d. Now consider the probability $\Pr(\mathcal{C}$ on top of $T_r)$ that the stacks define a set of labeled cycles $\mathcal{C}$ on top of T_r. Again due to the i.i.d. nature of the stacks and since $\mathcal{C}$ implies a unique labeling of T_r, this probability can be written as

$$\Pr(\mathcal{C} \text{ on top of } T_r) = k(\mathcal{C}) \cdot \Pr(G_{vis} = T_r) = k(\mathcal{C}) \cdot C_r \prod_{v \in V} \frac{1}{C_v} \cdot \mathrm{weight}_C(T_r)$$

where $k(\mathcal{C})$ only depends on $\mathcal{C}$. Applying the law of total probability to all possible combinations of sets of labeled cycles on top of T_r, we see that

$$\Pr(\text{Wilson's algorithm produces } T_r) = k \cdot C_r \prod_{v \in V} \frac{1}{C_v} \cdot \mathrm{weight}_C(T_r)$$

where k is a constant that results from summing all $k(\mathcal{C})$, i.e. the probability of producing a certain rooted spanning tree T_r is proportional to its weight. □

Remark. By Lemma 3.25, we can use Wilson's algorithm to produce (unrooted) random spanning trees with distribution $\mu_{\mathcal{N}}$.

4 Electrical networks

In this chapter we will consider another way of interpreting a network (G, C), in particular we want to investigate electrical networks. To do this, we interpret each edge as a conductor with *conductance* $C(e)$ and a *resistance* $R(e) := 1/C(e)$.

The structure of this chapter roughly follows a paper by Benjamini, Lyons, Peres, and Schramm [1], although we will adjust some notations to be more consistent with the rest of this work.

4.1 Hilbert spaces on V and E

Note that if not otherwise stated, we will assume the considered networks to be finite.

Definition 4.1. Let $\mathcal{N} = (V, E, C)$ be a possibly infinite network.

1. For $f, g : V \to \mathbb{R}$ let
$$(f, g)_{\mathcal{N},C} := \sum_{v \in V} C_v f(v) g(v)$$
and
$$l^2_{\mathcal{N}}(V) := \{f : V \to \mathbb{R} \mid (f, f)_{\mathcal{N},C} < \infty\} .$$
2. For $I, J : E \to \mathbb{R}$ such that $I(\widehat{e}) = -I(e)$ and $J(\widehat{e}) = -J(e)$ for all $e \in E$ let
$$(I, J)_{\mathcal{N},R} := \frac{1}{2} \sum_{e \in E} R(e) I(e) J(e)$$
and
$$l^2_{\mathcal{N},-}(E) := \{I : E \to \mathbb{R} \mid I(\widehat{e}) = -I(e) \ \forall\ e \in E, (J, J)_{\mathcal{N},R} < \infty\} .$$

Remarks.

1. If $\mathcal{N}$ is finite, then so are V and E, and $l^2_{\mathcal{N}}(V)$ depends neither on C nor on E. Analogously, $l^2_{\mathcal{N},-}(E)$ does not depend on C or V if $\mathcal{N}$ is finite.
2. We will often completely omit $\mathcal{N}$ in our notation since the context will make it clear which network we are considering.
3. The order of summation in the definition of $(f, g)_{\mathcal{N},C}$ and $(I, J)_{\mathcal{N},R}$ can be chosen arbitrarily because $C_v > 0$ and $C(e) > 0$ for all $v \in V$, $e \in E$.

Proposition 4.2 (BLPS [1]). *For a possibly infinite network $\mathcal{N} = (V, E, C)$, both $(l^2(V), (\cdot, \cdot)_C)$ and $(l^2_-(E), (\cdot, \cdot)_R)$ are Hilbert spaces. We denote by $\|\cdot\|_C$ and $\|\cdot\|_R$ the associated norms.*

Proof. We will only prove that $(l^2(V), (\cdot, \cdot)_C)$ is a Hilbert space since the proof for $l^2_-(E)$ is completely analogous.

For $f \in l^2(V)$, we have $\lambda \cdot f \in l^2(V)$ for all $\lambda \in \mathbb{R}$ because $(\lambda f, \lambda f)_C = \lambda^2 (f, f)_C < \infty$. Let $f, g \in l^2(V)$. Then

$$2f(v)g(v) \leq 2f(v)g(v) + (f(v) - g(v))^2 = f(v)^2 + g(v)^2.$$

Using this, and applying $(f, f)_C < \infty$, $(g, g)_C < \infty$, and $C_v > 0$ for all $v \in V$, we can compute

$$\begin{aligned}\infty > \sum_{v \in V} C_v f(v)^2 + \sum_{v \in V} C_v g(v)^2 &= \sum_{v \in V} C_v (f(v)^2 + g(v)^2) \\ \geq \sum_{v \in V} C_v \cdot 2f(v)g(v) &= 2 \sum_{v \in V} C_v f(v) g(v).\end{aligned}$$

This implies

$$\begin{aligned}\infty &> \sum_{v \in V} C_v f(v)^2 + 2 \sum_{v \in V} C_v f(v) g(v) + \sum_{v \in V} C_v g(v)^2 \\ &= \sum_{v \in V} (f(v) + g(v))^2 = (f + g, f + g)_C\end{aligned}$$

because all three sums on the left side of the equation are finite. Hence, $f + g \in l^2(V)$.

It is obvious that $(\cdot, \cdot)_C$ is an inner product on $l^2(V)$. The only thing left to be shown is the completeness of $l^2(V)$ with respect to $(\cdot, \cdot)_C$: Let $(f_n) \subseteq l^2(V)$ be a Cauchy sequence, i.e.

$$\forall \, \varepsilon > 0 \; \exists \, N \in \mathbb{N} \; \forall \, m, n > N : \|f_n - f_m\|_C^2 = \sum_{v \in V} C_v (f_n(v) - f_m(v))^2 < \varepsilon.$$

Since $C_v > 0$ for all $v \in V$, this implies that $(f_n(v))_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for each $v \in V$. Hence, it has a limit $f(v)$, i.e. $(f_n)_{n \in \mathbb{N}}$ has the point-wise limit f. Applying the triangle inequality of $\|\cdot\|_C$ we also get

$$|\|f_m\|_C - \|f_n\|_C| \leq \|f_m - f_n\|_C$$

Hence, $(\|f_n\|_C)_{n \in \mathbb{N}}$ is a also a Cauchy sequence in $\mathbb{R}$, in particular it is bounded. Let

$$\mu := \sum_{v \in V} C_v \cdot \delta_v$$

where δ_v denotes the Dirac measure concentrated in $\{v\}$, $v \in V$. Then $(V, 2^V, \mu)$ is a measure space such that

$$\int g \, \mathrm{d}\mu = \sum_{v \in V} C_v g(v)$$

for any $g : V \to \mathbb{R}$. We can now apply Fatou's Lemma (see [8], p. 264-265) to (f_n^2) since $f_n^2 \to f^2$ μ-almost surely, which gives us

$$\begin{aligned}(f, f)_C = \sum_{v \in V} C_v f(v)^2 &= \int f^2 \, \mathrm{d}\mu \leq \liminf_{n \to \infty} \int f_n^2 \, \mathrm{d}\mu \\ &= \liminf_{n \to \infty} \sum_{v \in V} C_v f_n(v)^2 = \liminf_{n \to \infty} \|f_n\|_C^2 < \infty.\end{aligned}$$

Thus, $f \in l^2(V)$. We know that $(f_m - f_n)^2 \overset{m\to\infty}{\to} (f - f_n)^2$ μ-almost surely. Again, applying Fatou's Lemma produces

$$\|f - f_n\|_C^2 = \int (f - f_n)^2 \, \mathrm{d}\mu \leq \liminf_{m\to\infty} \int (f_m - f_n)^2 \, \mathrm{d}\mu = \liminf_{m\to\infty} \|f_m - f_n\|_C^2 .$$

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\|f_m - f_n\|_C^2 < \varepsilon$ for all $m, n > N$. Hence,

$$\|f - f_n\|_C^2 \leq \liminf_{m\to\infty} \|f_m - f_n\|_C^2 \leq \varepsilon$$

for all $n > N$, i.e. $\|f - f_n\|_C \overset{n\to\infty}{\to} 0$. □

Definition 4.3 (Orientation of a graph)**.** Let $G = (V, E)$ be a graph and $L := \{e \in E \mid e = \widehat{e}\}$. An *orientation* of G is a set of edges $E' \subseteq E \setminus L$ such that

$$e \in E' \Leftrightarrow \widehat{e} \notin E' \quad \forall\, e \in E \setminus L.$$

Remark. An orientation of a graph chooses a direction for each edge while disregarding loops.

Lemma 4.4. *Let (V, E, C) be a possibly infinite network and E' be an orientation of (V, E). The Hilbert spaces $l^2(V)$ and $l^2_-(E)$ are seperable with orthonormal basis (ONB) $\left\{\frac{1}{\sqrt{C_x}}\mathbb{1}_x \mid x \in V\right\}$ and $\left\{\frac{1}{\sqrt{C(e)}}(\mathbb{1}_e - \mathbb{1}_{\widehat{e}}) \mid e \in E'\right\}$, respectively. In particular, $\{\mathbb{1}_x \mid x \in V\}$ and $\{\mathbb{1}_e - \mathbb{1}_{\widehat{e}} \mid e \in E'\}$ are orthogonal bases of $l^2(V)$ and $l^2_-(E)$, respectively.*

Proof.

1. $\left\{\frac{1}{\sqrt{C_x}}\mathbb{1}_x \mid x \in V\right\}$ is an ONB of $l^2(V)$:
   Let $b_x := 1/\sqrt{C_x} \cdot \mathbb{1}_x$. Then $b_x \in l^2(V)$ and for $f \in l^2(V)$ we have

   $$(b_x, f)_C = \sum_{z\in V} C_z \cdot \frac{1}{\sqrt{C_x}} \mathbb{1}_x(z) \cdot f(z) = \sqrt{C_x} f(x).$$

   Hence, $(b_x, b_x)_C = 1$, $(b_x, b_y)_C = 0$ for all $x \neq y$ and

   $$f = \sum_{x\in V} f(x) \cdot \mathbb{1}_x = \sum_{x\in V} \sqrt{C_x} f(x) \cdot \frac{1}{\sqrt{C_x}} \mathbb{1}_x = \sum_{x\in V} (b_x, f)_C \cdot b_x \;\forall\, f \in l^2(V).$$

2. Let $B_e := \sqrt{C(e)} \cdot (\mathbb{1}_e - \mathbb{1}_{\widehat{e}})$. Then $B_e \in l^2_-(E)$ and for $I \in l^2_-(E)$, we have

   $$(B_e, I)_R = \frac{1}{2} \sum_{f\in E} \frac{1}{C(f)} \cdot \sqrt{C(e)} (\mathbb{1}_e(f) - \mathbb{1}_{\widehat{e}}(f)) \cdot I(f) = \frac{1}{\sqrt{C(e)}} I(e).$$

Hence, $(B_e, B_e)_R = 1$, $(B_f, B_f)_R = 0$ for all $e, f \in E', e \neq f$ and

$$I = \sum_{e \in E'} I(e) \cdot (\mathbb{1}_e - \mathbb{1}_{\hat{e}}) = \sum_{e \in E'} (B_e, I)_R \cdot B_e \quad \forall\, I \in l^2_-(E).$$

□

Definition 4.5 (Divergence, Gradient and Laplacian)**.** Let (V, E, C) be a possibly infinite network. We define

1. the *divergence operator* $\operatorname{div} : l^2_-(E) \to l^2(V)$ by

$$(\operatorname{div} I)(x) := \frac{1}{C_x} \sum_{e \in E(x)} I(e) \ , \ x \in V \ , \ I \in l^2_-(E).$$

2. the *gradient operator* $\nabla : l^2(V) \to l^2_-(E)$ by

$$(\nabla f)(e) := C(e)(f(\overline{e}) - f(\underline{e})) \ , \ f \in l^2(V).$$

3. the *Laplacian* $\Delta : l^2(V) \to l^2(V)$ by

$$(\Delta f)(x) := (\operatorname{div} \nabla f)(x) = \sum_{y \in V} \frac{C(x,y)}{C_x} f(y) - f(x) \ , \ x \in V \ , \ f \in l^2(V).$$

We say a function $f \in l^2(V)$ is *harmonic* in $x \in V$ if $(\Delta f)(x) = 0$.

Remarks.

1. The next lemma will show that the ranges of div and ∇ are indeed subsets of $l^2_-(E)$ and $l^2(V)$, respectively.
2. Obviously, div, ∇ and Δ are linear.
3. The second equality of the definition of Δ is a simple computation:

$$(\operatorname{div} \nabla f)(x) = \frac{1}{C_x} \sum_{e \in E(x)} C(e)(f(\overline{e}) - f(x)) = \frac{1}{C_x} \sum_{y \in V} \sum_{e \in E(x,y)} C(e)(f(y) - f(x))$$
$$= \frac{1}{C_x} \sum_{y \in V} C(x,y)(f(y) - f(x)) = \sum_{y \in V} \frac{C(x,y)}{C_x} f(y) - f(x).$$

Lemma 4.6 (BLPS [1])**.** *Let $\mathcal{N} = (V, E, C)$ be a possibly infinite network. Then*

1. $\|\operatorname{div} I\|_C \leq \sqrt{2} \cdot \|I\|_R$ *for all* $I \in l^2_-(E)$.
2. $\|\nabla f\|_R \leq \sqrt{2} \cdot \|f\|_C$ *for all* $f \in l^2(V)$.

Proof.

1. Fix $x \in V$ and let $I \in l^2_-(E)$. Using the Cauchy-Schwarz inequality we get

$$\left|\sum_{e\in E(x)} I(e)\right| \leq \sum_{e\in E(x)} |I(e)| = \sum_{e\in E(x)} \frac{1}{C(e)} \cdot |I(e)| \cdot C(e)$$
$$\leq \sqrt{\sum_{e\in E(x)} C(e)} \cdot \sqrt{\sum_{e\in E(x)} \frac{1}{C(e)} I(e)^2} = \sqrt{C_x} \cdot \sqrt{\sum_{e\in E(x)} \frac{1}{C(e)} I(e)^2}.$$

It follows that

$$|(\operatorname{div} I)(x)| = \left|\frac{1}{C_x} \sum_{e\in E(x)} I(e)\right| \leq \frac{1}{\sqrt{C_x}} \sqrt{\sum_{e\in E(x)} \frac{1}{C(e)} I(e)^2}.$$

In particular,

$$|(\operatorname{div} I)(x)| \leq \sqrt{\frac{\mathcal{E}(I)}{C_x}}$$

and

$$\|\operatorname{div} I\|_C^2 = \sum_{x\in V} C_x (\operatorname{div} I)(x)^2 \leq \sum_{x\in V} \sum_{e\in E(x)} \frac{1}{C(e)} I(e)$$
$$= \sum_{e\in E} \frac{1}{C(e)} I(e)^2 = 2\,\|I\|_R^2 .$$

2. For $a, b \in \mathbb{R}$, we have $0 \leq (a+b)^2 = a^2 + 2ab + b^2$ and therefore

$$-2ab \leq a^2 + b^2.$$

Hence, $(a-b)^2 = a^2 - 2ab + b^2 \leq 2(a^2 + b^2)$ for all $a, b \in \mathbb{R}$. It follows that

$$\|\nabla f\|_R^2 = \frac{1}{2} \sum_{e\in E} C(e)(f(\overline{e}) - f(\underline{e}))^2 \leq \frac{1}{2} \sum_{e\in E} C(e) \cdot 2(f(\overline{e})^2 + f(\underline{e})^2)$$
$$= \sum_{e\in E} C(e) f(\overline{e})^2 + \sum_{e\in E} C(e) f(\underline{e})^2$$
$$= \sum_{x\in V} \left(\sum_{e\in E(x)} C(e)\right) \cdot f(x)^2 + \sum_{x\in V} \left(\sum_{\widehat{e}\in E(x)} C(e)\right) \cdot f(x)^2$$
$$= 2 \sum_{x\in V} C_x \cdot f(x)^2$$
$$= 2\,\|f\|_C^2 .$$

Changing the order of summation can in each step be justified by the absolute convergence of each sum. □

Lemma 4.7 (BLPS [1])**.** *Let (V, E, C) be a possibly infinite network. The operators $-\nabla$ and* div *are adjoints of each other, making Δ self-adjoint.*

Proof. Let $f \in l^2(V)$, $I \in l^2_-(E)$. Using the Cauchy-Schwarz inequality for Hilbert spaces (Proposition 5.40) we get

$$|(-\nabla f, I)_R| \leq \|-\nabla f\|_R \cdot \|I\|_R \leq \sqrt{2}\,\|f\|_C\,\|I\|_R < \infty$$

and

$$|(f, \operatorname{div} I)_C| \leq \|f\|_C \cdot \|\operatorname{div} I\|_C \leq \sqrt{2}\,\|f\|_C\,\|I\|_R < \infty.$$

This implies that all sums in the following computation converge.

$$\begin{aligned}
(-\nabla f, I)_R &= \frac{1}{2}\sum_{e\in E} R(e)\cdot C(e)(f(\underline{e}) - f(\overline{e}))\cdot I(e) \\
&= \frac{1}{2}\left(\sum_{e\in E} f(\underline{e})I(e) - \sum_{e\in E} f(\overline{e})I(e)\right) \\
&= \frac{1}{2}\left(\sum_{e\in E} f(\underline{e})I(e) - \sum_{e\in E} f\left(\overline{\widehat{e}}\right) I(\widehat{e})\right) \\
&= \frac{1}{2}\left(\sum_{e\in E} f(\underline{e})I(e) + \sum_{e\in E} f(\underline{e})I(e)\right) \\
&= \sum_{e\in E} f(\underline{e})I(e) = \sum_{x\in V}\sum_{e\in E(x)} f(\underline{e})I(e) = \sum_{x\in V} f(x) \sum_{e\in E(x)} I(e) \\
&= \sum_{x\in V} C_x \cdot f(v)\cdot \left(\frac{1}{C_x}\sum_{e\in E(x)} I(e)\right) = (f, \operatorname{div} I)_C.
\end{aligned}$$

Hence,

$$(\Delta f, g)_C = (-\operatorname{div}(-\nabla f), g)_C = (-\nabla f, \nabla g)_C = (f, (\operatorname{div}\nabla g))_C = (f, \Delta g)_C.$$

□

Important: For the remainder of this chapter, all considered networks will be assumed to be finite.

Definition 4.8 (Directed flow, Kirchhoff's first law)**.** Let $x, y \in V$, $x \neq y$. $I \in l^2_-(E)$ is called a *directed flow from x to y* if

$$\sum_{e\in E(v)} I(e) = 0 \;\forall\; v \in V \setminus \{x, y\}$$

and

$$0 \leq \sum_{e \in E(x)} I(e) = -\sum_{e \in E(y)} I(e) =: A(I).$$

We call x the *source*, y the *drain* and $A(I)$ the *amount* of I.

Remark (Interpretation)**.** We interpret $I(e)$, $e \in E$ as the amount of whatever is flowing through our network (water, gas, electrical current, etc.) flowing through the edge e in its direction. If there is actual flow along e, i.e. $I(e) \neq 0$, then the flow along the reversed edge $\widehat{e}$ is negative but of the same absolute value, i.e. $I(\widehat{e}) = -I(e)$. The definition formalizes Kirchhoff's first law (also called *Kirchhoff's Current Law* or *KCL*, see Thomassen [10], p. 89):

> If $x \in V$ is neither source nor drain then the sum of flow entering x equals the sum of flow leaving x.

Lemma 4.9. *If $I \in l^2_-(E)$ is a directed flow from x to y, then*

$$\operatorname{div} I = \frac{A(I)}{C_x}\mathbb{1}_x - \frac{A(I)}{C_y}\mathbb{1}_y.$$

Proof. Let $v \in V$. Then

$$(\operatorname{div} I)(v) = \frac{1}{C_v} \sum_{e \in E(v)} I(e) \overset{\text{Def. 4.8}}{=} \frac{1}{C_v} \cdot \begin{cases} 0 & , x \neq v \neq y \\ A(I) & , v = x \\ -A(I) & , v = y \end{cases}.$$

□

Definition 4.10 (Unit flow, cycle)**.**

1. For $e \in E$ we call $\chi^e := \mathbb{1}_e - \mathbb{1}_{\widehat{e}} \in l^2_-(E)$ the *unit flow along e.*
2. Let $(e_1, \ldots, e_n)$ be a cycle in G. Then we call $\sum_{i=1}^n \chi^{e_i} \in l^2_-(E)$ a *cycle.*

Remarks.

1. χ^e is called unit flow along e because it is a directed flow from $\underline{e}$ to $\overline{e}$ such that $\chi^e(f) = 0$ for all $f \in E \setminus \{e, \widehat{e}\}$ and $A(\chi^e) = 1$.
2. $e \in E$ is a loop if and only if $\chi^e \equiv 0$.

Lemma 4.11 (BLPS [1])**.** *Let $I \in l^2_-(E)$ be a cycle. Then* $\operatorname{div} I \equiv 0$.

Proof. Let $I = \sum_{i=1}^n \chi^{e_i}$. By Lemma 4.9, we have

$$\operatorname{div} I = \sum_{i=1}^n \operatorname{div} \chi^{e_i} = \sum_{i=1}^n \frac{1}{C_{\underline{e_i}}} \mathbb{1}_{\underline{e_i}} - \frac{1}{C_{\overline{e_i}}} \mathbb{1}_{\overline{e_i}} .$$

Since $(e_1, \dots, e_n)$ is a cycle in G, we have $\overline{e_i} = \underline{e_{i+1}}$ for all $i = 1, \dots, n-1$ and $\underline{e_1} = \overline{e_n}$. Hence,

$$\operatorname{div} I = \sum_{i=1}^n \frac{1}{C_{\underline{e_i}}} \mathbb{1}_{\underline{e_i}} - \frac{1}{C_{\overline{e_i}}} \mathbb{1}_{\overline{e_i}} = \sum_{i=1}^n \frac{1}{C_{\underline{e_i}}} \mathbb{1}_{\underline{e_i}} - \frac{1}{C_{\underline{e_{i+1}}}} \mathbb{1}_{\underline{e_{i+1}}} = \frac{1}{C_{\underline{e_1}}} \mathbb{1}_{\underline{e_1}} - \frac{1}{C_{\overline{e_n}}} \mathbb{1}_{\overline{e_n}} \equiv 0.$$

□

Definition 4.12 (Subspaces of stars and cycles)**.** We denote by

1. $\bigstar := \nabla l^2(V)$ the subspace of $l^2_-(E)$ spanned by the *stars* $-\nabla \mathbb{1}_x = \sum_{e \in E(x)} C(e)\chi^e$.
2. $\diamondsuit$ the subspace of $l^2_-(E)$ spanned by all cycles.

Proposition 4.13 (BLPS [1])**.** *$l^2_-(E)$ has the orthogonal decomposition*

$$l^2_-(E) = \bigstar \oplus \diamondsuit$$

or equivalently

$$\bigstar = \diamondsuit^\perp.$$

Proof. First, we show $\bigstar \perp \diamondsuit$, i.e. $\bigstar \subseteq \diamondsuit^\perp$:
Since $(\cdot, \cdot)_R$ is bilinear and $l^2_-(E)$ is finitely generated, we only need to show $(I, J)_R = 0$ for spanning elements I and J of $\bigstar$ and $\diamondsuit$, respectively. Let $x \in V$, $I = -\nabla \mathbb{1}_x$ and J be any cycle. Then by Lemma 4.7 and Lemma 4.11, we have

$$(-\nabla \mathbb{1}_v, J)_R = (\mathbb{1}_v, \operatorname{div} J)_C = (\mathbb{1}_v, 0)_C = 0.$$

Now we show $\diamondsuit^\perp \subseteq \bigstar$: Let $I \perp \diamondsuit$. Fix a vertex $x \in V$ and define

$$J(y) := \sum_{j=1}^n R(e_j) I(e_j) \ , \ y \in V$$

where $(e_1, \dots, e_n)$ is a path from x to y. This definition is independent of the actual choice of the path because I is orthogonal to the cycles $\diamondsuit$. Now let $e \in E$, $(e_1, \dots, e_n)$ be a path from x to $\underline{e}$ and $e_{n+1} = e$. Then $(e_1, \dots, e_{n+1})$ is a path from x to $\overline{e}$, and we get

$$\begin{aligned}(\nabla J)(e) &= C(e)(J(\overline{e}) - J(\underline{e})) \\ &= C(e)\left(\underbrace{\sum_{i=1}^{n+1} R(e_i)I(e_i)}_{x\to\overline{e}} - \underbrace{\sum_{i=1}^{n} R(e_i)I(e_i)}_{x\to\underline{e}}\right) \\ &= C(e)R(e)I(e) = I(e).\end{aligned}$$

Hence, $I = \nabla J \in \nabla l^2(V) = \bigstar$. □

Corollary 4.14 (BLPS [1]). *For all $I \in l^2_-(E)$, we have* $\operatorname{div} I \equiv 0$ *if and only if* $I \in \diamondsuit$.

Proof.
<u>"$\Leftarrow$":</u> Since div is linear and $\diamondsuit$ is finitely generated, it suffices to show $\operatorname{div} I \equiv 0$ for any cycle I. However, this is just the statement of Lemma 4.11.

<u>"$\Rightarrow$":</u> Let $x \in V$. By Lemma 4.7, we have

$$(I, -\nabla \mathbb{1}_x)_R = (\operatorname{div} I, \mathbb{1}_x)_C = 0.$$

Using that $\bigstar$ is spanned by $\{-\nabla \mathbb{1}_x\}_{x\in V}$ and Proposition 4.13 we get

$$I \in \bigstar^\perp = (\diamondsuit^\perp)^\perp = \diamondsuit$$

because $l^2_-(E)$ has finite dimension. □

Definition 4.15 (Electrical current, Kirchoff's second law). Let $x, y \in V, x \neq y$ and I be a directed flow from x to y. I is called *electrical current* if for any cycle $c = \sum_{i=1}^{n} \chi^{e_i}$, we have

$$(I, c)_R = \sum_{i=1}^{n} R(e_i)I(e_i) = 0,$$

i.e. if and only if $I \in \diamondsuit^\perp$. $(I, c)_R$ is called *voltage drop* of I along the cycle c.

Remark (Interpretation). From basic physics we know that the voltage drop between the endpoints of a conductor equals the electrical current flowing through the conductor multiplied by its resistance. Interpreting $I(e)$ as the electrical current flowing through e and $R(e)$ as e's resistance, our definition formalizes Kirchhoff's second law (also called *Kirchhoff's Voltage Law or KVL*, see Thomassen [10], p. 89):

The voltage drop along any cycle equals zero.

Note however that every electrical current needs to be a directed flow, i.e. it has to satisfy both of Kirchhoff's laws.

Lemma 4.16. *Let $x, y \in V$ and $(e_1, \dots, e_n)$, $(\widetilde{e}_1, \dots, \widetilde{e}_m)$ be two paths from x to y. Furthermore, let*

$$J := \sum_{i=1}^{n} \chi^{e_i} \ , \ \widetilde{J} := \sum_{i=1}^{m} \chi^{\widetilde{e}_i} \ .$$

Then $P_\bigstar J = P_\bigstar \widetilde{J}$ where $P_\bigstar$ denotes the orthogonal projection onto $\bigstar$.

Proof. Clearly $\widetilde{J} - J \in \diamondsuit$ (it might not be a cycle, but it is definitely a sum of cycles or zero). Hence, by Proposition 4.13, $P_\bigstar(\widetilde{J} - J) = 0$ and thus $P_\bigstar J = P_\bigstar \widetilde{J}$. □

Remark. For any linear subspace $X \subseteq l^2_-(E)$, we will denote by P_X the orthogonal projection onto X and by $P_X^\perp$ the orthogonal projection onto $X^\perp$.

Definition 4.17 (Unit current)**.**

1. Let $e \in E$. We call $I^e := P_\bigstar \chi^e$ the *unit current through* e.
2. Let $x, y \in V$ and $(e_1, \dots, e_n)$ be any path from x to y. We call

$$I^{xy} := P_\bigstar \left(\sum_{i=1}^{n} \chi^{e_i} \right) = \sum_{i=1}^{n} I^{e_i}$$

unit current from x to y. By Lemma 4.16, this definition does not depend on the chosen path from x to y.

Remark. Note that $I^e = I^{\underline{e}\overline{e}}$ for all $e \in E$.

Lemma 4.18. *Let $e \in E$, $x, y \in V$. Then the unit currents I^e and I^{xy} actually are electrical currents (with respect to Definition 4.15) if and only if e is no loop and $x \neq y$, respectively.*

Proof. I^e is a flow: $\operatorname{div}(\chi^e - I^e) \equiv 0$ because $\chi^e - I^e \in \bigstar^\perp = \diamondsuit$. Hence,

$$\operatorname{div} I^e = \operatorname{div} \chi^e = \frac{1}{C_{\underline{e}}} \mathbb{1}_{\underline{e}} - \frac{1}{C_{\overline{e}}} \mathbb{1}_{\overline{e}} \ .$$

Thus,

$$\sum_{f\in E(v)} I^e(f) = C_v(\operatorname{div} I^e)(v) = \begin{cases} 0 & , \underline{e} \neq v \neq \overline{e} \\ 1 & , v = \underline{e} \\ -1 & , v = \overline{e} \end{cases}$$

if e is no loop, i.e. $\overline{e} \neq \underline{e}$. By Definition 4.8, I^e is a flow from $\underline{e}$ to $\overline{e}$ with amount $A(I^e) = 1$.
I^e is an electrical current because $I^e = P_\bigstar \chi^e \in \bigstar$. The analogous statement for I^{xy} follows by linearity of div and $P_\bigstar$. □

Remark (Interpretation)**.** $I^{xy}(f)$ is the amount of current flowing through the edge f (in the same direction as f) when we hook up a battery between x and y such that unit current flows through the entire network.

4.2 The effects of contracting edges on $l^2_-(E)$

We fix a finite connected network $\mathcal{N} = (G, C) = (V, E, C)$ and a set of directed edges $A \subseteq E$, $\widehat{A} = A$, containing no cycle. Let $\mathcal{N}/A := (G/A, C)$ be the contracted network. Since we identify E with the set of directed edges of G/A (see Section 2.3), contracting the edges in A will not change $l^2_-(E)$ on a global scale. However, a change can be noticed regarding the subspaces $\bigstar$ and $\diamondsuit$, which we will examine in this section.

Definition 4.19. Analogously to Definition 4.12, we define:

1. $\bigstar_A := \nabla l^2(V_{G/A})$ to be the subspace of $l^2_-(E)$ spanned by the stars $-\nabla \mathbb{1}_{v'} = \sum_{e\in E(v')} C(e)\chi^e$ in G/A.
2. $\diamondsuit_A$ the subspace of $l^2_-(E)$ spanned by all cycles in G/A (recall Definition 4.10).
3. $\langle \chi^A \rangle := \operatorname{span}\{\chi^e \mid e \in A\}$.

Proposition 4.20 (BLPS [1])**.** $l^2_-(E) = \bigstar_A \oplus P_\bigstar \langle \chi^A \rangle \oplus \diamondsuit$.

Proof. One can easily verify that $\diamondsuit_A = \diamondsuit + \langle \chi^A \rangle$. Hence, $\diamondsuit_A \supseteq \diamondsuit$ and $\bigstar_A \subseteq \bigstar$. Clearly, $\bigstar \cap \diamondsuit_A = P_\bigstar(\bigstar \cap \diamondsuit_A) \subseteq P_\bigstar \diamondsuit_A$ holds. Furthermore, let $I \in P_\bigstar \diamondsuit_A$ and $I = P_\bigstar J$ for some $J \in \diamondsuit_A$. For all $K \in \bigstar_A = (\diamondsuit_A)^\perp$, we then have

$$(I, K)_R = (P_\bigstar J, K)_R = (J, P_\bigstar K)_R = (J, K)_R = 0.$$

It follows that $P_\bigstar \diamondsuit_A \subseteq (\bigstar_A)^\perp = \diamondsuit_A$. Since $P_\bigstar \diamondsuit_A \subseteq \bigstar$ also holds, we have $P_\bigstar \diamondsuit_A \subseteq \bigstar \cap \diamondsuit_A$. Hence, $P_\bigstar \diamondsuit_A = \bigstar \cap \diamondsuit_A$. This gives us

$$(\diamondsuit \oplus \bigstar_A)^\perp = \bigstar \cap \diamondsuit_A = P_\bigstar \diamondsuit_A = \underbrace{P_\bigstar \diamondsuit}_{=0} + P_\bigstar \langle \chi^A \rangle .$$

Hence,

$$\bigstar = \bigstar_A \oplus P_\bigstar \left\langle \chi^A \right\rangle \ , \ \Diamond_A = \Diamond \oplus P_\bigstar \left\langle \chi^A \right\rangle .$$

□

Definition 4.21. Let $e \in E$ be a directed edge. Analogously to Definition 4.17, we define

$$I_A^e := P_{\bigstar_A}\chi^e = P^\perp_{P_\bigstar\langle\chi^A\rangle} P_\bigstar \chi^e = P^\perp_{P_\bigstar\langle\chi^A\rangle} I^e.$$

Remark. Note that any edge $e \in E$ that is a loop or is contained in A or forms a cycle with edges in A becomes a loop in G/A. Hence, it will produce $I_A^e \equiv 0$.

4.3 The Current Matrix Theorem

We will now use the theory of electrical networks established in sections 4.1 and 4.2 to make a very interesting connection between electrical networks and spanning tree measures.

Let $\mathcal{N} = (V, E, C) = (G, C)$ be a finite connected network. Furthermore, let E' be an orientation of G. Then $\{\chi^e \mid e \in E'\}$ is an orthogonal basis of $l^2_-(E)$ since $I(e) = 0$ for all $I \in l^2_-(E)$ and all loops $e \in E$. The matrix of $P_\bigstar$ with respect to this orthonormal basis is called *transfer impedance matrix* and is given by

$$(P_\bigstar \chi^e, \chi^f)_R = (I^e, \chi^f)_R = R(f)I^e(f) \ \ , \ e, f \in E'.$$

Definition 4.22 (Transfer current matrix)**.** The matrix

$$TCM := (I^e(f))_{e,f\in E'} \in M(E' \times E')$$

is called *transfer current matrix.*

Lemma 4.23. *For $e, f \in E$,*

$$I^f(e) = \frac{C(e)}{C(f)} I^e(f)$$

holds.

Proof. Since $P_\bigstar$ is an orthogonal projection, it is self-adjoint and, in particular, symmetric. It follows that

$$R(e)I^f(e)(P_\bigstar \chi^f, \chi^e) = (P_\bigstar \chi^e, \chi^f) = R(f)I^e(f)$$

and therefore
$$I^f(e) = \frac{R(f)}{R(e)} I^e(f).$$
$R(e) = 1/C(e)$ implies
$$I^f(e) = \frac{C(e)}{C(f)} I^e(f)$$
for all $e, f \in E'$. Since $I^e = -I^{\widehat{e}}$ and $I^e(f) = -I^e(\widehat{f})$, this holds for all $e, f \in E$. □

Remark. Note that the (e, f)-entry of $P_\bigstar$ is $R(f)I^e(f)$, which can be interpreted as the voltage developing along f when a battery is hooked up at the endpoints of e such that unit current flows. Thus, the symmetry of $P_\bigstar$ is a version of a phenomenon known as the Reciprocity Law, which we will prove more generally in Proposition 5.14.

Proposition 4.24 (Connection between random spanning trees and I^e, BLPS [1]). *Let*

$$\beta(e, f) := \mu_{\mathcal{N}}(\{T \in ST(G) \mid \textit{The unique path } \underline{e} \to \overline{e} \textit{ in } T \textit{ uses } f\})$$

be the probability that the path from $\underline{e}$ to $\overline{e}$ in a randomly chosen spanning tree passes through f in the same direction as f. Furthermore, recall Definition 3.18 and let
$$J^e(f) := J_f^{\underline{e}\overline{e}} - J_{\widehat{f}}^{\underline{e}\overline{e}}$$
be the expected number of times a RNW starting in $\underline{e}$ uses the edge f before reaching $\overline{e}$ minus the expected number of times it uses $\widehat{f}$. Then

$$\beta(e, f) - \beta(e, \widehat{f}) = J^e(f) = I^e(f).$$

Proof. Let $t := \tau_{\overline{e}}$, $Z := LE(X_1, \ldots, X_t) = (Z_1, \ldots, Z_n)$. Note that t and n are random variables. Using Wilson's algorithm 3.26 set $\overline{e} = r$, $v_1 = \underline{e}$ and the enumeration of the remaining vertices $v_2, \ldots, v_n$ arbitrarily. By Theorem 3.27, we have

$$\beta(e, f) - \beta(e, \widehat{f}) = \mathbb{P}_{\underline{e}}[Z \text{ uses } f] - \mathbb{P}_{\underline{e}}[Z \text{ uses } \widehat{f}].$$

Since a loop-erased RNW uses every edge either once or not at all, it follows that

$$\beta(e, f) - \beta(e, \widehat{f}) = \mathbb{E}_{\underline{e}}\left[\sum_{k=1}^{n} \mathbb{1}_f(Z_k) - \mathbb{1}_{\widehat{f}}(Z_k)\right].$$

The only difference between Z and X is that X may use cycles in between the steps that Z takes. By Lemma 3.17, it follows that X uses every cycle in each direction an equal number of times in expectation. Hence,

$$\beta(e,f)-\beta(e,\widehat{f})=\mathbb{E}_{\underline{e}}\left[\sum_{k=1}^{\tau_{\overline{e}}}\mathbb{1}_f(X_k)-\mathbb{1}_{\widehat{f}}(X_k)\right]=J^e(f).$$

By Lemma 3.20, we know that $J_f^{\underline{e}\overline{e}}=U_{\underline{f}}^{\underline{e}\overline{e}}\cdot p(f)=\sum_{x\in V}\sum_{g\in E(x)}U_x^{\underline{e}\overline{e}}\cdot\mathbb{1}_g(f)\cdot\frac{C(f)}{C_x}$. The second equality holds because $\mathbb{1}_g(f)=1$ occurs exactly once and implies $x=\underline{g}=\underline{f}$. Hence,

$$\begin{aligned}J^e(f)=J_f^{\underline{e}\overline{e}}-J_{\widehat{f}}^{\underline{e}\overline{e}}&=\sum_{x\in V}\sum_{g\in E(x)}U_x^{\underline{e}\overline{e}}\frac{C(f)}{C_x}(\mathbb{1}_g(f)-\mathbb{1}_g(\widehat{f}))\\&=\sum_{x\in V}\frac{U_x^{\underline{e}\overline{e}}}{C_x}\sum_{g\in E(x)}C(g)(\mathbb{1}_g(f)-\mathbb{1}_{\widehat{g}}(f))=\sum_{x\in V}\frac{U_x^{\underline{e}\overline{e}}}{C_x}\sum_{g\in E(x)}C(g)\chi^g(f)\\&=\sum_{x\in V}\frac{U_x^{\underline{e}\overline{e}}}{C_x}(-\nabla\mathbb{1}_x)(f).\end{aligned}$$

This implies $J^e\in\bigstar$. Now note that

$$\begin{aligned}(\operatorname{div}J^e)(x)&=\frac{1}{C_x}\sum_{f\in E(x)}J^e(f)=\frac{1}{C_x}\cdot\begin{cases}1 & ,\ x=\underline{e}\\-1 & ,\ x=\overline{e}\\0 & ,\ \text{otherwise}\end{cases}\\&=\frac{1}{C_{\underline{e}}}\mathbb{1}_{\underline{e}}(x)-\frac{1}{C_{\overline{e}}}\mathbb{1}_{\overline{e}}(x)=(\operatorname{div}I^e)(x).\end{aligned}$$

Indeed for $x\neq\overline{e}$, Lemmas 3.20 and 3.19 imply

$$\begin{aligned}\sum_{f\in E(x)}J^e(f)&=\sum_{f\in E(x)}\left(U_x^{\underline{e}\overline{e}}p(f)-U_{\overline{f}}^{\underline{e}\overline{e}}p(\widehat{f})\right)\\&=U_x^{\underline{e}\overline{e}}-\sum_{v\in V}U_v^{\underline{e}\overline{e}}\frac{C(v,x)}{C_v}=\begin{cases}1 & ,\ x=\underline{e}\\0 & ,x\in V\setminus\{\underline{e},\overline{e}\}\end{cases}.\end{aligned}$$

It follows that $(\operatorname{div}J^e)(\underline{e})=1/C_{\underline{e}}$ and $(\operatorname{div}J^e)(x)=0$ for $x\in V\setminus\{\underline{e},\overline{e}\}$. Furthermore,

$$0=(0,J^e)_R=(\nabla 1,J^e)_R=(1,\operatorname{div}J^e)_C=\sum_{x\in V}C_x(\operatorname{div}J^e)(x)=1+C_{\overline{e}}(\operatorname{div}J^e)(\overline{e})$$

and therefore $(\operatorname{div}J^e)(\overline{e})=-1/C_{\overline{e}}$.

Hence, $\operatorname{div}(I^e-J^e)=0$ which implies $I^e-J^e\in\diamondsuit=\bigstar^\perp$ by Corollary 4.14. It follows that $I^e-J^e\in\bigstar\cap\bigstar^\perp=\{0\}$, i.e. $I^e=J^e$. □

Corollary 4.25 (BLPS [1])**.** *Let $e\in E$. Then $\mu_{\mathcal{N}}[e\in T]=I^e(e)$.*

Proof. If e is a loop, then $I^e(e) = 0 = \mu_{\mathcal{N}}[e \in T]$. If e is no loop, then $\beta(e, \widehat{e}) = 0$ and by Proposition 4.24

$$\mu_{\mathcal{N}}[e \in T] = \beta(e,e) = \beta(e,e) - \beta(e,\widehat{e}) = I^e(e).$$

□

Theorem 4.26 (Burton and Pemantle [2])**.** *Let $\mathcal{N}$ be a finite connected network and E' an orientation of G. For any distinct edges $e_1, \ldots, e_k \in E'$, we have*

$$\mu_{\mathcal{N}}[e_1, \ldots, e_k \in T] = \det[I^{e_i}(e_j)]_{1 \le i,j \le k}.$$

Proof (BLPS [1]).

1. Suppose a cycle can be formed using the edges $e_1, \ldots, e_k$, namely $\sum_j a_j \chi^{e_j} \in \diamondsuit$ where $a_j \in \{-1, 0, 1\}$. Then

$$\sum_j a_j R(e_j) I^{e_i}(e_j) = \sum_j a_j (I^{e_i}, \chi^{e_j})_R = \left(I^{e^i}, \sum_j a_j \chi^{e_j} \right)_R = 0$$

because $I^{e_i} = P_\bigstar \chi^{e_i} \perp \diamondsuit$. This is a linear combination of the columns of $(I^{e_i}(e_j))_{1 \le i,j \le k}$. Hence, both sides of the desired equation equal zero.

2. Now we assume that one cannot form a cycle using only the edges $e_1, \ldots, e_k$. Since $P_\bigstar$ is an orthogonal projection, we have

$$I^e(f) = C(f)(P_\bigstar \chi^e, \chi^f)_R = C(f)(P_\bigstar \chi^e, P_\bigstar \chi^f)_R = C(f)(I^e, I^f)_R.$$

Hence,

$$\det[I^{e_i}(e_j)]_{1 \le i,j \le k} = \left(\prod_{i=1}^k C(e_i) \right) \det \begin{pmatrix} (I^{e_1}, I^{e_1})_R & \ldots & (I^{e_1}, I^{e_k})_R \\ \vdots & \ddots & \vdots \\ (I^{e_k}, I^{e_1})_R & \ldots & (I^{e_k}, I^{e_k})_R \end{pmatrix}.$$

The matrix on the right hand side is a Gram matrix. Its determinant is the square volume of the parallelepiped spanned by its determining vectors, i.e.

$$\sqrt{\det \begin{pmatrix} (v_1, v_1) & \ldots & (v_1, v_n) \\ \vdots & \ddots & \vdots \\ (v_n, v_1) & \ldots & (v_n, v_n) \end{pmatrix}} = vol(v_1, \ldots, v_n).$$

To compute that determinant, let Z_i be the linear span of $I^{e_1}, \ldots, I^{e_{i-1}}$. Then we have

$$\det[I^{e_i}(e_j)]_{1 \le i,j \le k} = \prod_{i=1}^k C(e_i) \left\| P_{Z_i}^\perp I^{e_i} \right\|_R^2.$$

Let $E_i := \{e_1, \ldots, e_{i-1}\}$. Using Proposition 3.4 and Proposition 4.24 we have

$$\mu_{\mathcal{N}}[e_i \in T \mid e_1, \ldots, e_{i-1} \in T] = \mu_{\mathcal{N}}[e_i \in T_{G/E_i}] = I^{e_i}_{E_i}(e_i).$$

Since $P_{\bigstar_{E_i}}$ is an orthogonal projection for each E_i, we can compute

$$I^{e_i}_{E_i}(e_i) = C(e_i)(P_{\bigstar_{E_i}}\chi^{e_i}, \chi^{e_i})_R = C(e_i)(I^{e_i}_{E_i}, I^{e_i}_{E_i})_R = C(e_i)\left\|P^{\perp}_{Z_i} I^{e_i}\right\|^2_R$$

because $P_{\bigstar}\left\langle \chi^{E_i}\right\rangle = P_{\bigstar}\,\mathrm{span}\{\chi^{e_j} \mid j = 1, \ldots, i-1\} = Z_i$. Hence,

$$\begin{aligned}
\mu_{\mathcal{N}}[e_1, \ldots, e_k \in T] &= \prod_{i=1}^{k} \mu_{\mathcal{N}}[e_i \in T \mid e_1, \ldots, e_{i-1} \in T] \\
&= \prod_{i=1}^{k} I^{e_i}_{E_i}(e_i) = \prod_{i=1}^{k} C(e_i)\left\|P^{\perp}_{Z_i} I^{e_i}\right\|^2 \\
&= \det[I^{e_i}(e_j)]_{1\le i,j\le k}.
\end{aligned}$$

□

4.4 Potential theory and energy

Until now we have focused our attention mostly on $l^2_-(E)$ and have seen that the electrical currents, which are the objects of $l^2_-(E)$ that we are most interested in, are in $\bigstar = \nabla l^2(V)$. Hence, we will now shift our attention towards $l^2(V)$.

Throughout this section let $\mathcal{N} = (V, E, C) = (G, C)$ be a finite connected network.

Definition 4.27 (Potential)**.** Let I be an electrical current, i.e. I satisfies Kirchhoff's first (4.8) and second law (4.15), with source $x \in V$ and drain $y \in V$. Since $I \in \bigstar = \nabla l^2(V)$, there exists $\phi_I \in l^2(V)$ such that

$$I(f) = (-\nabla\phi_I)(f) = C(f)(\phi_I(\underline{f}) - \phi_I(\overline{f}))$$

and $\phi_I(y) = 0$. We call ϕ_I *potential* of I.

Remark (Ohm's Law)**.** Multiplying the characteristic equation of the potential ϕ_I by $R(f) = 1/C(f)$ we get

$$\phi_I(\underline{f}) - \phi_I(\overline{f}) = R(f) \cdot I(f).$$

In physics this is called *Ohm's Law* where the left side of the equation is interpreted as the voltage drop between $\underline{f}$ and $\overline{f}$.

Proposition 4.28 (Computing the potential)**.** *For an electrical current I with source x and drain y, ϕ_I is the unique solution of the PDE*

$$\begin{aligned} -\Delta\phi_I &= \frac{A(I)}{C_x}\mathbb{1}_x - \frac{A(I)}{C_y}\mathbb{1}_y \\ \phi_I(y) &= 0 \end{aligned}$$

or equivalently of the linear equation system

$$\begin{aligned} \phi_I(x) - \sum_{w\in V} \frac{C(x,w)}{C_x}\phi_I(w) &= \frac{A(I)}{C_x} \\ \phi_I(z) - \sum_{w\in V} \frac{C(z,w)}{C_z}\phi_I(w) &= 0 \quad \forall\ x \neq z \neq y \\ \phi_I(y) &= 0. \end{aligned}$$

Proof. By Lemma 4.9, we know that

$$-\Delta\phi_I = \operatorname{div}(-\nabla\phi_I) = \operatorname{div} I = \frac{A(I)}{C_x}\mathbb{1}_x - \frac{A(I)}{C_y}\mathbb{1}_y$$

and by Definition 4.27, $\phi_I(y) = 0$. Hence, ϕ_I solves the given PDE. Using the definition of the Laplacian (see Definition 4.5), the PDE without boundary condition $\phi_I(y) = 0$ is equivalent to the linear equation system

$$\begin{aligned} \phi_I(x) - \sum_{w\in V} \frac{C(x,w)}{C_x}\phi_I(w) &= \frac{A(I)}{C_x} \\ \phi_I(z) - \sum_{w\in V} \frac{C(z,w)}{C_z}\phi_I(w) &= 0 \quad \forall\ x \neq z \neq y \\ \phi_I(y) - \sum_{w\in V} \frac{C(y,w)}{C_y}\phi_I(w) &= -\frac{A(I)}{C_y}\ . \end{aligned}$$

This linear equation system contains $n := |V|$ variables and n equations. However, its coefficient matrix

$$(\mathcal{L})_{zw} = \begin{cases} -\frac{C(z,w)}{C_z} & , z \neq w \\ 1 - \frac{C(z,z)}{C_z} & , z = w \end{cases}, \ \ z, w \in V,$$

which is the matrix of Δ, does not satisfy $\det \mathcal{L} \neq 0$ since $\sum_{w\in V}(\mathcal{L}_{zw}) = 0$ for all $z \in V$. Thus, the system does not have a unique solution. However, replacing the last equation with $\phi_I(y) = 0$ will result in another system that has a unique solution and is equivalent to the PDE with boundary condition $\phi_I(y) = 0$. Indeed, let ϕ and ϕ' be two solutions of this altered linear equation system. Then $(\Delta\phi)(z) = (\Delta\phi')(z)$

for all $z \neq y$. However, it also follows that

$$\begin{aligned} 0 &= (0, \phi - \phi')_C = (\Delta 1, \phi - \phi')_C = (1, \Delta(\phi - \phi'))_C \\ &= \sum_{z \in V} C_z \cdot \Delta(\phi - \phi')(z) = \Delta(\phi - \phi')(y) \end{aligned}$$

by Lemma 4.7. Thus,

$$0 = \Delta(\phi - \phi') = \operatorname{div}(\nabla(\phi - \phi'))$$

holds. Hence, $\nabla(\phi - \phi') \in \bigstar$ and $\nabla(\phi - \phi') \in \diamondsuit$ by Corollary 4.14. It follows that $\nabla(\phi - \phi') = 0$ and therefore $\phi - \phi' \equiv r$ for some $r \in \mathbb{R}$. The altered equation implies $r = (\phi - \phi')(y) = \phi(y) - \phi'(y) = 0$, i.e. $\phi = \phi'$. Furthermore, ϕ solves the PDE since

$$C_x \cdot (-\Delta\phi)(x) + C_y \cdot (-\Delta\phi)(y) = (1, -\Delta\phi)_C = (\nabla 1, \nabla\phi)_R = (0, \nabla\phi)_R = 0.$$

Hence, $(-\Delta\phi)(y) = -C_x/C_y \cdot (-\Delta\phi)(x) = -A(I)/C_y$. ☐

Definition 4.29. We denote by

1. ϕ^e the potential of the unit current through $e \in E$.
2. ϕ^{xy} the potential of the unit current from x to y.

Lemma 4.30 (Superposition principle)**.** *Let $x, y \in V$ and $(e_1, \ldots, e_n)$ be a path $x \to y$. Then there exists $c \in \mathbb{R}$ such that*

$$\phi^{xy} = \sum_{i=1}^{n} \phi^{e_i} + c.$$

Proof. Let $\phi := \sum_{i=1}^{n} \phi^{e_i}$. Then

$$-\Delta\phi = \sum_{i=1}^{n} -\Delta\phi^{e_i} = \frac{1}{C_{\underline{e_1}}}\mathbb{1}_{\underline{e_1}} - \frac{1}{C_{\overline{e_n}}}\mathbb{1}_{\overline{e_n}} = \frac{1}{C_x}\mathbb{1}_x - \frac{1}{C_y}\mathbb{1}_y.$$

Furthermore, let

$$c = -\sum_{i=1}^{n} \phi^{e_i}(y) = -\phi(y).$$

Then $-\Delta(\phi + c) = -\Delta\phi$, and we obtain

$$\begin{aligned} -\Delta(\phi + c) &= \frac{1}{C_x}\mathbb{1}_x - \frac{1}{C_y}\mathbb{1}_y \\ (\phi + c)(y) &= 0. \end{aligned}$$

By Proposition 4.28, it follows that $\phi + c = \phi^{xy}$. ☐

Definition 4.31.

1. Let $J \in l^2_-(E)$. We call

$$\mathcal{E}_{\mathcal{N}}(J) := (J,J)_R = \frac{1}{2}\sum_{e\in E} R(e)J(e)^2 \geq 0$$

the *(dissipated) energy* of J and $\mathcal{E}_{\mathcal{N}}$ the *energy form* of $\mathcal{N}$.

2. For $f \in l^2(V)$, we define $\mathcal{E}_{\mathcal{N}}(f) := \mathcal{E}_{\mathcal{N}}(\nabla f)$.

Remarks.

1. We will omit the index $\mathcal{N}$ if the context makes it clear which network we want to consider.
2. Note that $\mathcal{E}_{\mathcal{N}}(J) = 0$ if and only if $J \equiv 0$ for all $J \in l^2_-(E)$.

Theorem 4.32 (Thomson principle, BLPS [1]). *Let I be an electrical current from x to y. Then I has minimal energy among all flows $J \in l^2_-(E)$ with $\operatorname{div} J = \operatorname{div} I$, i.e.*

$$\mathcal{E}(I) = \min\left\{\mathcal{E}(J) \mid J \in l^2_-(E), \operatorname{div} J = \operatorname{div} I\right\}.$$

Furthermore, $\mathcal{E}(J) = \mathcal{E}(I)$ if and only if $J = I$, i.e. the electrical current is the unique solution of the variational problem of minimizing energy for a given divergence.

Proof. Let $J \in l^2_-(E)$ with $\operatorname{div} J = \operatorname{div} I = \frac{A(I)}{C_x}\mathbb{1}_x - \frac{A(I)}{C_y}\mathbb{1}_y$. Then $\operatorname{div}(J - I) = 0$, i.e. $J - I \in \diamondsuit$ and since $I \in \bigstar$, we have $I \perp J - I$ by Proposition 4.13. Hence,

$$\begin{aligned}\mathcal{E}(J) &= (J,J)_R = (I + J - I, I + J - I)_R \\ &= (I,I)_R + (J - I, I)_R + (I, J - I)_R + (J - I, J - I)_R \\ &= \mathcal{E}(I) + \mathcal{E}(J - I) \geq \mathcal{E}(I).\end{aligned}$$

Furthermore, we have $\mathcal{E}(J) = \mathcal{E}(I)$ if and only if $\mathcal{E}(J - I) = 0$, which is equivalent to $J = I$. □

Lemma 4.33 (Energy of currents). *Let $I = -\nabla\phi_I$ be an electrical current from x to y. Then*

$$\mathcal{E}(I) = A(I) \cdot (\phi_I(x) - \phi_I(y)).$$

In particular, $\mathcal{E}(I^e) = R(e) \cdot I^e(e)$ for all $e \in E$.

Proof.

$$\mathcal{E}(I) = \mathcal{E}(-\nabla\phi_I) = (-\nabla\phi_I, -\nabla\phi_I)_R \overset{4.7}{=} (\phi_I, -\Delta\phi_I)_C \overset{4.28}{=} A(I)\cdot(\phi_I(x) - \phi_I(y))$$

Hence, $\mathcal{E}(I^e) = A(I^e)\cdot(\phi^e(\underline{e}) - \phi^e(\overline{e})) = 1\cdot R(e)\cdot I^e(e)$ by Ohm's Law. □

5 Effective resistance of finite networks

Suppose there is an electrical current flowing through a network from a single source x to a single drain y. If we interpret the whole network as a conductor with endpoints x and y, we can determine its resistance. This value is called the network's effective resistance between x and y. We will discover that it not only has strong connections to electrical currents but also has a nice probabilistic interpretation as well as a connection to the Green's function of networks Laplacian. However, before we define and examine the effective resistance in Section 5.2, we will investigate the properties of harmonic functions on networks.

5.1 Harmonic functions on V

Let $\mathcal{N} = (V, E, C) = (G, C)$ be any connected network, finite or infinite. Recall the definition of harmonicity of a function on V (Definition 4.5). We will show three basic but also very useful principles regarding harmonic functions.

Proposition 5.1 (Existence of harmonic functions). *Let $W \subsetneq V$ and $f : W \to \mathbb{R}$ be any bounded function. Then there exists $h : V \to \mathbb{R}$ such that h is harmonic on $V \setminus W$ and $h \restriction_W = f$.*

Proof. Let $h(x) := \mathbb{E}_x[f(Y_{\tau_W})\cdot \mathbb{1}_{\{\tau_W<\infty\}}]$. For $x \in W$, we have $\tau_W = 0$ and $h(x) = \mathbb{E}_x[f(Y_0)] = f(x)$. For $x \in V \setminus W$, we compute

$$\begin{aligned}
h(x) = \mathbb{E}_x[f(Y_{\tau_W})\cdot \mathbb{1}_{\{\tau_W<\infty\}}] &= \sum_{n=1}^{\infty} \mathbb{E}_x[f(Y_n)\mathbb{1}_{\{\tau_W=n\}}] \\
&= \sum_{n=1}^{\infty}\sum_{y\in W} \mathbb{P}_x[Y_n = y, \tau_W = n]\cdot f(y) \\
&= \sum_{y\in W} \mathbb{P}_x[Y_1 = y, \tau_W = 1]\cdot f(y) + \sum_{n=2}^{\infty}\sum_{y\in W} \mathbb{P}_x[Y_n = y, \tau_W = n]\cdot f(y) \\
&= \sum_{y\in W} \frac{C(x,y)}{C_x} f(y) + \sum_{n=2}^{\infty}\sum_{y\in W}\sum_{z\in V\setminus W} \mathbb{P}_x[Y_n = y, \tau_W = n \mid Y_1 = z]\mathbb{P}_x[Y_1 = z] f(y) \\
&= \sum_{y\in W} \frac{C(x,y)}{C_x} f(y) + \sum_{n=2}^{\infty}\sum_{y\in W}\sum_{z\in V\setminus W} \mathbb{P}_z[Y_{n-1} = y, \tau_W = n-1]\frac{C(x,z)}{C_x} f(y)
\end{aligned}$$

$$= \sum_{y \in W} \frac{C(x,y)}{C_x} f(y) + \sum_{z \in V \setminus W} \frac{C(x,z)}{C_x} \sum_{n=1}^{\infty} \sum_{y \in W} \mathbb{P}_z[Y_n = y, \tau_W = n] f(y)$$

$$= \sum_{y \in W} \frac{C(x,y)}{C_x} h(y) + \sum_{z \in V \setminus W} \frac{C(x,z)}{C_x} h(z) = \sum_{y \in V} \frac{C(x,y)}{C_x} \cdot h(y).$$

Hence, h is harmonic on $V \setminus W$. □

Definition 5.2 (Neighborhood)**.** Let $v \in V$ and $W \subseteq V$. Then we define

$$B(v) := \{x \in V \mid C(v,x) > 0\} \;,\; B(W) := \bigcup_{w \in W} B(w)$$

to be the *neighborhood* of v and W, respectively.

Lemma 5.3 (Maximum principle)**.** *Let $W \subseteq V$ such that the subgraph $G(W)$ of G generated by W is connected, and let $f : V \to \mathbb{R}$ be harmonic on W. If*

$$\sup_{v \in V} f(v) = f(w_0)$$

for some $w_0 \in W$, then $f \restriction_{B(W)}$ is constant.

Proof. We have $f(w_0) \geq f(v)$ for all $v \in V$ and

$$0 = (-\Delta f)(w_0) = f(w_0) - \sum_{v \in V} \frac{C(v,w_0)}{C_{w_0}} f(v)$$
$$= \sum_{v \in B(w_0)} \frac{C(v,w_0)}{C_{w_0}} f(w_0) - \sum_{v \in B(w_0)} \frac{C(v,w_0)}{C_{w_0}} f(v)$$
$$= \sum_{v \in B(w_0)} \underbrace{\frac{C(v,w_0)}{C_{w_0}}}_{>0} \underbrace{(f(w_0) - f(v))}_{\geq 0}.$$

It follows that $f(w_0) = f(v)$ for all $v \in B(w_0)$. Now let $w \in B(W)$. Then there exists a path $w_0 \to w$ in G. Applying the above argument to each vertex in this path produces $f(w_0) = f(w)$. □

Corollary 5.4 (Minimum principle)**.** *Let $W \subseteq V$ such that the subgraph $G(W)$ of G generated by W is connected, and let $f : V \to \mathbb{R}$ be harmonic on W. If*

$$\inf_{v\in V} f(v) = f(w_0)$$

for some $w_0 \in W$*, then* $f \restriction_{B(W)}$ *is constant.*

Proof. Applying Lemma 5.3 to $g := -f$, which is harmonic on W if and only if f is harmonic on W, gives the desired result. □

Corollary 5.5 (Extrema of harmonic functions on finite networks)**.** *Let* $\mathcal{N}$ *be finite,* $\emptyset \neq W \subseteq V$*, and let* $f : V \to \mathbb{R}$ *be harmonic on* $V \setminus W$*. Then*

$$\min_{w\in W} f(w) \le f(v) \le \max_{w\in W} f(w)$$

holds for all $v \in V$*, i.e.* f *attains its extrema on* W*.*

Proof. We only show $\max_W f = \max_V f$ since $\min_W f = \min_V f$ can be proven analogously. Since V is finite, we know that f attains its maximum at some vertex $v^* \in V$. If $v^* \in W$, then we are done. If $v^* \in V \setminus W$, then let $G^* = (V^*, E^*)$ be the connected component of $G(V \setminus W)$ that contains v^*. The Maximum principle implies that f is constant on $B(V^*)$. Since G is connected, $B(V^*) \neq V^*$, i.e. $B(V^*) \cap W \neq \emptyset$. Hence, there exists $w^* \in B(V^*) \cap W$ such that $f(w^*) = \max_V f$. It follows that

$$\max_{w\in W} f(w) \le \max_{v\in V} f(v) = f(w^*) \le \max_{w\in W} f(w)$$

and therefore $\max_W f = \max_V f$. □

Example 5.6. Let $\mathcal{N} = (V, E, C) = (G, C)$ be a finite network and $x, y \in V$, $x \neq y$. Corollary 5.5 implies that the potential ϕ^{xy} of the unit current from x to y in $\mathcal{N}$ satisfies

$$\sup_{v\in V} \phi^{xy}(v) = \phi^{xy}(x) \ , \quad \inf_{v\in V} \phi^{xy}(v) = \phi^{xy}(y) = 0$$

because Lemma 4.33 states that $0 \le \mathcal{E}(I^{xy}) = \phi^{xy}(x) - \phi^{xy}(y) = \phi^{xy}(x)$ holds.

Proposition 5.7 (Uniqueness of harmonic functions on finite networks)**.** *Let* $\mathcal{N}$ *be finite and* $x, y \in V$*. Furthermore, let* $f, g : V \to \mathbb{R}$ *be harmonic on* $V \setminus \{x, y\}$ *satisfying* $f(x) = g(x)$ *and* $f(y) = g(y)$*. Then* $f = g$ *follows.*

Proof. Consider $h := f - g$ and let $G_i = (V_i, E_i), i = 1, \dots, m$ be the connected components of the subgraph $G(V \setminus \{x, y\})$ of G generated by $V \setminus \{x, y\}$. h is then harmonic on each V_i and satisfies $h(x) = h(y) = 0$. From the Maximum principle and the Minimum principle it follows that $h \restriction_{V_i} \equiv 0$ for all $i = 1, \dots, m$. Hence, $h \equiv 0$ on and therefore $f = g$. □

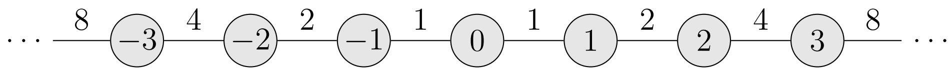

Figure 5.1: The network $(G(\mathbb{Z}), C)$.

Example 5.8. Unfortunately, the Uniqueness principle for harmonic functions is in general false on infinite networks: Let $E_{\mathbb{Z}} = \{(z, z+1, 1), (z+1, z, 1) \mid z \in \mathbb{Z}\}$, $G(\mathbb{Z}) := (\mathbb{Z}, E_{\mathbb{Z}})$ and consider the network $(G(\mathbb{Z}), C)$ where

$$C(e) = 2^{\min(|\overline{e}|, |\underline{e}|)}.$$

A part of this network's diagram is shown in Figure 5.1. Now let

$$f(z) := \operatorname{sign}(z) \cdot \left(2 - 2^{1-|z|}\right).$$

It follows that

$$f(0) = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = \frac{C(0,-1)}{C_0} f(-1) + \frac{C(0,1)}{C_0} f(1) = \sum_{z \in \mathbb{Z}} \frac{C(0,z)}{C_0} f(z).$$

This implies $(\Delta f)(0) = 0$. For $z > 0$, we have $C_z = 2^{z-1} + 2^z$ and

$$\begin{aligned}
\sum_{y \in Z} \frac{C(z,y)}{C_z} f(y) &= \frac{C(z,z-1)}{C_z} f(z-1) + \frac{C(z,z+1)}{C_z} f(z+1) \\
&= \frac{2^{z-1}}{2^{z-1} + 2^z}(2 - 2^{2-z}) + \frac{2^z}{2^{z-1} + 2^z}(2 - 2^{-z}) \\
&= \frac{1}{3}(2 - 2^{2-z}) + \frac{2}{3}(2 - 2^{-z}) = 2 - \frac{1}{3}(2^{2-z} + 2^{1-z}) \\
&= 2 - \frac{1}{3} \cdot 2^{1-z} \cdot 3 = 2 - 2^{1-z} = f(z).
\end{aligned}$$

Hence, $(\Delta f)(z) = 0$ for all $z > 0$. For $z < 0$ we have

$$C(z, z-1) = C(-z, -z+1)\ , f(z-1) = -f(-z+1)$$

$$C(z, z+1) = C(-z, -z-1),\ f(z+1) = -f(-z-1)$$

which implies $(\Delta f)(z) = -(\Delta f)(-z) = 0$. Hence, f is harmonic on $\mathbb{Z}$ and satisfies $f(-1) = f(1) = 1$. It follows that both f and the constant function 1 are harmonic on $\mathbb{Z} \setminus \{-1, 1\}$ and agree on $\{-1, 1\}$, but they are obviously not the same function.

5.2 Effective resistance and the Green's function

The first part of this section follows Tetali's work [9] on the effective resistance of finite networks. In the second part we adopt a theory Kasue develops in [4] about the effective resistance on metric graphs and its connection to certain Green's functions.

Throughout this section let $\mathcal{N} = (V, E, C) = (G, C)$ be a finite connected network.

Definition 5.9 (Effective resistance)**.** For $x, y \in V$, let I^{xy} be the unit current from x to y, and define the *effective resistance between x and y* by

$$R_{\mathcal{N}}(x, y) = \mathcal{E}(I^{xy}) = \phi^{xy}(x).$$

Remark (Physical interpretation)**.** Fix two vertices $x, y \in V$ and consider a battery being hooked up between them such that unit current flows through the entire network from x to y, and let $\mathcal{V}$ denote the voltage needed to do this. If we now consider a network consisting only of two vertices x and y and one edge e connecting them, one could pose the following question: What resistance $\mathcal{R}$ must e have so that if we hook up a battery with voltage $\mathcal{V}$ to this network, unit current flows from x to y?

The answer is provided by Ohm's Law. It states that the current $\mathcal{I}$ flowing through this one-conductor network satisfies

$$\mathcal{V} = \mathcal{R} \cdot \mathcal{I}.$$

Since we want to achieve unit current ($\mathcal{I} = 1$), we get

$$\mathcal{R} = \mathcal{V} = \phi^{xy}(x) - \phi^{xy}(y) = \phi^{xy}(x) = R_{\mathcal{N}}(x, y).$$

Hence, $R_{\mathcal{N}}(x, y)$ can be interpreted as the resistance of the whole network $\mathcal{N}$ between x and y.

Examples 5.10.

1. We want to compute the effective resistance within the simple series circuit

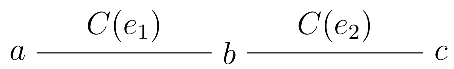

Figure 5.2: Two resistors connected in series.

which is shown in Figure 5.2. By definition of ϕ^{ac} (Definition 4.27), we have

$$\begin{aligned} R(a, c) = \phi^{ac}(a) = \phi^{ac}(a) - \phi^{ac}(c) &= \phi^{ac}(a) - \phi^{ac}(b) + \phi^{ac}(b) - \phi^{ac}(c) \\ &= R(e_1)I^{ac}(e_1) + R(e_2)I^{ac}(e_2) = R(e_1) + R(e_2). \end{aligned}$$

The last equality uses Kirchhoff's Current Law 4.8

$$1 = A(I^{ac}) = \sum_{e \in E(a)} I^{ac}(e) = I^{ac}(e_1)$$

and

$$1 = A(I^{ac}) = -\sum_{e \in E(c)} I^{ac}(e) = -(-I^{ac}(e_2)) = I^{ac}(e_2).$$

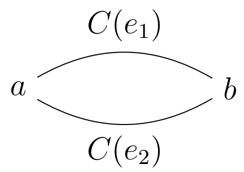

Figure 5.3: Two resistor connected in parallel.

2. We want to compute the effective resistance of the simple parallel circuit which is shown in Figure 5.3. Kirchhoff's Current Law 4.8 implies

$$\begin{aligned}1 = A(I^{ab}) &= \sum_{e \in E(a)} I^{ab}(e) = I^{ab}(e_1) + I^{ab}(e_2) \\ &= \frac{\phi^{ab}(a) - \phi^{ab}(b)}{R(e_1)} + \frac{\phi^{ab}(a) - \phi^{ab}(b)}{R(e_2)} = \left(\frac{1}{R(e_2)} + \frac{1}{R(e_2)}\right) \cdot \phi^{ab}(a)\end{aligned}$$

Hence,

$$R(a,b) = \phi^{ab}(a) = \frac{1}{\frac{1}{R(e_1)} + \frac{1}{R(e_1)}} .$$

Proposition 5.11 (Tetali [9]). *Let U_z^{xy} be the expected number of times a RNW starting in x hits z before hitting y (see Definition 3.18). Then*

$$R(x,y) = \frac{U_x^{xy}}{C_x}.$$

Proof. If $x = y$, then $U_x^{xx} = 0$ by Definition 3.18 and $R(x,x) = 0$. It follows that the desired equality holds.

Now assume $x \neq y$. Then we have $U_y^{xy} = 0$ and, by Lemma 3.19,

$$U_z^{xy} - \sum_{w \in V} \frac{C(w,z)}{C_w} U_w^{xy} \quad \forall\, x \neq z \neq y$$

and

$$U_x^{xy} - \sum_{w \in V} \frac{C(w,x)}{C_w} U_w^{xy} = 1.$$

Hence, $f(z) := \frac{U_z^{xy}}{C_z}$ solves the linear equation system which defines ϕ^{xy} uniquely (compare Proposition 4.28). It follows that $\phi^{xy} = f$. ☐

Remark. If $C(e) = 1$ for all edges $e \in E$, the effective resistance $R(x,y)$ can be interpreted as the expected number of traversals out of x along any specific edge e.

Proposition 5.12 (Tetali [9])**.** *Let $z \in V$. Then*

$$R(x,y) = \frac{U_z^{xy} + U_z^{yx}}{C_z} \ .$$

Proof. Let $\phi(z) := \phi^{xy}(z) + \phi^{yx}(z)$ and $(e_1, \dots, e_n)$ be a path $x \to y$. By Definition 4.17, we have

$$I^{xy} = P_\star \left(\sum_{i=1}^{n} \chi^{e_i} \right) = P_\star \left(\sum_{i=1}^{n} -\chi^{\widehat{e_i}} \right) = -I^{yx}.$$

Hence,

$$-\nabla\phi = -\nabla(\phi^{xy} + \phi^{yx}) = -\nabla\phi^{xy} - \nabla\phi^{yx} = I^{xy} + I^{yx} = 0.$$

This implies that ϕ is constant on V. For instance, $\phi(x) = \phi^{xy}(x) + \phi^{yx}(x) = \phi^{xy}(x)$. In the proof of Proposition 5.11 we saw that $\frac{U_z^{xy}}{C_z} = \phi^{xy}(z)$. Hence,

$$\frac{U_z^{xy} + U_z^{yx}}{C_z} = \frac{U_z^{xy}}{C_z} + \frac{U_z^{yx}}{C_z} = \phi^{xy}(z) + \phi^{yx}(z) = \phi(z) = \phi^{xy}(x) = R(x,y).$$

□

Remark (Interpretation)**.** We consider a RNW starting in x, traveling to y and then returning to x. The expected number of times such a round-trip hits z is exactly $U_z^{xy} + U_z^{yx}$.

Corollary 5.13 (Tetali [9])**.** *Let $x, y \in V$ and let $Com(x,y) := \mathbb{E}_x[\tau_y] + \mathbb{E}_y[\tau_x]$ be the* commute time *between x and y, i.e. the expected time a RNW needs to travel from x to y and back to x again. Then*

$$Com(x,y) = R(x,y) \cdot C_G$$

where $C_G := \sum_{z \in V} C_z = \sum_{e \in E} C(e)$.

Proof (Tetali [9]). By Proposition 5.12, it follows that

$$\begin{aligned}
Com(x,y) = \mathbb{E}_x[\tau_y] + \mathbb{E}_y[\tau_x] &= \mathbb{E}_x\left[\sum_{k=0}^{\tau_y - 1} 1\right] + \mathbb{E}_y\left[\sum_{k=0}^{\tau_x - 1} 1\right] \\
&= \mathbb{E}_x\left[\sum_{k=0}^{\tau_y - 1} \sum_{z \in V} \mathbb{1}_z(Y_k)\right] + \mathbb{E}_y\left[\sum_{k=0}^{\tau_x - 1} \sum_{z \in V} \mathbb{1}_z(Y_k)\right] \\
&= \sum_{z \in V} \left(\mathbb{E}_x\left[\sum_{k=0}^{\tau_y - 1} \mathbb{1}_z(Y_k)\right] + \mathbb{E}_y\left[\sum_{k=0}^{\tau_x - 1} \mathbb{1}_z(Y_k)\right] \right) \\
&= \sum_{z \in V} (U_z^{xy} + U_z^{yx}) = \sum_{z \in V} C_z \cdot R(x,y) = R(x,y) \cdot C_G.
\end{aligned}$$

□

Proposition 5.14 (Reciprocity Law)**.** *Let $x, x', y, y' \in V$ be distinct vertices. Then*

$$\phi^{xy}(x') - \phi^{xy}(y') = \phi^{x'y'}(x) - \phi^{x'y'}(y).$$

Proof. By Lemma 4.23, we have

$$\phi^f(\underline{e}) - \phi^f(\overline{e}) = R(e)I^f(e) = R(f)I^e(f) = \phi^e(\underline{f}) - \phi^e(\overline{f}) \quad \forall\, e, f \in E$$

Now let $(e_1, \ldots, e_m)$ be a path $x \to y$ and $(f_1, \ldots, f_n)$ be a path $x' \to y'$. By the Superposition principle (Lemma 4.30), there exist $c, d \in \mathbb{R}$ such that

$$\phi^{xy} = \sum_{i=1}^{m} \phi^{e_i} + c\ ,\ \ \phi^{x'y'} = \sum_{j=1}^{n} \phi^{f_j} + d.$$

Hence,

$$\begin{aligned}
\phi^{xy}(x') - \phi^{xy}(y') &= \sum_{j=1}^{n} \phi^{xy}(\underline{f_j}) - \phi^{xy}(\overline{f_j}) = \sum_{j=1}^{n}\sum_{i=1}^{m} \phi^{e_i}(\underline{f_j}) - \phi^{e_i}(\overline{f_j}) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \phi^{f_j}(\underline{e_i}) - \phi^{f_j}(\overline{e_i}) = \sum_{i=1}^{m} \phi^{x'y'}(\underline{e_i}) - \phi^{x'y'}(\overline{e_i}) \\
&= \phi^{x'y'}(x) - \phi^{x'y'}(y).
\end{aligned}$$

□

Remark (Physical interpretation)**.** We introduce a single current source to our network between the vertices x and y and measure the occurring voltage between x' and y'. The Reciprocity Law states that this will yield the same result as introducing the same current source between x' and y' and measuring the voltage between x and y.

Corollary 5.15 (Tetali [9])**.** *For $x, y, z \in V$, we have*

$$\frac{U_z^{xy}}{C_z} = \frac{U_x^{zy}}{C_x}\ .$$

Proof. As seen in the proof of Proposition 5.11, we have $\phi^{xy}(z) = \frac{U_z^{xy}}{C_z}$. Using the Reciprocity Law, we obtain

$$\frac{U_z^{xy}}{C_z} = \phi^{xy}(z) = \phi^{xy}(z) - \phi^{xy}(y) = \phi^{zy}(x) - \phi^{zy}(y) = \phi^{zy}(x) = \frac{U_x^{zy}}{C_x}\ .$$

□

Theorem 5.16 (Tetali [9])**.** *The effective resistance $R(x,y)$ is a metric.*

Proof (Tetali [9]).

1. $R(x,x) = 0$ for all $x \in V$ is clear.
2. $R(x,y) = \frac{U_z^{xy}+U_z^{yx}}{C_z} = R(y,x)$ is due to Proposition 5.12.
3. The triangle inequality: Consider three vertices x, y, z. Then

$$\begin{aligned} R(x,z) + R(z,y) - R(x,y) &\overset{5.12}{=} \frac{U_y^{xz}+U_y^{zx}}{C_y} + \frac{U_x^{zy}+U_x^{yz}}{C_x} - \frac{U_z^{xy}+U_z^{yx}}{C_z} \\ &\overset{5.15}{=} \frac{U_y^{xz}}{C_y} + \frac{U_z^{yx}}{C_z} + \frac{U_z^{xy}}{C_z} + \frac{U_x^{yz}}{C_x} - \frac{U_z^{xy}}{C_z} - \frac{U_z^{yx}}{C_z} \\ &= \frac{U_y^{xz}}{C_y} + \frac{U_x^{yz}}{C_x} \overset{5.15}{=} 2\frac{U_y^{xz}}{C_y} \geq 0. \end{aligned}$$

Hence,

$$R(x,z) + R(z,y) \geq R(x,y) \quad \forall\, x,y,z \in V.$$

□

Corollary 5.17. *Let x, y, z be distinct vertices. Then*

$$R(x,z) + R(z,y) = R(x,y) \Leftrightarrow \mathbb{P}_x(\tau_z < \tau_y) = 1.$$

Proof. By Theorem 5.16, we get the equality

$$R(x,z) + R(z,y) - R(x,y) = 2\frac{U_y^{xz}}{C_y}.$$

This implies

$$R(x,z) + R(z,y) = R(x,y) \Leftrightarrow U_y^{xz} = 0.$$

U_y^{xz} is the expected number of times we visit y on a RNW from x to z. This can only equal zero if y is almost surely never visited by any RNW from x to z. In other words, every RNW from x to y will visit z with probability 1 before reaching y. □

Definition 5.18 (Following Kasue [4])**.** For $x, y, z \in V$, we define

1. $G_z(x,y) := \frac{1}{2}\left(R(x,z) + R(z,y) - R(x,y)\right)$.
2. $g_{x,z}(y) := G_z(x,y)$.

Lemma 5.19 (Basic properties of the Green's function)**.** *Let $x, y, z \in V$. Then*

1. $g_{x,z}(y) = G_z(x,y) = G_z(y,x) = g_{y,z}(x)$.
2. $g_{x,z}(y) = G_z(x,y) = \frac{U_y^{xz}}{C_y}$.

3. $g_{x,z}(z) = G_z(x,z) = 0.$
4. $\Delta g_{x,z} = \mathbb{1}_z \frac{1}{C_z} - \mathbb{1}_x \frac{1}{C_x}.$

Proof.

1. By Theorem 5.16, it follows that

$$G_z(x,y) = \frac{1}{2}(R(x,z) + R(z,y) - R(x,y)) = \frac{1}{2}(R(y,z) + R(z,x) - R(y,x))$$
$$= G_z(y,x).$$

2. Following the proof of the triangle inequality for $R(x,y)$ (Theorem 5.16) and using the Reciprocity property (Corollary 5.15), we get

$$R(x,z) + R(z,y) - R(x,y) = \frac{U_y^{xz}}{C_y} + \frac{U_x^{yz}}{C_x} = 2\frac{U_y^{xz}}{C_y}$$

and hence

$$g_{x,z}(y) = G_z(x,y) = \frac{U_y^{xz}}{C_y} \ .$$

3. By definition, we have $g_{x,z}(z) = G_z(x,z) = \frac{1}{2}\left(R(x,z) + R(z,z) - R(x,z)\right) = 0.$
4. First, note that by properties 3 and 1, we have

$$g_{z,z}(x) = G_z(z,x) = G_z(x,z) = 0 \quad \forall\ x \in V$$

and hence $\Delta g_{z,z} \equiv 0$. Thus, we can use the more complicated term

$$\Delta g_{z,z} = \mathbb{1}_z \frac{1}{C_z} - \mathbb{1}_z \frac{1}{C_z} \ .$$

Now assume $x \neq z$, and let $y \in V$. Then

$$\begin{aligned}
(\Delta g_{x,z})(y) &= \sum_{w \in V} \frac{C(y,w)}{C_y} g_{x,z}(w) - g_{x,z}(y) \\
&= \sum_{w \in V} \frac{C(y,w)}{C_y} \frac{U_w^{xz}}{C_w} - \frac{U_y^{xz}}{C_y} \\
&= \frac{1}{C_y} \left(\sum_{w \in V} \frac{C(w,y)}{C_w} U_w^{xz} - U_y^{xz} \right).
\end{aligned}$$

By Lemma 3.19, we have

$$\sum_{w \in V} \frac{C(w,y)}{C_w} U_w^{xz} - U_y^{xz} = 0 \quad \forall\ x \neq y \neq z$$

and

$$\sum_{w \in V} \frac{C(w,x)}{C_w} U_w^{xz} - U_x^{xz} = -1.$$

Hence,

$$(\Delta g_{x,z})(y) = 0 \quad \forall\ x \neq y \neq z.$$

and

$$(\Delta g_{x,z})(x) = -\frac{1}{C_x}\ .$$

Using both equalities, we compute

$$\begin{aligned} 0 &= (0, \nabla g_{x,z})_R = (-\nabla 1, \nabla g_{x,z})_R = (1, \Delta g_{x,z})_C = \sum_{w \in V} C_w \cdot \Delta g_{x,z}(w) \\ &= C_x \cdot \Delta g_{x,z}(x) + C_z \cdot \Delta g_{x,z}(z) = -1 + C_z \cdot \Delta g_{x,z}(z). \end{aligned}$$

It follows that

$$\Delta g_{x,z}(z) = \frac{1}{C_z}\ .$$

□

Corollary 5.20. *The function $g_{x,z}$ is harmonic on $V \setminus \{x, z\}$.*

Proof. This is exactly property 4 of Lemma 5.19. □

Theorem 5.21. *$g_{x,z}$ is the Green's function for the PDE*

$$\begin{aligned} -\Delta u &= f \quad \text{on } V \setminus \{z\} \\ u(z) &= 0. \end{aligned}$$

Proof. Let $u(x) := (g_{x,z}, f)_C$. It follows that

$$u(x) = (g_{x,z}, f)_C = \sum_{y \in V} C_y \cdot g_{x,z}(y) \cdot f(y) = \sum_{y \in V} C_y \cdot g_{y,z}(x) \cdot f(y)$$

for all $x \in V$. Hence,

$$u(z) = \sum_{y \in V} C_y \cdot \underbrace{g_{y,z}(z)}_{=0} \cdot f(y) = 0$$

and

$$-\Delta u = -\sum_{y \in V} C_y \cdot \Delta g_{y,z} \cdot f(y) = \sum_{y \in V} C_y \cdot \left(\frac{1}{C_y} \mathbb{1}_y - \frac{1}{C_z} \mathbb{1}_z \right) \cdot f(y)$$

hold, i.e. for $x \neq z$, we have

$$(\Delta u)(x) = C_x \cdot \frac{1}{C_x} \cdot f(x) = f(x).$$

□

Proposition 5.22 (The Green's function as a potential)**.** *Let $x, z \in V$. Then $g_{x,z} = \phi^{xz}$ and*

$$-\nabla g_{x,z} = I^{xz}.$$

Proof. By Lemma 5.19, we have

$$-\Delta g_{x,z} = \frac{1}{C_x}\mathbb{1}_x - \frac{1}{C_z}\mathbb{1}_z$$

and $g_{x,z}(z) = 0$. Proposition 4.28 implies $g_{x,z} = \phi^{xy}$. □

Corollary 5.23. *Let $x, y \in V$, $e \in E$. Then*

$$I^{xy}(e) = \frac{C(e)}{2}\left(R(x,\overline{e}) - R(x,\underline{e}) - (R(y,\overline{e}) - R(y,\underline{e}))\right).$$

In particular, if $e \in E(x,y)$, then

$$I^{xy}(e) = I^e(e) = C(e) \cdot R(x,y).$$

Proof. Using Proposition 5.22 and the definition of $g_{x,y}$, we compute

$$\begin{aligned} I^{xy}(e) &= (-\nabla g_{x,y})(e) = C(e)(g_{x,y}(\underline{e}) - g_{x,y}(\overline{e})) \\ &= \frac{C(e)}{2}(R(x,y) + R(y,\underline{e}) - R(x,\underline{e}) - (R(x,y) + R(y,\overline{e}) - R(x,\overline{e})) \\ &= \frac{C(e)}{2}(R(x,\overline{e}) - R(x,\underline{e}) - (R(y,\overline{e}) - R(y,\underline{e}))). \end{aligned}$$

□

5.3 The simple structure of a network

Let $\mathcal{N} = (V, E, C) = (G, C)$ be a finite connected network. By Theorem 5.16, we know that the effective resistance is a metric on V. It is thus an obvious question how much of the graph's structure is encoded within its effective resistance.

Lemma 5.24. *Loops do not affect the effective resistance.*

Proof. We fix $v \in V$ and $r > 0$ and add a loop at v with edge weight r. This means we consider the network $\mathcal{N}' := (V, E', C')$ where

$$E' = E \cup \{(v, v, n)\}$$

for some $n \in \mathbb{N}$ such that $(v, v, n) \notin E$ and

$$C'(e) = \begin{cases} C(e) & , e \in E \\ r & , e = (v, v, n) \end{cases}.$$

Let $x, y \in V$, $x \neq y$, and let ϕ^{xy} potential of I^{xy} in $\mathcal{N}$. We will show that ϕ^{xy} solves the linear equation system in Proposition 4.28 for the network $\mathcal{N}'$. It then follows that ϕ^{xy} is the potential of the unit current from x to y in $\mathcal{N}'$ and therefore

$$R_{\mathcal{N}}(x, y) = \phi^{xy}(x) = R_{\mathcal{N}'}(x, y).$$

We have $\phi^{xy}(y) = 0$ by the definition of ϕ^{xy}. For $z \neq v$ we have $C'(z, w) = C(z, w)$ for all $w \in V$ and $C'_z = C_z$. Furthermore, $C'(v, v) = C(v, v) + r$, $C'(v, w) = C(v, w)$ for all $w \neq v$ and $C'_v = C_v + r$. Hence, for $z \neq v$ we have

$$\phi^{xy}(z) - \sum_{w \in V} \frac{C'(z, w)}{C'_z} \phi^{xy}(w) = \phi^{xy}(z) - \sum_{w \in V} \frac{C(z, w)}{C_z} \phi^{xy}(w)$$

and

$$\begin{aligned}
\phi^{xy}(v) - \sum_{w \in V} \frac{C'(v, w)}{C'_v} \phi^{xy}(w) &= \frac{C_v}{C'_v} \left(\frac{C'_v}{C_v} \phi^{xy}(v) - \sum_{w \in V} \frac{C'(v, w)}{C_v} \phi^{xy}(w) \right) \\
= \frac{C_v}{C'_v} \left(\left(1 + \frac{r}{C_v} \right) \phi^{xy}(v) - \sum_{w \in V} \frac{C(v, w)}{C_v} \phi^{xy}(w) - \frac{r}{C_v} \phi^{xy}(v) \right) \\
= \frac{C_v}{C'_v} \left(\phi^{xy}(v) - \sum_{w \in V} \frac{C(v, w)}{C_v} \phi^{xy}(w) \right).
\end{aligned}$$

Hence, if $x \neq v$, it follows that

$$\begin{aligned}
\phi^{xy}(x) - \sum_{w \in V} \frac{C'(x, w)}{C'_x} \phi^{xy}(w) &= \phi^{xy}(x) - \sum_{w \in V} \frac{C(x, w)}{C_x} \phi^{xy}(w) = \frac{1}{C_x} = \frac{1}{C'_x}, \\
\phi^{xy}(z) - \sum_{w \in V} \frac{C'(z, w)}{C'_z} \phi^{xy}(w) &= \phi^{xy}(z) - \sum_{w \in V} \frac{C(z, w)}{C_z} \phi^{xy}(w) = 0 \quad \forall\, z \in V \setminus \{x, v\}, \\
\phi^{xy}(v) - \sum_{w \in V} \frac{C'(v, w)}{C'_v} \phi^{xy}(w) &= \frac{C_v}{C'_v} \left(\phi^{xy}(v) - \sum_{w \in V} \frac{C(v, w)}{C_v} \phi^{xy}(w) \right) = \frac{C_v}{C'_v} \cdot 0 = 0.
\end{aligned}$$

On the other hand, if $x = v$, we have

$$\begin{aligned}
\phi^{vy}(v) - \sum_{w \in V} \frac{C'(v, w)}{C'_v} \phi^{vy}(w) &= \frac{C_v}{C'_v} \left(\phi^{vy}(v) - \sum_{w \in V} \frac{C(v, w)}{C_v} \phi^{vy}(w) \right) = \frac{C_v}{C'_v} \frac{1}{C_v} = \frac{1}{C'_v}, \\
\phi^{vy}(z) - \sum_{w \in V} \frac{C'(z, w)}{C'_z} \phi^{vy}(w) &= \phi^{vy}(z) - \sum_{w \in V} \frac{C(z, w)}{C_z} \phi^{vy}(w) = 0 \quad \forall\, z \in V \setminus \{v\}.
\end{aligned}$$

The claim follows. □

Example 5.25. Consider the networks depicted in Figure 5.4. Both induce the same metric space $(\{a, b\}, R)$ where $R(a, b) = 1$ since loops do not affect the effective resistance.

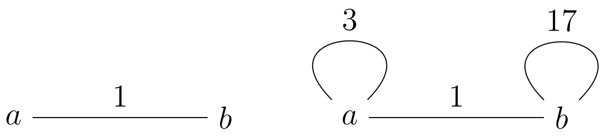

Figure 5.4: Two networks with the same effective resistance.

Lemma 5.26. *Let (V,E,C) be a finite network, $y \in V$ and $V^{(y)} := V \setminus \{y\}$. Then the matrix $\phi^{\bullet y} \in M(V^{(y)} \times V^{(y)})$ defined by*

$$(\phi^{\bullet y})_{x,z} := \phi^{xy}(z)\ ,\ \ x,z \in V^{(y)}$$

satisfies $\det \phi^{\bullet y} \neq 0$.

Proof. Let $\mathcal{L} \in M(V \times V)$ be the matrix of the Laplacian of (G,C), namely

$$(\mathcal{L})_{zw} = \begin{cases} -\frac{C(z,w)}{C_z} & ,z \neq w \\ 1 - \frac{C(z,z)}{C_z} & ,z = w \end{cases}\ ,\ \ z,w \in V.$$

Furthermore, let $\mathcal{L}^{(y)}$ be matrix resulting from $\mathcal{L}$ by omitting the y-th row and column. Using $\phi^{xy}(y) = 0$ at $(*)$, we have

$$\begin{aligned} \phi^{\bullet y} \cdot \mathcal{L}^{(y)} &= \left(\phi^{xy}(z) - \sum_{w \in V^{(y)}} \frac{C(z,w)}{C_z} \phi^{xy}(w) \right)_{x,z \in V^{(y)}} \\ &\overset{(*)}{=} \left(\phi^{xy}(z) - \sum_{w \in V} \frac{C(z,w)}{C_z} \phi^{xy}(w) \right)_{x,z \in V^{(y)}} \\ &= \left(\delta_{xz} \frac{1}{C_x} \right)_{x,z \in V^{(y)}}. \end{aligned}$$

Hence,

$$\det \phi^{\bullet y} \cdot \det \mathcal{L}^{(y)} = \det \left(\phi^{\bullet y} \cdot \mathcal{L}^{(y)} \right) = \prod_{x \in V^{(y)}} \frac{1}{C_x} \neq 0.$$

It follows that $\det \phi^{\bullet y} \neq 0$. □

Theorem 5.27. *Let $R = (R(x,y))_{x,y \in V}$ be the effective resistance matrix of any finite connected network $\mathcal{N} = (V,E,C)$. Then there exists a (up to a isomorphism) unique simple network $\mathcal{N}' = (V,E',C')$ with effective resistance R.*

Proof. Let $G = (V,E)$ and $G' = (V,E')$. Without loss of generality, let $V = \{1,\dots,n\}$. By Theorem 5.16, we have

$$\phi^{xy}(z) = \frac{U_z^{xy}}{C_z} = \frac{1}{2}\left(R(x,y) + R(y,z) - R(x,z)\right)\ \forall\ x,y,z \in V.$$

Note that the mapping $(R(x,y))_{x,y\in V} \mapsto (\phi^{xy}(z))_{x,y,z\in V}$ is injective since $\phi^{xy}(x) = R(x,y)$, i.e. we can reconstruct $(\phi^{xy}(z))_{x,y,z\in V}$ uniquely from the effective resistance. If (G', C') is a simple network with effective resistance R, then the potential of a unit current I^{xy} flowing through G' is ϕ^{xy}. Hence, for a fixed $y \in V$, we get

$$1 = \sum_{e\in E(y)} I^{xy}(\hat{e}) = \sum_{w\in V} C(y,w)(\phi^{xy}(w) - \underbrace{\phi^{xy}(y)}_{=0}) = \sum_{w\in V} C(y,w)\phi^{xy}(w)$$

for all $x \in V \setminus \{y\}$. Define $A_y \in M(V \times V)$ and $b_y \in \mathbb{R}^n$ by

$$(A_y)_{x,w} := \begin{cases} 0 & , \ x = y \neq w \\ 1 & , \ x = y = w \\ \phi^{xy}(w) & , \ x \neq y \end{cases} \ , \ (b_y)_x := 1 - \delta_{xy} \ \ \forall\, x, w \in V.$$

Since G' is supposed to be simple, we have $C'(y,y) = 0$. Hence, the vector $C'(y,\cdot) = (C'(y,w))_{w\in V}$ has to be a solution of the linear equation system

$$A_y \cdot \mathfrak{X} = b_y.$$

Since the y-th row and the y-th column of A_y are of the form $(\delta_{xy})_{x\in V}$, we can use the Laplace expansion and obtain

$$\det A_y = 1 \cdot \det\left((\phi^{xy}(w))_{\substack{x,w\in V \\ x\neq y\neq w}} \right) = \det \phi^{\bullet y}.$$

Using Lemma 5.26, we get $\det A_y = \det \phi^{\bullet y} \neq 0$. Hence, for each $y \in V$ we obtain a linear equation system with a unique solution $C'(y,\cdot)$, i.e. it is proven that $(C'(x,y))_{x,y\in V}$ is uniquely determined by R if we assume that $C'(x,x) = 0$. A simple network (G', C') is also uniquely determined by the matrix $(C'(x,y))_{x,y\in V}$ since $E' = \{(x,y,1) \mid C'(x,y) \neq 0\}$. □

Remark. It is noteworthy that not all finite metric spaces can be realized as the effective resistance of a finite network, see Example 5.28.

Example 5.28. Consider the metric space (V, d) where $V = \{a,b,c,d\}$ and

$$(d(x,y))_{x,y\in V} = \begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 0 & 2 & 5 \\ 3 & 2 & 0 & 3 \\ 4 & 5 & 3 & 0 \end{pmatrix}.$$

If we solve the linear equation systems for $(C(x,y))_{x,y\in V}$ that arise in the proof of Theorem 5.27, we get the solutions

$$(C(x,y))_{x,y\in V} = \begin{pmatrix} 0 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{2} & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

Hence, there is no network (with $C(x,y) \geq 0$ for all $x, y \in V$) admitting d as its effective resistance.

Definition 5.29. We define the *simplification* $S(\mathcal{N}) := (V, E_S, C_S)$ of $\mathcal{N}$ by

$$E_S := \{(x,y,1),(y,x,1) \in V \times V \times \mathbb{N} \mid C(x,y) > 0\}$$

and

$$C_S((x,y,1)) = C_S((y,x,1)) := C(x,y).$$

Proposition 5.30. *Let $\mathcal{N}_1 = (V, E_1, C_1)$ and $\mathcal{N}_2 = (V, E_2, C_2)$ be two finite networks. Then $R_{\mathcal{N}_1} = R_{\mathcal{N}_2}$ if and only if $S(\mathcal{N}_1) = S(\mathcal{N}_2)$.*

Proof. Let $S(\mathcal{N}_i) = (V, E'_i, C'_i)$, $i = 1, 2$. By Proposition 4.28, the effective resistance of a network can be computed using only $(C(x,y))_{x,y \in V}$. Since $C_i(x,y) = C'_i(x,y)$ for all $x, y \in V$, it follows that $R_{\mathcal{N}_i} = R_{S(\mathcal{N}_i)}$, $i = 1, 2$.

Suppose $R_{\mathcal{N}_1} = R_{\mathcal{N}_2}$. It follows that $R_{S(\mathcal{N}_1)} = R_{S(\mathcal{N}_2)}$. Hence, $S(\mathcal{N}_1)$ and $S(\mathcal{N}_2)$ are two simple networks with the same effective resistance. By Theorem 5.27, it follows that $S(\mathcal{N}_1)$ and $S(\mathcal{N}_2)$ are isomorphic and therefore equal.

Now let $S(\mathcal{N}_1) = S(\mathcal{N}_2)$. Then

$$C_1(x,y) = C'_1(x,y) = C'_2(x,y) = C_2(x,y)$$

for all $x, y \in V$. It follows from Proposition 4.28 that $R_{\mathcal{N}_1} = R_{\mathcal{N}_2}$. □

Proposition 5.31 (Effective resistance and STMs)**.** *Let $\mathcal{N}_1 = (V, E_1, C_1)$ and $\mathcal{N}_2 = (V, E_2, C_2)$ be two finite networks with the same vertex set V. If $R_{\mathcal{N}_1} = R_{\mathcal{N}_2}$, then*

$$\mu_{\mathcal{N}_1}[x \text{ and } y \text{ share an edge in } T] = \mu_{\mathcal{N}_2}[x \text{ and } y \text{ share an edge in } T]$$

for all $x, y \in V$.

Proof. Let $E_i(x,y) := \{e \in E_i \mid \underline{e} = x, \overline{e} = y\}$, $i = 1, 2$, and $R_{\mathcal{N}_1} = R_{\mathcal{N}_2}$. Using Theorem 4.26 and Corollary 5.23, we compute

$$\begin{aligned}\mu_{\mathcal{N}_i}[x \text{ and } y \text{ share an edge in } T] &= \mu_{\mathcal{N}_i}\left[\bigvee_{e \in E_i} e \in T\right] = \sum_{e \in E_i} \mu_{\mathcal{N}_1}[e \in T] \\ &= \sum_{e \in E_i} I^e_{\mathcal{N}_i}(e) = \sum_{e \in E_i} C_i(e) R_{\mathcal{N}_i}(x,y) \\ &= C_i(x,y) \cdot R_{\mathcal{N}_i}(x,y)\end{aligned}$$

for $i = 1, 2$. Proposition 5.30 implies $C_1(x, y) = C_2(x, y)$ for all $x, y \in V$, and the claim follows. □

Remark. The inverse implication of Proposition 5.31 is in general not true. Let $\lambda \in \mathbb{R}^+$, $\lambda \neq 1$ and consider two networks $\mathcal{N}_1 = (V, E, C)$ and $\mathcal{N}_2 = (V, E, C \cdot \lambda)$. By Lemma 3.3, it follows that $\mu_{\mathcal{N}_1} = \mu_{\mathcal{N}_2}$ but $R_{\mathcal{N}_1} \neq R_{\mathcal{N}_2}$.

5.4 Inequalities and estimates

Lemma 5.32 (An upper bound for the effective resistance). *Let $x, y \in V$ and $(e_1, \ldots, e_n)$ be a path from x to y. Then*

$$R(x, y) \leq \sum_{i=1}^{n} R(e_i) = \sum_{i=1}^{n} \frac{1}{C(e_i)} \ .$$

Proof. Let $J = \sum_{i=1}^{n} \chi^{e_n}$. Then $\operatorname{div} J = \frac{1}{C_{\underline{e_1}}} \mathbb{1}_{\underline{e_1}} - \frac{1}{C_{\overline{e_n}}} \mathbb{1}_{\overline{e_n}} = \frac{1}{C_x} \mathbb{1}_x - \frac{1}{C_y} \mathbb{1}_y = \operatorname{div} I^{xy}$. Using the Thomson principle 4.32, we obtain

$$\sum_{i=1}^{n} \frac{1}{C(e_i)} = \mathcal{E}(J) \geq \mathcal{E}(I^{xy}) = R(x, y).$$

□

Remark. In particular,

$$R(x, y) \leq d_{\mathcal{N}}(x, y) := \min \left\{ \sum_{i=1}^{n} R(e_i) \mid (e_1, \ldots, e_n) \text{ path from } x \text{ to } y \right\}.$$

$d_{\mathcal{N}}(x, y)$ is called the *canonical* or *geodesic metric* on $\mathcal{N}$.

Lemma 5.33 (A lower bound for the effective resistance). *Let $\mathcal{N} = (V, E, C)$ be a network. Then*

$$R(x, y) \geq \frac{1}{C_x} \quad \forall\, x, y \in V,\ x \neq y.$$

Proof. By Proposition 5.11, we have

$$R(x, y) = \phi^{xy}(x) = \frac{U_x^{xy}}{C_x} \geq \frac{1}{C_x}.$$

□

Theorem 5.34 (Rayleigh's monotonicity principle, Thomassen [10])**.** *Let* $\mathcal{N} = (V, E, C)$ *and* $\mathcal{N}' = (V, E, C')$ *be finite networks such that* $C \le C'$*. Then*

$$R_{\mathcal{N}}(x, y) \ge R_{\mathcal{N}'}(x, y) \quad \forall\, x, y \in V.$$

Proof. Let $I_{\mathcal{N}}^{xy}$ and $I_{\mathcal{N}'}^{xy}$ be the unit currents from x to y in $\mathcal{N}$ and $\mathcal{N}'$, respectively. We then have $R'(e) \le R(e)$ for all $e \in E$ and

$$\mathcal{E}_{\mathcal{N}'}(J) = \frac{1}{2} \sum_{e \in E} R'(e) J(e)^2 \le \frac{1}{2} \sum_{e \in E} R(e) J(e)^2 = \mathcal{E}_{\mathcal{N}}(J) \quad \forall\, J \in l^2_-(E).$$

By using the Thomson principle 4.32, we get

$$R_{\mathcal{N}'}(x, y) = \mathcal{E}_{\mathcal{N}'}(I_{\mathcal{N}'}^{xy}) \le \mathcal{E}_{\mathcal{N}'}(I_{\mathcal{N}}^{xy}) \le \mathcal{E}_{\mathcal{N}}(I_{\mathcal{N}}^{xy}) = R_{\mathcal{N}}(x, y)$$

since $\operatorname{div} I_{\mathcal{N}}^{xy} = \operatorname{div} I_{\mathcal{N}'}^{xy}$ (in $\mathcal{N}'$). □

Proposition 5.35. *Let* (V, E, C) *be a finite network with effective resistance* R *and* (V, E') *a connected subgraph. Then the effective resistance* R' *of* (V, E', C') *where* $C' = C \restriction_{E'}$ *satisfies*

$$R(x, y) \le R'(x, y) \quad \forall\, x, y \in V.$$

Proof. Let I_C^{xy} and $I_{C'}^{xy}$ be the unit currents from x to y in (V, E, C) and (V, E', C'), respectively. Define

$$I(e) := \begin{cases} I_{C'}^{xy}(e) & , e \in E' \\ 0 & , e \in E \setminus E' \end{cases}.$$

Then $I \in l^2_-(E)$, $I \restriction_{E'} = I_{C'}^{xy}$ and $\operatorname{div}_C I = \frac{1}{C_x} \mathbb{1}_x - \frac{1}{C_y} \mathbb{1}_y = \operatorname{div}_C I_C^{xy}$. Hence,

$$\begin{aligned} R'(x, y) = \mathcal{E}_{C'}(I_{C'}^{xy}) &= \frac{1}{2} \sum_{e \in E'} R'(e) I_{C'}^{xy}(e)^2 \\ &= \frac{1}{2} \sum_{e \in E} R(e) I(e)^2 = \mathcal{E}_C(I) \ge \mathcal{E}_C(I_C^{xy}) = R(x, y). \end{aligned}$$ □

Remark. Proposition 5.35 is an extension of Rayleigh's monotonicity principle if we interpret C' as a weight function on E such that

$$C'(e) = 0 \quad \forall\, e \in E \setminus E'.$$

Example 5.36 (The complete graph with n vertices)**.** For $n \in \mathbb{N}$, $n \ge 2$ let $G(n) = (V_n, E_n) := (\{1, \dots, n\}, \{1, \dots, n\}^2 \times \{1\})$ be the *complete graph with* n *vertices.* Let $\lambda > 0$ and set $C_\lambda(e) := \lambda$ for all $e \in E$. Now consider the network $\mathcal{N}(n,\lambda) := (G(n), C_\lambda)$ and let $R_\lambda^{(n)}$ denote its effective resistance. Obviously, $R_\lambda^{(n)}(x,y) = R_\lambda^{(n)}(x',y')$ for all $x,x',y,y' \in \{1,\dots,n\}$, $x \neq y$, $x' \neq y'$. By Definition 5.9, $R_\lambda^{(n)}(1,2) = \phi^{12}(1)$ where ϕ^{12} is the potential of the unit current from 1 to 2 in $\mathcal{N}(n,\lambda)$. Using Proposition 4.28, we obtain the linear equation system

$$\begin{aligned}
\phi^{12}(1) - \sum_{w\in V_n} \frac{C_\lambda(1,w)}{(C_\lambda)_1}\phi^{12}(w) &= \frac{1}{(C_\lambda)_1} \\
\phi^{12}(2) &= 0 \\
\phi^{12}(z) - \sum_{w\in V_n} \frac{C_\lambda(z,w)}{(C_\lambda)_z}\phi^{12}(w) &= 0 \quad \forall\ z > 2.
\end{aligned}$$

For $x \in V_n$, we have $(C_\lambda)_x = \lambda \cdot (n-1)$ and $C_\lambda(x,y) = (1-\delta_{xy}) \cdot \lambda$ for all $y \in V_n$. Hence, we need to solve the linear equation system

$$\begin{pmatrix}
1 & -\frac{1}{n-1} & -\frac{1}{n-1} & -\frac{1}{n-1} & \dots & -\frac{1}{n-1} \\
0 & 1 & 0 & 0 & \dots & 0 \\
-\frac{1}{n-1} & -\frac{1}{n-1} & 1 & -\frac{1}{n-1} & \dots & -\frac{1}{n-1} \\
\vdots & \vdots & & \ddots & & \vdots \\
-\frac{1}{n-1} & -\frac{1}{n-1} & -\frac{1}{n-1} & \dots & 1 & -\frac{1}{n-1} \\
-\frac{1}{n-1} & -\frac{1}{n-1} & -\frac{1}{n-1} & \dots & -\frac{1}{n-1} & 1
\end{pmatrix} \cdot \begin{pmatrix} \phi^{12}(1) \\ \phi^{12}(2) \\ \vdots \\ \phi^{12}(n) \end{pmatrix} = \begin{pmatrix} \frac{1}{\lambda(n-1)} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

It is easy to check that the only solution is

$$(\phi^{12}(1), \phi^{12}(2), \dots, \phi^{12}(n)) = \left(\frac{2}{\lambda\cdot n}, 0, \frac{1}{\lambda\cdot n}, \dots, \frac{1}{\lambda \cdot n}\right).$$

Hence,

$$R_\lambda^{(n)}(x,y) = (1-\delta_{xy})\frac{2}{\lambda\cdot n}.$$

Corollary 5.37. *Let (V,E,C) be a finite network and R its effective resistance. Then*

$$R(x,y) \geq \frac{2}{\max_{e\in E} C(e)\cdot |V|} \quad \forall\ x,y \in V.$$

Proof. Let $\lambda := \max_{e\in E} C(e)$ and $C_\lambda : E \to \mathbb{R}$, $C_\lambda(e) := \lambda$. If we set $\mathcal{N}' := (V,E,C_\lambda)$, then it follows from Theorem 5.34 that

$$R(x,y) \geq R_{\mathcal{N}'}(x,y)\ \forall\ x,y \in V.$$

Hence, Proposition 5.35 and Example 5.36 imply that

$$R(x,y) \geq R_{\mathcal{N}'}(x,y) \geq R_{\mathcal{N}(|V|,\lambda)}(x,y) = \frac{2}{|V|\cdot\lambda}\ \forall\ x,y \in V$$

holds, where $\mathcal{N}(|V|,\lambda)$ is the complete graph over V with edge weight λ. □

Proposition 5.38. *Let $\mathcal{N} = (V, E, C)$ be a finite network and $x, y \in V$, $x \neq y$. For*

$$M_{xy} := \inf \left\{ \frac{\mathcal{E}(I)}{I_x^2} \mid I \in l_-^2(E), \operatorname{div} I \restriction_{V \setminus \{x,y\}} \equiv 0, I_x \neq 0 \right\}$$

where $I_x := C_x(\operatorname{div} I)(x) = \sum_{e \in E(x)} I(e)$ and

$$S_{xy} := \sup \left\{ \frac{|u(x) - u(y)|^2}{\mathcal{E}(u)} \mid u \in l^2(V), \mathcal{E}(u) > 0 \right\},$$

we have $M_{xy} = R(x, y) = S_{xy}$.

Proof. Let $I \in l_-^2(E)$ such that $(\operatorname{div} I)(v) = 0$ for all $v \neq x, y$ and $I_x \neq 0$. Then

$$0 = (\underbrace{-\nabla 1}_{=0}, I)_R = (1, \operatorname{div} I)_C = \sum_{v \in V} C_v (\operatorname{div} I)(v) = C_x (\operatorname{div} I)(x) + C_y (\operatorname{div} I)(y),$$

i.e. $C_x(\operatorname{div} I)(x) = -C_y(\operatorname{div} I)(y)$. Let $I' := \frac{1}{C_x(\operatorname{div} I)(x)} \cdot I$. Then by linearity of div we have,

$$\operatorname{div} I' = \frac{1}{C_x} \mathbb{1}_x - \frac{1}{C_y} \mathbb{1}_y \ ,$$

i.e. $I'_x = 1$ and since $\mathcal{E}$ is a quadratic form, we get

$$\frac{\mathcal{E}(I)}{I_x^2} = \mathcal{E}\left(\frac{I}{I_x}\right) = \mathcal{E}\left(\frac{I}{C_x(\operatorname{div} I)(x)}\right) = \mathcal{E}(I') = \frac{\mathcal{E}(I')}{(I'_x)^2}.$$

Hence,

$$M_{xy} = \inf \left\{ \mathcal{E}(I) \mid I \in l_-^2(E), \operatorname{div} I = \frac{1}{C_x} \mathbb{1}_x - \frac{1}{C_y} \mathbb{1}_y \right\} \overset{4.32}{=} \mathcal{E}(\phi^{xy}) = R(x, y).$$

Regarding the second equality: Using Ohm's law, the definition of the effective resistance and Lemma 4.33, we obtain

$$\mathcal{E}(\phi^{xy}) = \frac{\mathcal{E}(\phi^{xy})^2}{\mathcal{E}(\phi^{xy})} = \frac{\overbrace{A(-\nabla \phi^{xy})^2}^{=1} (\phi^{xy}(x) - \phi^{xy}(y))^2}{\mathcal{E}(\phi^{xy})} = \frac{|\phi^{xy}(x) - \phi^{xy}(y)|^2}{\mathcal{E}(\phi^{xy})}.$$

Hence,

$$S_{xy} = R(x, y) \Leftrightarrow S_{xy} = \frac{|\phi^{xy}(x) - \phi^{xy}(y)|^2}{\mathcal{E}(\phi^{xy})}.$$

For $u \in l^2(V), \mathcal{E}(u) > 0$, let $u_1 := u - u(y)$ and $u_2 := -u$. Then

$$|u_1(x) - u_1(y)|^2 = |u(x) - u(y)|^2 = |u_2(x) - u_2(y)|^2$$

and

$$\begin{aligned}\mathcal{E}(u_1) &= \mathcal{E}(-\nabla u_1) = \mathcal{E}(-\nabla u - \underbrace{\nabla(u(y))}_{=0}) = \mathcal{E}(u) \\ &= (-\nabla u, -\nabla u)_R = (\nabla u, \nabla u)_R = \mathcal{E}(u_2).\end{aligned}$$

Hence,

$$S_{xy} = \sup\left\{\frac{u(x)^2}{\mathcal{E}(u)} \mid u \in l^2(V), u(x) \geq 0, u(y) = 0, \mathcal{E}(u) > 0\right\}.$$

Now let $k \in \mathbb{R}_{\geq 0}$ and $A_k := \{u \in l^2(V) \mid u(x) = k, u(y) = 0, \mathcal{E}(u) > 0\}$. Then

$$S_{xy} = \sup_{k\geq 0}\sup_{u\in A_k} \frac{k^2}{\mathcal{E}(u)}.$$

By Propositions 5.1 and 5.7, there exists a unique u_k^* in each A_k, which is harmonic on $V \setminus \{x,y\}$, i.e. $\Delta u_k^* \restriction_{V\setminus\{x,y\}} = 0$. The following computation will show that $\inf_{u\in A_k}\mathcal{E}(u) = \mathcal{E}(u_k^*)$ for all $k \geq 0$, and it will follow that

$$S_{xy} = \sup_{k\geq 0} \frac{k^2}{\mathcal{E}(u_k^*)}\ .$$

Let $u \in A_k$. Then

$$\begin{aligned}\mathcal{E}(u) &= \mathcal{E}(u_k^*) + 2(-\nabla u_k^*, -\nabla(u - u_k^*))_R + \mathcal{E}(u - u_k^*) \\ &= \mathcal{E}(u_k^*) + 2\underbrace{(-\Delta u_k^*, u - u_k^*)_C}_{=0} + \underbrace{\mathcal{E}(u - u_k^*)}_{\geq 0} \geq \mathcal{E}(u_k^*).\end{aligned}$$

The fact that the inner product of $-\Delta u_k^*$ and $u-u_k^*$ vanishes is due to $-\Delta u_k^* \restriction_{V\setminus\{x,y\}}\equiv 0$ and $u, u_k^* \in A_k$.

The expression $k^2/\mathcal{E}(u_k^*)$ however does not actually depend on k (if $k > 0$) since

$$\frac{k^2}{\mathcal{E}(u_k^*)} = \frac{k^2}{C_x \cdot k \cdot (-\Delta u_k^*)(x)} = \left(C_x\left(-\Delta\left(\frac{u_k^*}{k}\right)(x)\right)\right)^{-1} = (C_x(-\Delta u_1^*)(x))^{-1}.$$

Thus, $S_{xy} = k^2/\mathcal{E}(u_k^*)$ for all $k > 0$. Choosing $k = \phi^{xy}(x)$ we get $u_k^* = \phi^{xy}$ (see Definition 4.27 and Proposition 4.28). Hence,

$$S_{xy} = \frac{\phi^{xy}(x)^2}{\mathcal{E}(\phi^{xy})} = \frac{|\phi^{xy}(x) - \phi^{xy}(y)|^2}{\mathcal{E}(\phi^{xy})} = \mathcal{E}(\phi^{xy}) = R(x,y).$$

□

Remark. Proposition 5.38 shows that the effective resistance can be realized as the solution to a variational problem on both $l^2_-(E)$ and $l^2(V)$.

5.5 Resistance forms

We follow Kasue's theory of resistance forms (see [5]) to introduce an equivalent definition of a finite networks effective resistance.

Definition 5.39 (Symmetric quadratic/bilinear form)**.** Let X be a real vector space.

1. A mapping $q : X \times X \to \mathbb{R}$ such that
   (a) $q(x,x) \geq 0 \;\forall\; x \in X$
   (b) $q(x,y) = q(y,x) \;\forall\; x,y \in X$
   (c) $q(\lambda x + \mu y, z) = \lambda q(x,z) + \mu q(y,z) \;\forall\; \lambda,\mu \in \mathbb{R}, x,y,z \in X$

   is called a *symmetric bilinear form* on X.
2. A mapping $Q : X \to \mathbb{R}$ such that

$$q(x,y) := \frac{1}{2}(Q(x+y) - Q(x) - Q(y))$$

   is a symmetric bilinear form is called a *symmetric quadratic form* on X.

Remarks.

1. From now on we will omit the word 'symmetric'. If we talk about bilinear or quadratic forms, they are supposed to be symmetric.
2. Because of 2, we will not differentiate between the bilinear form q and the quadratic form Q but instead use $q(\cdot,\cdot)$ in the bilinear case and $q(\cdot)$ for the quadratic form.

Proposition 5.40 (Cauchy-Schwarz inequality)**.** *Let X be real vector space and $q : X \times X \to \mathbb{R}$ a bilinear form on X such that $N := \{x \in X \mid q(x,x) = 0\}$ is a linear subspace of X. Then*

$$q(x,y)^2 \leq q(x,x) \cdot q(y,y) \quad \forall\; x,y \in X.$$

Proof. Let $x, y \in X$. If $x, y \in N$, then $x + y \in N$ and

$$q(x,y) = \frac{1}{2}(\underbrace{q(x+y,x+y)}_{=0} - \underbrace{q(x,x)}_{=0} - \underbrace{q(y,y)}_{=0}) = 0.$$

Hence,

$$q(x,y)^2 = 0 = q(x,x) \cdot q(y,y)$$

Now assume $x \notin N$ and set $\lambda := q(x,y)/q(x,x)$. Then

$$\begin{aligned} 0 \leq q(y-\lambda x, y-\lambda x) &= q(y,y) - 2\lambda q(x,y) + \lambda^2 q(x,x) \\ &= q(y,y) - 2\frac{q(x,y)^2}{q(x,x)} + \frac{q(x,y)^2}{q(x,x)} = q(y,y) - \frac{q(x,y)^2}{q(x,x)} \;. \end{aligned}$$

Hence, $q(x,y)^2 \leq q(x,x) \cdot q(y,y)$. $\square$

Definition 5.41 (Resistance form)**.** Let X be a set. A pair $(\mathcal{F}, D[\mathcal{F}])$ of a linear subspace $D[\mathcal{F}] \subseteq l(X) := \{f \mid f : X \to \mathbb{R}\}$ and a quadratic form $\mathcal{F} : D[\mathcal{F}] \to \mathbb{R}$ is called a *resistance form on* X if the following hold:

(RF-1) $1 \in D[\mathcal{F}]$ and $\mathcal{F}(f) = 0$ if and only if f is constant on X.

(RF-2) For all $x \in X$, $(D[\mathcal{F}], \mathcal{F} + o_x)$ is a Hilbert space. Here, o_x denotes the bilinear form given by $o_x(f,g) = f(x)g(x)$.

(RF-3) For any finite subset $V \subseteq X$ and for any $f \in l(X)$, there exists $g \in D[\mathcal{F}]$ such that $g \restriction_V = f$.

(RF-4) For any $x, y \in X$, $x \neq y$, we have

$$0 < R_{\mathcal{F}}(x,y) := \sup \left\{ \frac{|f(x) - f(y)|^2}{\mathcal{F}(f)} \mid f \in D[\mathcal{F}], \mathcal{F}(f) > 0 \right\} < \infty.$$

(RF-5) If $f \in D[\mathcal{F}]$, then $\widetilde{f} \in D[\mathcal{F}]$ where $\widetilde{f}(x) = \min(1, \max(0, f(x)))$ and $\mathcal{F}(\widetilde{f}) \leq \mathcal{F}(f)$.

Remark.

1. Note that if X is finite, (RF-3) implies $D[\mathcal{F}] = l(X)$. Indeed, let $f \in l(X)$. Then there exists $g \in D[\mathcal{F}]$ such that $g = g \restriction_X = f$. Hence, $f \in D[\mathcal{F}]$.
2. By defining $\mathcal{F}(f) = \infty$ for $f \in l(X) \setminus D[\mathcal{F}]$, we may consider every resistance form a functional on $l(X)$.

Lemma 5.42. *Let X be a set and $(\mathcal{F}, D[\mathcal{F}])$ a resistance form on X. Then the following hold:*

1. $\mathcal{F}(f, c) = 0$ *for all* $f \in D[\mathcal{F}], c \in \mathbb{R}$.
2. $\mathcal{F}(f + c) = \mathcal{F}(f)$ *for all* $f \in D[\mathcal{F}], c \in \mathbb{R}$.
3. *Let* $x, y \in X$, $x \neq y$. *If* $\mathbb{1}_x, \mathbb{1}_y \in D[\mathcal{F}]$, *then* $\mathcal{F}(\mathbb{1}_x, \mathbb{1}_y) \leq 0$.
4. *If* $\mathbb{1}_x \in D[\mathcal{F}]$ *for all* $x \in X$, *then*

$$\forall\, x \in X \;\exists\, y \in X : \mathcal{F}(\mathbb{1}_x, \mathbb{1}_y) < 0.$$

5. *If* $|X| < \infty$, *then*

$$\mathcal{F}(f) = \frac{1}{2} \sum_{x \in X} \sum_{y \in Y} (-\mathcal{F}(\mathbb{1}_x, \mathbb{1}_y)) \cdot (f(x) - f(y))^2 \;\forall\, f \in l(X).$$

6. If $|X| < \infty$, then

$$\forall\, x \in X\ \forall\, y \in Y\ \exists\, n \in \mathbb{N}_0\ \exists\, x_1, \ldots, x_n \in X : x_1 = x, x_n = y, \ \textit{and}$$

$$\mathcal{F}(\mathbb{1}_{x_i}, \mathbb{1}_{x_{i+1}}) < 0 \quad \forall\, i = 1, \ldots, n-1.$$

Proof.

1. Using the Cauchy-Schwarz inequality (Proposition 5.40) and (RF-1), we get

$$0 \le \mathcal{F}(f,c)^2 \le \mathcal{F}(f,f)\mathcal{F}(c,c) = 0.$$

   Hence, $\mathcal{F}(f,c) = 0$.

2. By 1 and (RF-1), we have

$$\mathcal{F}(f+c) = \mathcal{F}(f+c, f+c) = \mathcal{F}(f) + 2\mathcal{F}(f,c) + \mathcal{F}(c) = \mathcal{F}(f).$$

3. Let $x \neq y$ and $\mathbb{1}_x, \mathbb{1}_y \in D[\mathcal{F}]$. For $t > 0$, let $f_t := \mathbb{1}_x - t \cdot \mathbb{1}_y \in D[\mathcal{F}]$. Then $\widetilde{f_t} = \mathbb{1}_x$, and by (RF-5), we have

$$\mathcal{F}(\mathbb{1}_x) - 2t\mathcal{F}(\mathbb{1}_x, \mathbb{1}_y) + t^2\mathcal{F}(\mathbb{1}_y) = \mathcal{F}(f_t) \ge \mathcal{F}(\widetilde{f_t}) = \mathcal{F}(\mathbb{1}_x).$$

   Hence,

$$\frac{t}{2}\underbrace{\mathcal{F}(\mathbb{1}_y)}_{\ge 0} \ge \mathcal{F}(\mathbb{1}_x, \mathbb{1}_y)\ \forall\, t > 0.$$

   For $t \to 0$, we obtain $\mathcal{F}(\mathbb{1}_x, \mathbb{1}_y) \le 0$.

4. Assume $\mathbb{1}_x \in D[\mathcal{F}]$ for all $x \in X$. Then $0 = \mathcal{F}(\mathbb{1}_x, 1) = \sum_{y \in X} \mathcal{F}(\mathbb{1}_x, \mathbb{1}_y)$ and

$$0 < \mathcal{F}(\mathbb{1}_x) = \sum_{y \neq x} \underbrace{(-\mathcal{F}(\mathbb{1}_x, \mathbb{1}_y))}_{\ge 0}.$$

   Hence, there exists $y \in V$ such that $\mathcal{F}(\mathbb{1}_x, \mathbb{1}_y) < 0$.

5. If X is finite, we have

$$\mathcal{F}(f,g) = \mathcal{F}\left(\sum_{x \in X} f(x)\mathbb{1}_x, \sum_{y \in X} g(y)\mathbb{1}_y\right) = \sum_{x \in X}\sum_{y \in Y} f(x)g(y) \cdot \mathcal{F}(\mathbb{1}_x, \mathbb{1}_y).$$

   Hence,

$$0 = -\frac{1}{2}\mathcal{F}(f^2, 1) = -\frac{1}{2}\sum_{x \in X}\sum_{y \in X} f^2(x)\mathcal{F}(\mathbb{1}_x, \mathbb{1}_y) = -\frac{1}{2}\sum_{x \in X}\sum_{y \in X} f^2(y)\mathcal{F}(\mathbb{1}_x, \mathbb{1}_y).$$

Using this we compute

$$
\begin{aligned}
\mathcal{F}(f) &= -\frac{1}{2}\mathcal{F}(f^2,1) + \mathcal{F}(f,f) - \frac{1}{2}\mathcal{F}(f^2,1) \\
&= -\frac{1}{2}(\mathcal{F}(f^2,1) - 2\mathcal{F}(f,f) + \mathcal{F}(f^2,1)) \\
&= -\frac{1}{2}\sum_{x\in X}\sum_{y\in X}(f^2(x) - 2f(x)f(y) + f^2(y))\mathcal{F}(\mathbb{1}_x,\mathbb{1}_y) \\
&= \frac{1}{2}\sum_{x\in X}\sum_{y\in X}(f(x)-f(y))^2\cdot(-\mathcal{F}(\mathbb{1}_x,\mathbb{1}_y)).
\end{aligned}
$$

6. Let

$$A(x) := \left\{ y \in X \mid \exists\, n \in \mathbb{N}_0\ \exists\, x_1,\ldots,x_n : x_1 = x, x_n = y, \mathcal{F}(\mathbb{1}_{x_i},\mathbb{1}_{x_{i+1}}) < 0 \right\}.$$

We know that $|X| < \infty$ implies $D[\mathcal{F}] = l(X)$. It follows by 5 that

$$\mathcal{F}(\mathbb{1}_v,\mathbb{1}_w) = 0$$

for all $v \in A(x), w \notin A(x)$. Indeed, since $v \in A(x)$, there exist $x_1,\ldots,x_n \in V$, $x_1 = x, x_n = v$, such that $\mathcal{F}(\mathbb{1}_{x_i},\mathbb{1}_{x_{i+1}}) < 0$ for all $i = 1,\ldots,n-1$. Assuming $\mathcal{F}(\mathbb{1}_v,\mathbb{1}_w) < 0$, it follows that $w \in A(x)$ because we can set $x_{n+1} := w$. Hence,

$$\mathcal{F}(\mathbb{1}_{A(x)}) = \frac{1}{2}\sum_{v\in X}\sum_{w\in X}(\mathbb{1}_{A(x)}(v) - \mathbb{1}_{A(x)}(w))^2\cdot(-\mathcal{F}(\mathbb{1}_v,\mathbb{1}_w)) = 0.$$

(RF-1) implies that $\mathbb{1}_{A(x)}$ is constant on X while it follows from 4 that $A(x) \neq \emptyset$. Hence, $A(x) = X$. □

Remark. Note that if $|X| < \infty$, then we have $\mathbb{1}_x \in D[\mathcal{F}]$ for all $x \in X$ by (RF-3). In this case, statements 3 and 4 hold without the if-conditions for all resistance forms on X.

Proposition 5.43 (Resistance forms on finite sets)**.** *Let X be a finite set and $\mathcal{F} : l(X) \to \mathbb{R}$ a quadratic form. Then $(\mathcal{F}, l(X))$ is a resistance form on X if and only if it satisfies the following three conditions:*

1. $\mathcal{F}(f) \geq$ *for all* $f \in l(X)$,
2. $\mathcal{F}(f) = 0 \Leftrightarrow f$ *is constant on* X *and*
3. $\mathcal{F}(\widetilde{f}) \leq \mathcal{F}(f)$ *for all* $f \in l(X)$.

Proof. If $\mathcal{F}$ is a resistance form on X, (RF-3) implies $D[\mathcal{F}] = l(X)$, as mentioned above. The three conditions are then satisfied as $\mathcal{F}$ is non-negative and (RF-1) and (RF-5) hold for all $f \in l(X)$.

Now let $\mathcal{F} : l(X) \to \mathbb{R}$ be a quadratic form satisfying the three given conditions and set $D[\mathcal{F}] := l(X)$. By condition 1, $\mathcal{F}$ is a non-negative, symmetric quadratic form on $D[\mathcal{F}]$ which satisfies (RF-3). (RF-1) and (RF-5) hold because of conditions 2 and 3. It follows that $\mathcal{F}$ has all the properties described in Lemma 5.42 because the proof of this lemma only uses (RF-1), (RF-3) and (RF-5).
(RF-2): Let $x \in X$. Since $\mathcal{F}(f) + o_x(f) = \mathcal{F}(f) + f(x)^2 \geq 0$, we know that $\mathcal{F} + o_x$ is non-negative. Suppose that $\mathcal{F}(f) + o_x(f) = 0$. It follows that $\mathcal{F}(f) = 0$ and $f(x)^2 = 0$. Hence, f is constant on X and satisfies $f(x) = 0$, i.e. $f \equiv 0$. This implies that $\mathcal{F} + o_x$ is an inner product on $l(X)$. Since X is finite, $l(X)$ is finite-dimensional and thus complete with respect to $\mathcal{F} + o_v$.
(RF-4): Let $x, y \in X$. If $x \neq y$, then $|X| \geq 2$, and therefore $\mathbb{1}_x$ is non-constant on X. In this case, it follows that $\mathcal{F}(\mathbb{1}_x) > 0$ and

$$R_{\mathcal{F}}(x,y) \geq \frac{|\mathbb{1}_x(x) - \mathbb{1}_x(y)|^2}{\mathcal{F}(\mathbb{1}_x)} = \frac{1}{\mathcal{F}(\mathbb{1}_x)} > 0.$$

Now let $x_1, \ldots, x_n \in X$ such that $x_1 = x$, $x_n = y$, $\mathcal{F}(\mathbb{1}_{x_i}, \mathbb{1}_{x_{i+1}}) < 0$ for all $i = 1, \ldots, n-1$, see Lemma 5.42.6. For $f \in l(X)$, f non-constant, we then have

$$\begin{aligned}
|f(x) - f(y)| &\leq \sum_{i=1}^{n-1} |f(x_i) - f(x_{i+1})| \\
&= \sum_{i=1}^{n-1} (-\mathcal{F}(\mathbb{1}_{x_i}, \mathbb{1}_{x_{i+1}})) \cdot |f(x_i) - f(x_{i+1})| \cdot \frac{1}{(-\mathcal{F}(\mathbb{1}_{x_i}, \mathbb{1}_{x_{i+1}}))} \\
&\leq \sqrt{\sum_{i=1}^{n-1} (-\mathcal{F}(\mathbb{1}_{x_i}, \mathbb{1}_{x_{i+1}})) \cdot (f(x_i) - f(x_{i+1})^2} \cdot \sqrt{\sum_{i=1}^{n-1} \frac{1}{(-\mathcal{F}(\mathbb{1}_{x_i}, \mathbb{1}_{x_{i+1}}))}} \\
&\leq \sqrt{\mathcal{F}(f)} \cdot \sqrt{\sum_{i=1}^{n-1} \frac{1}{(-\mathcal{F}(\mathbb{1}_{x_i}, \mathbb{1}_{x_{i+1}}))}}.
\end{aligned}$$

Hence,

$$\frac{|f(x) - f(y)|^2}{\mathcal{F}(f)} \leq \sqrt{\sum_{i=1}^{n-1} \frac{1}{(-\mathcal{F}(\mathbb{1}_{x_i}, \mathbb{1}_{x_{i+1}}))}} < \infty$$

and therefore $R_{\mathcal{F}}(x,y) < \infty$. □

Proposition 5.44. *Let $\mathcal{N} = (V, E, C)$ be a finite network and $\mathcal{E}$ its energy form. For $u, v \in l^2(V)$, we use the notation $\mathcal{E}(u,v) = \mathcal{E}(\nabla u, \nabla v)$. Then $(\mathcal{E}, l^2(V))$ is a resistance form on V and $R_{\mathcal{N}}(x,y) = R_{\mathcal{E}}(x,y)$ holds for all $x, y \in V$.*

Proof. Note that $R(e) = 1/C(e) > 0$ for all $e \in E$. We can explicitly check all conditions from Proposition 5.43:

1. $\mathcal{E}(u,u) = (\nabla u, \nabla u)_R = \frac{1}{2}\sum_{e\in E} R(e)(\nabla u)(e)^2 \geq 0$.
2. Suppose $\mathcal{E}(u,u) = \frac{1}{2}\sum_{e\in E} R(e)(\nabla u)(e)^2 = 0$. Then $(\nabla u)(e) = C(e)(u(\overline{e}) - u(\underline{e})) = 0$ and $u(\overline{e}) = u(\underline{e})$ for all $e \in E$. Since (V,E,C) is connected, we get $u(x) = u(y)$ for all $x,y \in V$.
3. It is quite easy to see that $|\widetilde{u}(x) - \widetilde{u}(y)| \leq |u(x) - u(y)|$ for all $x,y \in V$. Then $|\nabla\widetilde{u}(e)| \leq |\nabla u(e)|$ follows for all $e \in E$. Hence,

$$\mathcal{E}(\widetilde{u},\widetilde{u}) = \frac{1}{2}\sum_{e\in E} R(e)(\nabla\widetilde{u})(e)^2 \leq \frac{1}{2}\sum_{e\in E} R(e)(\nabla u)(e)^2 = \mathcal{E}(u,u).$$

It follows that $(\mathcal{E}, l(V))$ is a resistance form on V. Furthermore, by Proposition 5.38, we have

$$R_{\mathcal{N}}(x,y) = \sup_{\substack{u\in l^2(V)\\ \mathcal{E}(u)>0}} \frac{|u(x)-u(y)|^2}{\mathcal{E}(u)} = R_{\mathcal{E}}(x,y).$$

□

Proposition 5.45. *Let (X,d), $|X| \geq 2$, be a finite metric space such that $d(x,y) = R_{\mathcal{F}}(x,y)$ for some resistance form $(\mathcal{F}, l(X))$ on X. Then there exists a finite, simple network $\mathcal{N} = (X,E,C)$ with energy form $\mathcal{E}$ such that $\mathcal{F}(u) = \mathcal{E}(\nabla u)$ for all $u \in l(X)$.*

Proof. For $x,y \in X$, $x \neq y$, let $C(x,x) := 0$ and $C(x,y) := -\mathcal{E}(\mathbb{1}_x, \mathbb{1}_y) \geq 0$. Furthermore, let $E := \{(x,y,1) \in X \times X \times \mathbb{N} \mid C(x,y) > 0\}$. We claim that $\mathcal{N} = (X,E,C)$ is now a finite connected network with energy form $\mathcal{E}$ such that $\mathcal{E}(\nabla u) = \mathcal{F}(u)$ for all $u \in l(X)$:

1. (X,E) is a well-defined finite graph since $C(x,y) = C(y,x)$, i.e. $e \in E$ if and only if $\widehat{e} \in E$.
2. Using $\mathcal{F}(\mathbb{1}_x, 1) = 0$ and $C(x,x) = 0$, we get

$$0 < \mathcal{F}(\mathbb{1}_x, \mathbb{1}_x) = -\sum_{y\neq x} \mathcal{F}(\mathbb{1}_x, \mathbb{1}_y) = \sum_{y\neq x} C(x,y) = C_x.$$

   In particular, this means that there exist no isolated vertices in (X,E,C).
3. $\mathcal{N}$ is connected: Let $x,y \in X$, $x \neq y$. By Lemma 5.42.6, we know that there exist $x_1,\ldots,x_n \in X$ such that $x_1 = x$, $x_n = y$ and $0 < -\mathcal{F}(\mathbb{1}_{x_i}, \mathbb{1}_{x_{i+1}}) = C(x_i, x_{i+1})$. Hence, $(x_i, x_{i+1}, 1) \in E$ for $i = 1,\ldots,n-1$. It follows that there exists a path $x \to y$ in $\mathcal{N}$.

4. Let $x, y \in X$. Then

$$\begin{aligned}\mathcal{E}(\nabla \mathbb{1}_x, \nabla \mathbb{1}_y) = (\nabla \mathbb{1}_x, \nabla \mathbb{1}_y)_R = -(\mathbb{1}_x, \Delta \mathbb{1}_y)_C &= -\sum_{z \in X} C_z \mathbb{1}_x(z)(\Delta \mathbb{1}_y)(z) \\ &= -C_x(\Delta \mathbb{1}_y)(x) = C_x \left(\mathbb{1}_y(x) - \sum_{z \in X} \frac{C(x,z)}{C_x} \mathbb{1}_y(z) \right) \\ &= C_x \delta_{xy} - C(x,y).\end{aligned}$$

Hence, $\mathcal{E}(\nabla \mathbb{1}_x, \nabla \mathbb{1}_x) = C_x - C(x,x) = C_x = \mathcal{F}(\mathbb{1}_x, \mathbb{1}_x)$ and $\mathcal{E}(\nabla \mathbb{1}_x, \nabla \mathbb{1}_y) = -C(x,y) = \mathcal{F}(\mathbb{1}_x, \mathbb{1}_y)$ for $x \neq y$. Since $\{\mathbb{1}_x\}_{x \in X}$ is a basis of $l(X)$, this implies $\mathcal{E}(\nabla u) = \mathcal{F}(u)$ for all $u \in l(X)$. $\square$

6 Infinite electrical networks

Throughout this chapter let $\mathcal{N} = (V, E, C)$ be a locally finite, infinite connected network and let $\mathcal{E}$ denote its energy form. We will roughly follow Chapters 5 and 7 in [1] to cover the basics of infinite electrical networks and give some examples.

6.1 Hilbert spaces

As we have proven in Proposition 4.2, $l^2_-(E)$ and $l^2(V)$ are Hilbert spaces. We will introduce another class of functions on V, the *Dirichlet functions*.

Definition 6.1 (Dirichlet functions)**.** Let

1. $D(\mathcal{N}) := \{f : V \to \mathbb{R} \mid \mathcal{E}(f) < \infty\}$ be the space of *Dirichlet functions.*
2. $HD(\mathcal{N}) := \{f \in D(\mathcal{N}) \mid \Delta f \equiv 0\}$ be the space of *harmonic Dirichlet functions.*

As always, we will drop the $\mathcal{N}$ and just write D or HD if it is clear which network we are considering at the time.

Remark. If f is a constant function on V, then $\nabla f \equiv 0$ and therefore $f \in HD$. This raises a question: Are there non-constant functions in HD or does $HD \cong \mathbb{R}$ hold? The answer to this question depends on the network and will provide an important structural property.

Example 6.2 (A network admitting non-constant harmonic functions)**.** Consider the network from Example 5.8, namely $(G(\mathbb{Z}), C)$ where

$$C(e) = 2^{\min(|\overline{e}|, |\underline{e}|)},$$

see Figure 6.1. Now let

$$f(z) := \operatorname{sign}(z) \cdot \left(2 - 2^{1-|z|}\right).$$

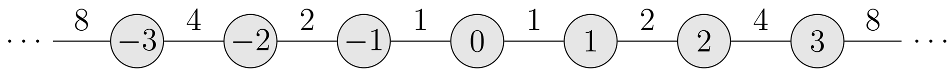

Figure 6.1: The network $(G(\mathbb{Z}), C)$.

Hence,

$$f(z+1) - f(z) = 2^{1-z} - 2^{-z} = 2^{-z} \quad \forall\ z \geq 0$$

and

$$f(z+1) - f(z) = 2^{2+z} - 2^{1+z} = 2^{1+z} \quad \forall\ z < 0.$$

It follows that

$$\begin{aligned}\mathcal{E}(f) &= \frac{1}{2}\sum_{e\in E} C(e)(f(\overline{e}) - f(\underline{e}))^2 = \sum_{z\in\mathbb{Z}} 2^{\min(|z|,|z+1|)}(f(z+1) - f(z))^2 \\ &= \sum_{z=-\infty}^{-1} 2^{-(z+1)} \cdot 4^{z+1} + \sum_{z=0}^{\infty} 2^z \cdot 4^{-z} = 2\sum_{z=0}^{\infty} 2^{-z} = 4 < \infty\end{aligned}$$

which means that $f \in D$. In Example 5.8 we have already seen that f is harmonic on Z and therefore $f \in HD$ holds. It is also easy to see that $\sum_{z\in\mathbb{Z}} C_z \cdot f(z)^2 = \infty$ since $f(z) \to \pm 2$ for $z \to \pm\infty$ and $C_z \geq 1$ for all $z \in \mathbb{Z}$. This means that f is a non-constant function in HD which does not belong to $l^2(V)$.

Example 6.3 (Networks whose harmonic functions are all constant)**.** Let $G(\mathbb{N}) := (\mathbb{N}, E_{\mathbb{N}})$ where $E_{\mathbb{N}} = \{(n, n+1, 1), (n+1, n, 1) \mid n \in \mathbb{N}\}$. Now consider the network $\mathcal{N} = (G(\mathbb{N}), C)$ where C is an arbitrary weight function and let $f \in HD$. We claim that $f(n) = f(1)\ \forall\ n \in \mathbb{N}$. This will be proven by induction as follows.

First, note that $C_1 = C(1,2)$ and $C_n = C(n, n-1) + C(n, n+1)$ for all $n > 1$. For $n = 1$, the claim is trivial, and for $n = 2$, we have

$$f(1) = \sum_{x\in\mathbb{N}} \frac{C(1,x)}{C_1} f(x) = \frac{C(1,2)}{C_1} f(2) = f(2)$$

because f is harmonic. Now let $n > 2$. We may assume that $f(k) = f(1)$ for all $k = 1, \ldots, n-1$. By the harmonicity of f at $n-1$, we have

$$\begin{aligned}f(1) = f(n-1) &= \sum_{x\in\mathbb{N}} \frac{C(n-1,x)}{C_{n-1}} f(x) \\ &= \frac{C(n-1,n-2)}{C_{n-1}} f(n-2) + \frac{C(n-1,n)}{C_{n-1}} f(n) \\ &= \frac{C(n-1,n-2)}{C_{n-1}} f(1) + \left(1 - \frac{C(n-1,n-2)}{C_{n-1}}\right) f(n).\end{aligned}$$

It follows that $f(n) = f(1)$. Hence, $HD(G(\mathbb{N}), C) \cong \mathbb{R}$ for every weight function C.

Definition 6.4 (Subspaces of $l^2_-(E)$)**.**

1. Let $\bigstar$ be the closure of the linear span of stars $-\nabla\mathbb{1}_x$, $x \in V$
$$\bigstar := \overline{\left\{\sum_{i=1}^{n} \lambda_i \cdot (-\nabla\mathbb{1}_{x_i}) \mid n \in \mathbb{N}, \lambda_i \in \mathbb{R}, x_i \in V\right\}}.$$
2. Analogously to the finite case, we call $\sum_{i=1}^{n} \chi^{e_i}$ a *cycle* if $(e_1, \ldots, e_n)$ is a cycle in (V, E).
3. Let $\diamondsuit$ be the closure of the linear span of cycles
$$\diamondsuit := \overline{\left\{\sum_{i=1}^{n} \lambda_i \cdot c_i \mid n \in \mathbb{N}, \lambda_i \in \mathbb{R}, c_i \in l^2_-(E) \text{ cycle}\right\}}.$$

Proposition 6.5 (BLPS [1])**.** *For a possibly infinite network, we have the orthogonal decomposition*
$$l^2_-(E) = \bigstar \oplus \nabla HD \oplus \diamondsuit.$$

Proof (BLPS [1]). $\bigstar \perp \diamondsuit$ holds analogously to the finite case because every star and every cycle are orthogonal, and $f_n \to f$ in $l^2_-(E)$ implies
$$(f_n, g)_R \to (f, g)_R \quad \forall\, g \in l^2_-(E).$$
Suppose $I \in (\bigstar \oplus \diamondsuit)^\perp = \bigstar^\perp \cap \diamondsuit^\perp$. Similar to the proof of Proposition 4.13 , $I \perp \diamondsuit$ implies that there exists a function $f \in D$ such that $I = \nabla f$, namely
$$f(y) = \sum_{i=1}^{n} R(e_i) I(e_i)\ ,\ y \in V$$
if $x \in V$ is a fixed vertex and $(e_1, \ldots, e_n)$ is a path $x \to y$. Since $I \perp \bigstar$, we also have
$$0 = (-\nabla\mathbb{1}_x, I)_R = (\mathbb{1}_x, \operatorname{div} I)_R = C_x \cdot (\operatorname{div} I)(x) \quad \forall\, x \in V.$$
Hence,
$$0 = (\operatorname{div} I)(x) = (\Delta f)(x)\ \forall\, x \in V,$$
i.e. $f \in HD$. Hence, $(\bigstar \oplus \diamondsuit)^\perp \subseteq \nabla HD$. Now assume that $I = \nabla f \in \nabla HD$, and let $c = \sum_{i=1}^{n} \chi^{e_i}$ be a cycle. Then
$$(I, c)_R = (\nabla f, c)_R = \sum_{i=1}^{n} (f(\overline{e_i}) - f(\underline{e_i})) = 0.$$

Furthermore, let $x \in V$. Then we have

$$(I, -\nabla \mathbb{1}_x)_R = (\operatorname{div} I, \mathbb{1}_x)_C = (\Delta f, \mathbb{1}_x)_C = 0.$$

It follows that $I \in (\bigstar \oplus \diamondsuit)^\perp$. □

Remark. Note that Proposition 4.13 is a special case of Proposition 6.5 since $\nabla HD(\mathcal{N}) = \{0\}$ for every finite network $\mathcal{N}$.

Definition 6.6 (Unit currents of infinite networks)**.** For $e \in E$, we define

1. $I_F^e := P_\diamondsuit^\perp \chi^e$ as the *free unit current* through e.
2. $I_W^e := P_\bigstar \chi^e$ as the *wired unit current* through e.

Lemma 6.7. *For $e \in E$, both I_F^e and I_W^e are electrical currents (with respect to Kirchhoff's laws 4.8 and 4.15).*

Proof. Since $\mathbb{1}_x \in l^2(V)$ for all $x \in V$, we have

$$\begin{aligned} C_x \cdot \operatorname{div}(I_F^e - \chi^e) &= (\operatorname{div}(I_F^e - \chi^e), \mathbb{1}_x)_C = (I_F^e - \chi^e, -\nabla \mathbb{1}_x)_R \\ &= (I_F^e - \chi^e, P_\diamondsuit^\perp(-\nabla \mathbb{1}_x))_R = (P_\diamondsuit^\perp(I_F^e - \chi^e), -\nabla \mathbb{1}_x)_R \\ &= (I_F^e - I_F^e, -\nabla \mathbb{1}_x)_R = 0 \end{aligned}$$

for all $x \in V$, and analogously $\operatorname{div}(I_W^e - \chi^e) = 0$. Hence,

$$\operatorname{div} I_F^e = \operatorname{div} I_F^e = \operatorname{div} \chi^e = \frac{1}{C_{\underline{e}}} \mathbb{1}_{\underline{e}} - \frac{1}{C_{\overline{e}}} \mathbb{1}_{\overline{e}} .$$

It follows that I_F^e and I_W^e satisfy Kirchhoff's current law 4.8. Since $\bigstar \subseteq \diamondsuit^\perp$, we have $I_F^e, I_W^e \in \diamondsuit^\perp$. Hence, I_F^e and I_W^e satisfy Kirchhoff's Voltage Law 4.15. □

Proposition 6.8 (Thomson principle, BLPS [1])**.**

$$\mathcal{E}(I_W^e) = \min \left\{ \mathcal{E}(I) \mid I \in l^2_-(E), \operatorname{div} I = \operatorname{div} \chi^e \right\} .$$

Furthermore, if $\operatorname{div} I = \operatorname{div} \chi^e$*, then $\mathcal{E}(I) = \mathcal{E}(I_W^e)$ holds if and only if $I = I_W^e$.*

Proof. The proof is analogous to its finite counterpart (Theorem 4.32). Let $I \in l^2_-(E)$ such that $\operatorname{div} I = \operatorname{div} \chi^e = \operatorname{div} I_W^e$. It follows that $\operatorname{div}(I - I_W^e) = 0$, i.e. $I - I_W^e \in \bigstar^\perp$, and therefore $I - I_W^e \perp I_W^e$. Hence,

$$\mathcal{E}(I) = \mathcal{E}(I - I_W^e + I_W^e) = \underbrace{\mathcal{E}(I - I_W^e)}_{\geq 0} - 2 \underbrace{(I - I_W^e, I_W^e)_R}_{=0} + \mathcal{E}(I_W^e) \geq \mathcal{E}(I_W^e).$$

Furthermore, $\mathcal{E}(I) = \mathcal{E}(I_W^e)$ holds if and only if $\mathcal{E}(I - I_W^e) = 0$, which is equivalent to $I = I_W^e$. □

Definition 6.9 (Exhaustion)**.**

1. Let $G = (V, E)$ be an infinite graph. An *exhaustion* of G is a sequence $(G_n)_{n\in\mathbb{N}}$ of finite, connected graphs $G_n = (V_n, E_n)$ such that $G_n = G(V_n)$, $G_n \subseteq G_{n+1}$ and
$$V = \bigcup_{n\in\mathbb{N}} V_n \ , \ E = \bigcup_{n\in\mathbb{N}} E_n \ .$$
When considering an infinite network $\mathcal{N} = (G, C)$, let $\mathcal{N}_n := (G_n, C\restriction_{E_n})$. We call $(\mathcal{N}_n)_{n\in\mathbb{N}}$ an *exhaustion* of $\mathcal{N}$.
2. Let $(G_n)_{n\in\mathbb{N}}$ be an exhaustion of $G = (V, E)$. Let $V_n^W = V_n \cup \{\omega_n\}$ for some new vertex ω_n. Furthermore, let $E_n^W \subseteq V_n^W \times V_n^W \times \mathbb{N}$ be a set of directed edges such that $\widehat{e} \in E_n^W$ for all $e \in E_n^W$ and there exists a bijection $\nu_n : E \to E_n^W$ satisfying $\eta_n^W(\widehat{e}) = \widehat{\eta_n^W(e)}$ for all $e \in E$ and
$$\underline{\nu_n(e)} = \begin{cases} \underline{e} & , \underline{e} \in V_n \\ \omega_n & , \underline{e} \notin V_n \end{cases} \ , \ \overline{\nu_n(e)} = \begin{cases} \overline{e} & , \overline{e} \in V_n \\ \omega_n & , \overline{e} \notin V_n \end{cases} \ .$$
We define $G_n^W := (V_n^W, E_n^W)$ and call $(G_n^W)_{n\in\mathbb{N}}$ a *wired exhaustion of* G. When considering an infinite network $\mathcal{N} = (G, C)$, let $\mathcal{N}_n^W := (G_n^W, C \circ \nu_n^{-1})$. We call $(\mathcal{N}_n^W)_{n\in\mathbb{N}}$ a *wired exhaustion* of $\mathcal{N}$.

Remarks.

1. G_n^W is not uniquely defined. However, using ν_n one can easily verify that two graphs G_n^W and $\widetilde{G_n^W}$, which both satisfy all requirements in the definition, are isomorphic.
2. The definition of G_n^W seems a little complicated, although the resulting object is quite intuitive: We merge all vertices of G that are not in G_n into ω_n. We basically keep all edges, adjusting them only slightly: If an endpoint of an edge $e \in E$ is not in V_n, we set it to ω_n.
3. The infinite network $\mathcal{N}_n^W = (G_n^W, C \circ \nu_n^{-1})$ is in general not locally finite since $C_{\omega_n} = \infty$ might occur. However, this does not pose a problem in our associated Hilbert spaces, because $C(\omega_n, v) < C_v < \infty$ for all $v \in V_n$, i.e. the 'infinite part' of edge weights only occurs in loops at ω_n.

Definition 6.10 (Increasing subspaces)**.** Let $\mathcal{H}$ be a Hilbert space, $X \subseteq \mathcal{H}$ a closed subspace and $(X_n)_{n\in\mathbb{N}}$ a sequence of closed subspaces of $\mathcal{H}$ such that $X_n \subseteq X_{n+1} \subseteq X$ for all $n \in \mathbb{N}$ and $\overline{\bigcup_{n\in\mathbb{N}} X_n} = X$ hold. Then we say that the sequence $(X_n)_{n\in\mathbb{N}}$ *increases to* X.

Proposition 6.11. *Let $\mathcal{H}$ be a Hilbert space and (X_n) a sequence of closed subspaces that increases to a closed subspace X of $\mathcal{H}$. Then*

$$\lim_{n\to\infty} \|P_{X_n}h - P_X h\|_{\mathcal{H}} = 0 \ \forall\ h \in \mathcal{H}$$

holds, where P_{X_n} and P_X denote the orthogonal projections onto X_n and X, respectively. This means that $\lim_{n\to\infty} P_{X_n}h = P_X h$ (in $\mathcal{H}$).

Proof. First, note that

$$\forall\ x \in X\ \forall\ \varepsilon > 0\ \exists\ n \in \mathbb{N}\ \exists\ x_n \in X_n : \|x_n - x\|_{\mathcal{H}} < \varepsilon$$

holds because for every $x \in X$, there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $\bigcup X_n$ such that $x_n \to x$ in $\mathcal{H}$. Since $X_n \subseteq X_{n+k}$ for all $k \in \mathbb{N}$, it follows that

$$\forall\ x \in X\ \forall\ \varepsilon > 0\ \exists\ n \in \mathbb{N}\ \forall\ k \geq n : \inf_{y\in X_k} \|y - x\|_{\mathcal{H}} < \varepsilon. \quad (*)$$

Now let $Y \subseteq \mathcal{H}$ be any closed subspace of $\mathcal{H}$. For every $h \in \mathcal{H}, y \in Y$, we have

$$\begin{aligned}\|y - h\|_{\mathcal{H}}^2 &= \|y - P_Y h + P_Y h - h\|_{\mathcal{H}}^2 \\ &= \|y - P_Y h\|_{\mathcal{H}}^2 + \underbrace{2\langle y - P_Y h, P_Y h - h\rangle_{\mathcal{H}}}_{=0} + \|P_Y h - h\|_{\mathcal{H}}^2 \\ &\geq \|P_Y h - h\|_{\mathcal{H}}^2\end{aligned}$$

because $P_Y h - h \in Y^\perp$. Hence, $\|P_Y h - h\|_{\mathcal{H}} = \inf_{y\in Y} \|y - h\|_{\mathcal{H}}$ for all $h \in \mathcal{H}$. Since $X_n \subseteq X$, it follows that $P_{X_n} = P_{X_n} P_X$ and therefore

$$\|P_{X_n}h - P_X h\|_{\mathcal{H}} = \|P_{X_n}P_X h - P_X h\|_{\mathcal{H}} = \inf_{y\in X_n} \|y - P_X h\|_{\mathcal{H}} \quad \forall\ h \in \mathcal{H}.$$

$(*)$ implies that

$$\forall\ h \in \mathcal{H}\ \forall\ \varepsilon > 0\ \exists\ n \in \mathbb{N}\ \forall\ k \geq n : \|P_{X_k}h - P_X h\|_{\mathcal{H}} < \varepsilon$$

or equivalently $\lim_{n\to\infty} \|P_{X_n}h - P_X h\|_{\mathcal{H}} = 0$ for all $h \in H$. □

Proposition 6.12 (BLPS [1])**.** *Let $(G_n)_{n\in\mathbb{N}}$ be an exhaustion of (V, E). The space $\diamondsuit_n$ of cycles in $\mathcal{N}_n$ and the space $\bigstar_n^W$ of stars of $\mathcal{N}_n^W$ can be embedded into $l^2_-(E)$. Due to this embedding, $(\diamondsuit_n)_{n\in\mathbb{N}}$ and $(\bigstar_n^W)_{n\in\mathbb{N}}$ are both sequences of closed subspaces in $l^2_-(E)$ that increase to $\diamondsuit$ and $\bigstar$, respectively.*

Proof. First, we prove the statement regarding the cycles and secondly, the statement regarding the stars.

1. We have $E_n \subseteq E$ and the inner product in $l^2_{\mathcal{N}_n,-}(E_n)$ is given by
$$(I,J)_{\mathcal{N}_n,R} = \frac{1}{2}\sum_{e\in E_n} R(e)I(e)J(e).$$
Hence, $l^2_{\mathcal{N}_n,-}(E_n)$ is isometrically isomorphic to the closed subspace
$$\left\{I \in l^2_-(E) \mid I\restriction_{E\setminus E_n} \equiv 0\right\}$$
of $l^2_-(E)$. Since every cycle $(e_1,\dots,e_n)$ in G_n is a cycle in G_{n+1} and G, it follows that
$$\diamondsuit_n \subseteq \diamondsuit_{n+1} \subseteq \diamondsuit.$$
Since $\bigcup E_n = E$, every cycle $(e_1,\dots,e_n)$ in G occurs in some G_n. Hence,
$$\bigcup_{n\in\mathbb{N}} \diamondsuit_n = \left\{\sum_{i=1}^n \lambda_i\cdot c_i \mid n\in\mathbb{N}, \lambda_i\in\mathbb{R}, c_i\in l^2_-(E) \text{ cycle}\right\}$$
and the claim follows.
2. Let $\nu_n : E \to E_n^W$ be the bijection described in Definition 6.9. For $I \in l^2_{\mathcal{N}_n^W,-}(E_n^W)$, let $\Psi_n(I) := I\circ\nu_n \in l^2_-(E)$. Then Ψ_n is a linear isometry between $l^2_{\mathcal{N}_n^W,-}(E_n^W)$ and $l^2_-(E)$. Indeed, for $I \in l^2_{\mathcal{N}_n^W,-}(E)$, we have
$$\|\Psi_n(I)\|^2_{\mathcal{N},R} = \frac{1}{2}\sum_{e\in E} C(e)I(\nu_n(e))^2 = \frac{1}{2}\sum_{e'\in E_n^W} C(\nu_n^{-1}(e'))I(e')^2 = \|I\|^2_{\mathcal{N}_n^W,R}\,.$$
Let ∇_n^W denote the gradient operator in $\mathcal{N}_n^W$. For $v\in V_n$, we have
$$\mathbb{1}_v(\overline{\nu_n(e)}) = \mathbb{1}_v(\overline{e})\ ,\ \mathbb{1}_v(\underline{\nu_n(e)}) = \mathbb{1}_v(\underline{e})$$
for all $e\in E$ by definition of ν_n. Hence,
$$\begin{aligned}\Psi_n(\nabla_n^W\mathbb{1}_v)(e) &= (\nabla_n^W\mathbb{1}_v)(\nu_n(e)) = C(\nu_n^{-1}(\nu_n(e)))\cdot(\mathbb{1}_v(\overline{\nu_n(e)}) - \mathbb{1}_v(\underline{\nu_n(e)}))\\ &= (\nabla\mathbb{1}_v)(e)\end{aligned}$$
for all $e\in E$, i.e. $\Psi_n(\nabla_n^W\mathbb{1}_v) = \nabla\mathbb{1}_v \in \bigstar$. Furthermore, we have
$$\mathbb{1}_{\omega_n}(\overline{\nu_n(e)}) = \mathbb{1}_{V\setminus V_n}(\overline{e})\ ,\ \mathbb{1}_{\omega_n}(\underline{\nu_n(e)}) = \mathbb{1}_{V\setminus V_n}(\underline{e}),$$
and therefore
$$\Psi_n(\nabla_n^W\mathbb{1}_{\omega_n}) = \sum_{v\in V\setminus V_n}\nabla\mathbb{1}_v \in \bigstar.$$
Since $\bigstar_n^W$ is the linear span of $\{\mathbb{1}_{\omega_n}, \nabla\mathbb{1}_v \mid v\in V_n\}$, it follows that $\bigstar_n^W \subseteq \bigstar$ for all $n\in\mathbb{N}$. Furthermore, the fact that
$$\begin{aligned}\Psi_n(\nabla_n^W\mathbb{1}_{\omega_n}) &= \sum_{v\in V\setminus V_n}\nabla\mathbb{1}_v = \sum_{v\in V\setminus V_{n+1}}\nabla\mathbb{1}_v + \sum_{v\in V_{n+1}\setminus V_n}\nabla\mathbb{1}_v\\ &= \Psi_{n+1}\left(\nabla_{n+1}^W\mathbb{1}_{\omega_{n+1}}\right) + \Psi_{n+1}\left(\sum_{v\in V_{n+1}\setminus V_n}\nabla_{n+1}^W\mathbb{1}_v\right) \in \bigstar_{n+1}^W\end{aligned}$$

holds, implies $\bigstar_n^W \subseteq \bigstar_{n+1}^W$ for all $n \in \mathbb{N}$. It is evident that

$$\bigcup_{n\in\mathbb{N}} \bigstar_n^W = \left\{ \sum_{i=1}^{n} \lambda_i \cdot (-\nabla \mathbb{1}_{x_i}) \mid n \in \mathbb{N}, \lambda_i \in \mathbb{R}, x_i \in V \right\}$$

because for every $v \in V$, there exists $n \in \mathbb{N}$ such that $v \in V_n$ and therefore $\nabla \mathbb{1}_v \in \bigstar_n^W$. It follows that $(\bigstar_n^W)_{n\in\mathbb{N}}$ increases to $\bigstar$. □

Proposition 6.13 (BLPS [1]). *Let $e \in E$ and $(G_n)_{n\in\mathbb{N}}$ be an exhaustion of G such that $e \in E_1$. Let $I^e_{\mathcal{N}_n}$ and $I^e_{\mathcal{N}_n^W}$ denote the unit currents through e in $\mathcal{N}_n$ and $\mathcal{N}_n^W$, respectively. Then*

1. $\left\| I^e_{\mathcal{N}_n} - I^e_F \right\|_R \overset{n\to\infty}{\to} 0$ *and* $I^e_F(e) = \mathcal{E}(I^e_F) \cdot R(e)$.
2. $\left\| I^e_{\mathcal{N}_n^W} - I^e_W \right\|_R \overset{n\to\infty}{\to} 0$ *and* $I^e_W(e) = \mathcal{E}(I^e_W) \cdot R(e)$.

Proof. Let $\diamondsuit_n$ and $\bigstar_n^W$ be as in Proposition 6.12 and let $\bigstar_n$ denote the space of stars in $\mathcal{N}$.

1. We have $I^e_{\mathcal{N}_n} = P_{\bigstar_n}\chi^e = \chi^e - P_{\diamondsuit_n}\chi^e$ by Proposition 4.13 and $I^e_F = P^{\perp}_{\diamondsuit}\chi^e = \chi^e - P_{\diamondsuit}\chi^e$. Using Propositions 6.11 and 6.12, we obtain

$$\left\| I^e_{\mathcal{N}_n} - I^e_F \right\|_R = \| P_{\diamondsuit}\chi^e - P_{\diamondsuit_n}\chi^e \|_R \to 0.$$

In $l^2_-(E)$, norm convergence implies point-wise convergence, i.e. it follows that $I^e_{\mathcal{N}_n}(f) \to I^e_F(f)$ for all $f \in E$. Since we also have $\mathcal{E}(I^e_{\mathcal{N}_n}) = \left\| I^e_{\mathcal{N}_n} \right\|_R^2 \to \| I^e_F \|_R^2 = \mathcal{E}(I^e_F)$, it follows that

$$\mathcal{E}(I^e_F) = \lim_{n\to\infty} \mathcal{E}(I^e_{\mathcal{N}_n}) = \lim_{n\to\infty} I^e_{\mathcal{N}_n}(e) \cdot R(e) = I^e_F(e) \cdot R(e).$$

2. We have $I^e_{G_n^W} = P_{\bigstar_n^W}\chi^e$ and $I^e_W = P_{\bigstar}\chi^e$. It follows that

$$\left\| I^e_{G_n^W} - I^e_W \right\|_R = \left\| P_{\bigstar_n^W}\chi^e - P_{\bigstar}\chi^e \right\|_R \to 0$$

and $\mathcal{E}(I^e_W) = \lim_{n\to\infty} \mathcal{E}(I^e_{G_n^W}) = \lim_{n\to\infty} I^e_{G_n^W}(e) \cdot R(e) = I^e_W(e) \cdot R(e)$. □

Remark. Note that the Thomson principle 6.8 implies that

$$I^e_W(e) = \mathcal{E}(I^e_W) \cdot R(e) \le \mathcal{E}(I^e_F) \cdot R(e) = I^e_F(e)$$

holds for all $e \in E$.

6.2 Spanning forrest measures

We introduce two probability measures on the set 2^E by taking the limits of spanning tree measures exhaustions of $\mathcal{N}$.

Lemma 6.14 (BLPS [1]). *Let $e \in E$ and $(G_n)_{n\in\mathbb{N}}$ be an exhaustion of G such that $e \in E_1$. Let $A \subseteq E_1$, $\widehat{A} = A$, be a set of directed edges such that $A \cup \{e, \widehat{e}\}$ does not contain any cycles and for any occuring network $\mathcal{M}$ let $I^e_{\mathcal{M}}$ denote the unit current through e in $\mathcal{M}$. Then*

1. $I^e_{\mathcal{N}_n}(e) \geq I^e_{\mathcal{N}_{n+1}}(e)$ *for all* $n \in \mathbb{N}$.
2. $I^e_{\mathcal{N}_n}(e) \geq I^e_{\mathcal{N}_n/A}(e)$ *for all* $n \in \mathbb{N}$.
3. $I^e_{\mathcal{N}_n/A}(e) \geq I^e_{\mathcal{N}_{n+1}/A}(e)$ *for all* $n \in \mathbb{N}$.
4. $I^e_{\mathcal{N}^W_n}(e) \leq I^e_{\mathcal{N}^W_{n+1}}(e)$ *for all* $n \in \mathbb{N}$.

Proof.

1. Since $G_n \subseteq G_{n+1}$, we can use Proposition 5.35 to obtain

$$I^e_{\mathcal{N}_n}(e) = C(e) \cdot \mathcal{E}_{\mathcal{N}_n}(I^e_{\mathcal{N}_n}) = C(e) \cdot R_{\mathcal{N}_n}(\underline{e}, \overline{e}) \geq C(e) \cdot R_{\mathcal{N}_{n+1}}(\underline{e}, \overline{e}) = I^e_{\mathcal{N}_{n+1}}(e).$$

2. As always, we identify the edges of G_n and G_n/A. Let $\bigstar_{\mathcal{N}_n}$ and $\bigstar_{\mathcal{N}_n/A}$ denote the subspaces of stars of $\mathcal{N}_n$ and $\mathcal{N}_n/A$. By Proposition 4.20, we have

$$\bigstar_{\mathcal{N}_n} = \bigstar_{\mathcal{N}_n/A} \oplus P_{\bigstar_{\mathcal{N}_n}}\left(\mathrm{span}\left\{\chi^a \mid a \in A\right\}\right).$$

Hence, $I^e_{\mathcal{N}_n} = I^e_{\mathcal{N}_n/A} + I$ for some $I \perp I^e_{\mathcal{N}_n/A}$. It follows that $\left\|I^e_{\mathcal{N}_n}\right\| = \left\|I^e_{\mathcal{N}_n/A}\right\| + \|I\|$ and therefore

$$I^e_{\mathcal{N}_n}(e) = C(e) \cdot \mathcal{E}_{\mathcal{N}_n}(I^e_{\mathcal{N}_n}) \geq C(e) \cdot \mathcal{E}_{\mathcal{N}_n/A}(I^e_{\mathcal{N}_n/A}) = I^e_{\mathcal{N}_n/A}(e).$$

3. This follows from 1. because $G_n \subseteq G_{n+1}$ implies $G_n/A \subseteq G_{n+1}/A$ by Lemma 2.25.
4. Let $B_n := \left\{e, \widehat{e} \mid e \in E^W_{n+1}, \underline{e} \in V_{n+1} \setminus V_n, \overline{e} = \omega_{n+1}\right\}$. For any $e \in E_1$, $B_n \cup \{e, \widehat{e}\}$ does not contain any cycle because otherwise $\underline{e}, \overline{e} \in V_{n+1} \setminus V_n$ would follow. Since $G^W_n = G^W_{n+1}/B_n$, statement 2 implies

$$I^e_{\mathcal{N}^W_n}(e) = I^e_{\mathcal{N}^W_{n+1}/B_n}(e) \leq I^e_{\mathcal{N}^W_{n+1}}(e).$$

□

Proposition 6.15 (BLPS [1]). *Let $A \subseteq E$, $\widehat{A} = A$, $|A| < \infty$, and $(G_n)_{n\in\mathbb{N}}$ be an exhaustion such that $A \subseteq E_1$. Let μ^F_n and μ^W_n denote the STM of $\mathcal{N}_n$ and $\mathcal{N}^W_n$, respectively. Then*

1. $\mu^F_n[A \subseteq T] \geq \mu^F_{n+1}[A \subseteq T]$ *for all* $n \in \mathbb{N}$, *and*
2. $\mu^W_n[A \subseteq T] \leq \mu^W_{n+1}[A \subseteq T]$ *for all* $n \in \mathbb{N}$.

Proof. We use the notation of Lemma 6.14 and prove both inequalities by induction on $|A|$. If A contains any cycles (or loops), $\mu_n^F[A \subseteq T] = \mu_n^W[A \subseteq T] = 0$ for all $n \in \mathbb{N}$ and there is nothing left to show. From now on we assume that A does not contain any cycles.

1. By Proposition 4.24, we have $\mu_n^F[e \in T] = I^e_{\mathcal{N}_n}(e)$ for all $n \in \mathbb{N}$ and all $e \in E_1$. Hence,

$$\mu_n^F[e \in T] = I^e_{\mathcal{N}_n}(e) \geq I^e_{\mathcal{N}_{n+1}}(e) = \mu_{n+1}^F[e \in T] \quad \forall\, n \in \mathbb{N}.$$

This implies the claim for all $A = \{e, \widehat{e}\} \subseteq E_1$.
Now suppose $|A| > 2$, and that the claim has already been proven for all $|A'| < |A|$. Let $e \in A$ and $A' := A \setminus \{e, \widehat{e}\}$. Then we can use Proposition 3.4 and Lemma 6.14.3 to compute

$$\begin{aligned}\mu_n^F[A \subseteq T] &= \mu_n^F[e \in T \mid A' \subseteq T] \cdot \mu_n^F[A' \subseteq T] \\ &= \mu_{\mathcal{N}_n/A'}[e \in T] \cdot \mu_n^F[A' \subseteq T] = I^e_{\mathcal{N}_n/A'}(e) \cdot \mu_n^F[A' \subseteq T] \\ &\geq I^e_{\mathcal{N}_{n+1}/A'}(e) \cdot \mu_{n+1}^F[A' \subseteq T] = \mu_{n+1}^F[e \in T \mid A' \subseteq T] \cdot \mu_{n+1}^F[A' \subseteq T] \\ &= \mu_{n+1}^F[A \subseteq T].\end{aligned}$$

2. Lemma 6.14.4 implies

$$\mu_n^W[e \in T] = I^e_{\mathcal{N}_n^W}(e) \leq I^e_{\mathcal{N}_{n+1}^W}(e) = \mu_{n+1}^W[e \in T] \;\forall\, e \in E_1,$$

i.e. the claim is proven for all $A = \{e, \widehat{e}\} \subseteq E_1$. Now suppose $|A| > 2$ and that the claim has already been proven for all $|A'| < |A|$. For some $e \in A$, let $A' := A \setminus \{e, \widehat{e}\}$ and let B_n as in Lemma 6.14. By Lemma 2.26, it follows that

$$\mathcal{N}_n^W/A' = (\mathcal{N}_{n+1}^W/B_n)/A' = \mathcal{N}_{n+1}^W/(B_n \cup A') = (\mathcal{N}_{n+1}^W/A')/B_n.$$

Lemma 6.14.2 now implies

$$I^e_{\mathcal{N}_n^W/A'}(e) = I^e_{(\mathcal{N}_{n+1}^W/A')/B_n}(e) \leq I^e_{\mathcal{N}_{n+1}^W/A'}(e)$$

for all $e \in E_1$. Hence,

$$\begin{aligned}\mu_n^W[A \subseteq T] &= \mu_n^W[e \in T \mid A' \subseteq T] \cdot \mu_n^W[A' \subseteq T] \\ &= \mu_{\mathcal{N}_n^W/A'}[e \in T] \cdot \mu_n^W[A' \subseteq T] = I^e_{\mathcal{N}_n^W/A'}(e) \cdot \mu_n^W[A' \subseteq T] \\ &\leq I^e_{\mathcal{N}_{n+1}^W/A'}(e) \cdot \mu_{n+1}^W[A' \subseteq T] = \mu_{n+1}^W[e \in T\;,\; A' \subseteq T] \cdot \mu_{n+1}^W[A' \subseteq T] \\ &= \mu_{n+1}^W[A \subseteq T]\end{aligned}$$

follows by induction. □

Proposition 6.16 (BLPS [1]). *Every exhaustion $(G_n)_{n\in\mathbb{N}}$ of G induces two probability measures μ^F and μ^W on 2^E such that*

$$\mu^F[A \subseteq T] = \lim_{n\to\infty} \mu_n^F[A \subseteq T]$$

and

$$\mu^W[A \subseteq T] = \lim_{n\to\infty} \mu_n^W[A \subseteq T]$$

for all finite $A \subseteq E$. Those measures are independent of the choice of exhaustion.

Proof. We will only proof the statement for μ^F since the proof for μ^W is completely analogous: Let $A \subseteq E$ be a finite set. Then $A \subseteq E_n$ for some $n \in \mathbb{N}$, and therefore

$$\mu^F[A \subseteq T] := \lim_{n\to\infty} \mu_n^F[A \subseteq T]$$

exists by Proposition 6.15. This limit is independent of the chosen exhaustion since for two exhaustions $(G_n)_{n\in\mathbb{N}}$ and $(G'_n)_{n\in\mathbb{N}}$ there exists an exhaustion $(\widetilde{G_n})_{n\in\mathbb{N}}$ that contains infinitely many G_n and G'_n. Indeed, we can define $\widetilde{G_n}$ inductively: Let $\widetilde{G_1} := G_1$ and suppose $n > 1$. If n is even, let $m = \min\left\{k \geq n \mid \widetilde{G_{n-1}} \subseteq G_k\right\}$ and set $\widetilde{G_n} := G_m$. If n is odd, let $m = \min\left\{k \geq n \mid \widetilde{G_{n-1}} \subseteq G'_k\right\}$ and set $\widetilde{G_n} := G'_m$. The fact that $\lim_{n\to\infty} \mu_n^F[A \subseteq T]$ exists for $(\widetilde{G_n})_{n\in\mathbb{N}}$ implies that the corresponding limits for $(G_n)_{n\in\mathbb{N}}$ and $(G'_n)_{n\in\mathbb{N}}$ are equal because they are both cluster points of a convergent sequence.

For $B \subseteq K \subsetneqq E$, $|K| < \infty$, let $\{T \cap K = B\} = \{T \subseteq E \mid T \cap K = B\}$. For a fixed K, we can define μ^F on those sets by induction on $m := |K \setminus B|$ as follows. For $m = 0$, we have $\{T \cap K = B\} = \{K \subseteq T\}$. Hence, we set $\mu^F[T \cap K = B] := \mu^F[K \subseteq T]$. Now let $m > 0$ and suppose that $\mu^F[T \cap K = B]$ is already defined for all $B' \subseteq K$ such that $|K \setminus B'| < m$. We define

$$\mu^F[T \cap K = B] := \mu^F[B \subseteq T] - \sum_{k=0}^{m-1} \sum_{x_1,\ldots,x_k \in K\setminus B} \mu^F[T \cap K = K \setminus \{x_1, \ldots, x_k\}].$$

In particular, $\mu^F[T \cap K = B] = \lim_{n\to\infty} \mu_n^F[T \cap K = B]$.

Let $\mathcal{C} := \{\{T \cap K = B\} \mid B \subseteq K \subsetneqq E, |K| < \infty\} \cup \{\emptyset\}$. Then $\mathcal{C}$ is semialgebra of sets. Indeed, for $B_i \subseteq K_i \subsetneqq E$, $|K_i| < \infty$, $i = 1, 2$, we have

$$\{T \cap K_1 = B_1\} \cap \{T \cap K_2 = B_2\} = \{T \cap (K_1 \cup K_2) = B_1 \cup B_2\} \in \mathcal{C}$$

if $B_1 \cap K_2 = B_2 \cap K_1$, and

$$\{T \cap K_1 = B_1\} \cap \{T \cap K_2 = B_2\} = \emptyset \in \mathcal{C}$$

otherwise. Furthermore, $\emptyset^C = 2^E = \{T \cap \emptyset = \emptyset\}$ and

$$\{T \cap K = B\}^C = \bigcup_{B' \subseteq K, B' \neq B} \{T \cap K = B'\}$$

for all $B \subseteq K \subsetneqq E$, $|K| < \infty$. Since μ^F is the point-wise limit of μ_n^F on $\mathcal{C}$, it follows that

$$\mu^F(A) = \sum_{i=1}^{\infty} \mu^F(A_i)$$

whenever $\bigcup A_n = A \in \mathcal{C}$ for $A_i \in \mathcal{C}$, $A_i \cap A_j = \emptyset$ for all $i \neq j$. Hence, μ^F is a measure on the algebra generated by $\mathcal{C}$ by Proposition 9 in [8] (p. 297-298), and therefore Caratheodory's Extension Theorem (see [8], p. 295) implies that μ^F has a unique extension to 2^E. □

Definition 6.17 (Spanning forrest measures)**.** We call μ^F the *free spanning forrest measure* of $\mathcal{N}$ and μ^W the *wired spanning forrest measure* of $\mathcal{N}$.

Proposition 6.18 (BLPS [1])**.** *Let $e_1, \ldots, e_k \in E$ be distinct edges. Then we have*

$$\mu^F[e_1, \ldots, e_k \in T] = \det\left((I_F^{e_i}(e_j))_{1 \leq i,j \leq k}\right)$$

and

$$\mu^W[e_1, \ldots, e_k \in T] = \det\left((I_W^{e_i}(e_j))_{1 \leq i,j \leq k}\right).$$

Proof (BLPS [1]). This is a direct consequence of Proposition 6.16, Theorem 4.26 and Proposition 6.13:

$$\begin{aligned}\mu^F[e_1, \ldots, e_k \in T] &= \lim_{n \to \infty} \mu_n^F[e_1, \ldots, e_k \in T] = \lim_{n \to \infty} \det((I_n^{e_i}(e_j))_{1 \leq i,j \leq k}) \\ &= \det((I_F^{e_i}(e_j))_{1 \leq i,j \leq k})\end{aligned}$$

The statement for μ^W is proven analogously. □

Remark. Note that Proposition 6.18 together with the Thomson principle (Proposition 6.8) implies that

$$\mu^F[e \in T] = I_F^e(e) \geq I_W^e(e) = \mu^W[e \in T] \quad \forall\, e \in T.$$

Theorem 6.19 (BLPS [1])**.** *The following statements are equivalent:*

(i) $\mu^F = \mu^W$.

(ii) $I_F^e = I_W^e$ *for all* $e \in E$.

(iii) $l^2_-(E) = \bigstar \oplus \Diamond$.
(iv) $HD(\mathcal{N}) \cong \mathbb{R}$.

Proof (BLPS [1]).
(i) ⇔ (ii): If $\mu^F = \mu^W$, then

$$I_F^e(e) = \mu^F[e \in T] = \mu^W[e \in T] = I_W^e(e) \quad \forall\, e \in E.$$

By Proposition 6.13, it follows that $\mathcal{E}(I_F^e) = \mathcal{E}(I_W^e)$ and therefore $I_F^e = I_W^e$ for all $e \in E$ by the Thomson principle (Proposition 6.8).

Suppose that $I_F^e = I_W^e$ for all $e \in E$. In the proof of Proposition 6.16 we have seen that it suffices to know the values of μ^F and μ^W on sets of the form $\{A \subseteq T\}$ for $A \subseteq E$, $|A| < \infty$. By the infinite version of the Current Matrix Theorem (Proposition 6.18), it follows that μ^F and μ^W coincide on these sets. Hence, $\mu^F = \mu^W$.

(ii) ⇔ (iii):
Let E' be an orientation of G. By Lemma 4.4, $\{\chi^e \mid e \in E'\}$ is an orthogonal basis of $l^2_-(E)$. Hence, $P_\Diamond^\perp \chi^e = I_F^e = I_W^e = P_\bigstar \chi^e$ for all $e \in E'$ is equivalent to $P_\Diamond^\perp = P_\bigstar$ which in turn is equivalent to $\Diamond^\perp = \bigstar$. Proposition 6.5 implies that this is equivalent to (iii).

(iii) ⇔ (iv):
By Proposition 6.5, (iii) is equivalent to $\nabla HD(\mathcal{N}) = \{0\}$ which in turn is equivalent to $HD = \{f : V \to \mathbb{R} \mid f \text{ constant on } V\}$, i.e. $HD \cong \mathbb{R}$. □

6.3 Effective resistance

Definition 6.20. For $x, y \in V$, we define the *free unit current* I_F^{xy} *from* x *to* y by

$$I_F^{xy} = \sum_{i=1}^{n} I_F^{e_i}$$

where $(e_1, \ldots, e_n)$ is a path $x \to y$.

Remark. As in the finite case, the definition of I_F^{xy} is independent from the actual choice of $(e_1, \ldots, e_n)$ because $I_F^{e_i}$ is orthogonal to $\Diamond$.

Definition 6.21 (Effective resistance of infinite networks)**.** Let $\mathcal{N} = (V, E, C)$ be an infinite network and $x, y \in V$. The *effective resistance* of $\mathcal{N}$ between x

and y is defined as

$$R_{\mathcal{N}}(x,y) := \mathcal{E}(I_F^{xy}).$$

If the context makes clear which network we are considering, we will drop the index and write $R(x,y)$.

Proposition 6.22. *Let $x, y \in V$ and $(G_n)_{n\in\mathbb{N}}$ be an exhaustion of G such that $x, y \in V_1$. Then*

$$R(x,y) = \lim_{n\to\infty} R_{\mathcal{N}_n}(x,y).$$

Proof. We know that $x, y \in V_1$ and G_1 is connected. Hence, there exists a path $(e_1, \ldots, e_m)$ from x to y in G_1. Then $e_1, \ldots, e_m \in E_n$ for all $n \in \mathbb{N}$. Let $I_{\mathcal{N}_n}^{xy}$ denote the unit current from x to y in $\mathcal{N}_n$. Proposition 6.13 implies

$$0 \leq \left\| I_{\mathcal{N}_n}^{xy} - I_F^{xy} \right\|_R = \left\| \sum_{i=1}^m \left(I_{\mathcal{N}_n}^{e_i} - I_F^{e_i} \right) \right\|_R \leq \sum_{i=1}^m \left\| I_{\mathcal{N}_n}^{e_i} - I_F^{e_i} \right\|_R \to 0.$$

Hence, $\left\| I_{\mathcal{N}_n}^{xy} - I_F^{xy} \right\|_R \to 0$ and therefore

$$R_{\mathcal{N}_n}(x,y) = \mathcal{E}_{\mathcal{N}_n}(I_{\mathcal{N}_n}^{xy}) = \mathcal{E}(I_{\mathcal{N}_n}^{xy}) = \left\| I_{\mathcal{N}_n}^{xy} \right\|_R^2 \to \left\| I_F^{xy} \right\|_R^2 = \mathcal{E}(I_F^{xy}) = R(x,y).$$

□

Remark. Proposition 6.22 implies that the inequalities in Lemma 5.32, Lemma 5.33, Theorem 5.34, and Proposition 5.35 hold for infinite networks as well.

Proposition 6.23. *Using the notation $\mathcal{E}(u) := \mathcal{E}(\nabla u)$ for $u \in D(\mathcal{N})$, we can interpret $(\mathcal{E}, D(\mathcal{N}))$ as a resistance form on V.*

Proof. We show (RF-1) - (RF-5) of Definition 5.41.

(RF-1) We have $1 \in D(\mathcal{N})$ since $\nabla 1 = 0$. Furthermore, $\mathcal{E}(u) = 0$ holds if and only if $\nabla u \equiv 0$. This is equivalent to u being constant on V.

(RF-2) It is easily verified that $D(\mathcal{N})$ is a real vector space and that $\mathcal{E} + o_v$ is an inner product on $D(\mathcal{N})$ for all $v \in V$. The completeness of $(D(\mathcal{N}), \mathcal{E} + o_v)$ follows from the completeness of $(l^2_-(E), \mathcal{E})$: If (f_n) is a Cauchy sequence in $D(\mathcal{N})$, then (∇f_n) is a Cauchy sequence in $l^2_-(E)$ and therefore has a limit I. Since $\diamondsuit^\perp$ is a closed subspace of $l^2_-(E)$, we have that $I = \nabla f$ where f is determined up to the addition of a constant. However, it follows that

$$(f_n(v) - f_m(v))^2 = o_v(f_n - f_m) \leq \mathcal{E}(f_n - f_m) + o_v(f_n - f_m) \to 0$$

because (f_n) is a Cauchy sequence in $D(\mathcal{N})$. Hence, $(f_n(v))$ has a unique limit $f(v)$ in $\mathbb{R}$, which determines the mentioned constant of f. It follows that $f \in D(\mathcal{N})$ (because $\nabla f = I \in l^2_-(E)$) and $\mathcal{E}(f_n - f) + o_v(f_n - f) \to 0$.

(RF-3) For a finite set $W \subseteq V$ and $u \in l(V)$, we have $g := u \cdot \mathbb{1}_W \in l^2(V) \subseteq D(\mathcal{N})$ and $g \restriction_W = u$.

(RF-4) For all $x \in V$, we have

$$0 < \frac{1}{C_x - C(x,x)} = \frac{1}{\mathcal{E}(\mathbb{1}_x)} \leq R_{\mathcal{E}}(x,y) \; \forall \; y \in V, x \neq y.$$

Now fix $x, y \in V$, $x \neq y$, and let $(e_1, \ldots, e_n)$ be a path $x \to y$ in $\mathcal{N}$. For $u \in D(\mathcal{N})$, u non-constant, we use the Cauchy-Schwarz inequality to compute

$$\begin{aligned} |u(x) - u(y)| &\leq \sum_{i=1}^{n} |u(\overline{e_i}) - u(\underline{e_i})| \leq \sum_{i=1}^{n} C(e_i) \cdot |u(\overline{e_i}) - u(\underline{e_i})| \cdot \frac{1}{C(e_i)} \\ &\leq \sqrt{\sum_{i=1}^{n} C(e_i) \cdot (u(\overline{e_i}) - u(\underline{e_i}))^2} \cdot \sqrt{\sum_{i=1}^{n} \frac{1}{C(e_i)}} \\ &\leq \sqrt{\mathcal{E}(u)} \cdot \sqrt{\sum_{i=1}^{n} \frac{1}{C(e_i)}} \; . \end{aligned}$$

Hence, $R_{\mathcal{E}}(x,y) \leq d_{(V,E)}(x,y) < \infty$ where $d_{(V,E)}(x,y)$ denotes the geodesic metric of the graph (V,E).

(RF-5) Analogously to the proof of Proposition 5.44. Let $u \in D(\mathcal{N})$. Then $|\widetilde{u}(x) - \widetilde{u}(y)| \leq |u(x) - u(y)|$ holds for all $x, y \in V$ and implies $|\nabla \widetilde{u}|(e) \leq |\nabla u|(e)$ for all $e \in E$. Hence, $\mathcal{E}(\widetilde{u}) \leq \mathcal{E}(u)$. In particular, $\widetilde{u} \in D(\mathcal{N})$ follows. □

Remark. Kasue shows at the beginning of Chapter 5 in [5] that

$$\lim_{n \to \infty} R_{\mathcal{E}_n}(x,y) = R_{\mathcal{E}}(x,y)$$

if $(\mathcal{N}_n)_{n \in \mathbb{N}}$ is an exhaustion of $\mathcal{N}$ and $\mathcal{E}_n$ is the energy form of $\mathcal{N}_n$. By Propositions 6.22 and 5.38, it follows that

$$\begin{aligned} \mathcal{E}(I_F^{xy}) &= R(x,y) = \lim_{n \to \infty} R_{\mathcal{N}_n}(x,y) \\ &= \lim_{n \to \infty} R_{\mathcal{E}_n}(x,y) = R_{\mathcal{E}}(x,y) = \sup_{\substack{u \in D \\ \mathcal{E}(\nabla u) > 0}} \frac{|u(x) - u(y)|^2}{\mathcal{E}(\nabla u)} \end{aligned}$$

where we interpret $\mathcal{E}_n$ and $\mathcal{E}$ as resistance forms on V_n and V, respectively.

7 Some limiting properties of networks

Kasue's works [4] and [5] show a connection between the Gamma convergence of the energy forms of a sequence of networks and the convergence of their effective resistances. In this chapter we define Gamma convergence and will see that for finite networks the Gamma convergence of energy forms is equivalent to the point-wise convergence of the effective resistance, see Proposition 7.7. This is a lemma Kasue states in [4] without a proof and it does not occur in his later, more extensive work [5] on this matter. At the end of the chapter we give two examples why this lemma cannot be easily extended to infinite networks.

7.1 Gamma convergence

Definition 7.1 (Gamma convergence)**.** Let (X, d) be a metric space. We say that a sequence of functions $F_n : X \to \mathbb{R}$ on X *converges to a function* $F : X \to \mathbb{R}$ *on* X *in the* Γ*-sense* if

(Γ1) For $x_n \to x$ in X we have $F(x) \leq \liminf_{n\to\infty} F_n(x_n)$.

(Γ2) For $x \in X$ $\exists$ $(x_n)_{n\in\mathbb{N}}$ in X such that $x_n \to x$ and $\limsup_{n\to\infty} F_n(x_n) \leq F(x)$.

We use the notation $F_n \stackrel{\Gamma}{\to} F$.

Remark.

1. A sequence obtained by property (Γ2) is called a *recovery sequence.*
2. Under (Γ1), (Γ2) is equivalent to
$$\forall\, x \in X\ \exists\ (x_n)_{n\in\mathbb{N}} \subseteq X, x_n \to x : F(x) = \lim_{n\to\infty} F_n(x_n).$$
This condition obviously implies (Γ2). If we assume (Γ1) and (Γ2) to be satisfied and (x_n) to be a recovery sequence for x, then
$$F(x) \stackrel{(\Gamma1)}{\leq} \liminf_{n\to\infty} F_n(x_n) \leq \limsup_{n\to\infty} F_n(x_n) \stackrel{(\Gamma2)}{\leq} F(x).$$
Hence, $F(x) = \liminf F_n(x_n) = \limsup F_n(x_n) = \lim F(x_n)$.
3. If X is a space of functions and we consider Γ-convergence of functionals, we will allow these functionals to have value $+\infty$.

Lemma 7.2 (Uniqueness of Γ-limits)**.** *Let* $F_n \stackrel{\Gamma}{\to} F_1$ *and* $F_n \stackrel{\Gamma}{\to} F_2$*. Then* $F_1 = F_2$.

Proof. Let $x \in X$ and $(x_n^{(1)})$ and $(x_n^{(2)})$ be recovery sequences for x with respect to the Γ-limits F_1 and F_2, respectively. Then
$$F_1(x) \stackrel{(\Gamma2)}{=} \lim_{n\to\infty} F_n(x_n^{(1)}) = \liminf_{n\to\infty} F_n(x_n^{(1)}) \stackrel{(\Gamma1)}{\leq} F_2(x)$$
and
$$F_2(x) \stackrel{(\Gamma2)}{=} \lim_{n\to\infty} F_n(x_n^{(2)}) = \liminf_{n\to\infty} F_n(x_n^{(2)}) \stackrel{(\Gamma1)}{\leq} F_1(x).$$
Hence, $F_1(x) = F_2(x)$ for all $x \in X$. □

Lemma 7.3 (Subsequences of Γ-convergent sequences)**.** *Let* $F_n \xrightarrow{\Gamma} F$*, and let* $(F_{n_k})_{k\in\mathbb{N}}$ *be a subsequence of* $(F_n)_{n\in\mathbb{N}}$*. Then* $F_{n_k} \xrightarrow{\Gamma} F$*.*

Proof. For any sequence (a_n) of real numbers and any subsequence (a_{n_k}), we have

$$\liminf_{n\to\infty} a_n \leq \liminf_{k\to\infty} a_{n_k} \ , \ \ \limsup_{k\to\infty} a_{n_k} \leq \limsup_{n\to\infty} a_n$$

because $\{a_{n_k} \mid k \geq n_0\} \subseteq \{a_n \mid n \geq n_0\}$ for all $n_0 \in \mathbb{N}$. Using this we can verify that $F_{n_k} \xrightarrow{\Gamma} F$:

(Γ1) Suppose $x_{n_k} \to x$ for $k \to \infty$ and set $x_n := x_{n_k}$ if $n = n_k$ for some $k \in \mathbb{N}$ and $x_n := x$ otherwise. Then $x_n \to x$, and

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n) \leq \liminf_{k\to\infty} F_{n_k}(x_{n_k}).$$

(Γ2) Let $x \in X$ and x_n be a recovery sequence for x with respect to (F_n). Then $x_{n_k} \to x$, and

$$\limsup_{k\to\infty} F_{n_k}(x_{n_k}) \leq \limsup_{n\to\infty} F_n(x_n) \leq F(x).$$

□

Lemma 7.4 (Minima and Γ-Convergence)**.** *Let* $F_n \xrightarrow{\Gamma} F$ *and* $x_n \in X$ *such that* $F_n(x_n) = \inf_{x'\in X} F_n(x')$ *and* $x_n \to x$*. Then*

1. $F(x) = \inf_{x'\in X} F(x')$.
2. $\lim_{n\to\infty} F_n(x_n) = F(x)$.

Proof.

1. Let $x' \in X$ and (x'_n) be a recovery sequence for x'. Then

$$F(x) \overset{\Gamma 1)}{\leq} \liminf F_n(x_n) \leq \liminf F_n(x'_n) \leq \limsup F_n(x'_n) \overset{\Gamma 2)}{\leq} F(x').$$

Hence, $F(x) \leq F(x')$ for all $x' \in X$.

2. Let x'_n be a recovery sequence for x. Then $F_n(x_n) \leq F_n(x'_n)$ for all $n \in \mathbb{N}$, and

$$\limsup F_n(x'_n) \overset{\Gamma 2)}{\leq} F(x) \overset{\Gamma 1)}{\leq} \liminf F_n(x_n) \leq \limsup F_n(x_n) \leq \limsup F_n(x'_n).$$

Hence, $F(x) = \liminf F_n(x_n) = \limsup F_n(x_n) = \lim F_n(x_n)$. □

Remark. Using Lemma 7.3, we can slightly generalize Lemma 7.4. If x_n is a sequence of minimizers of F_n that has a convergent subsequence $x_{n_k} \to x \in X$, we still have $F(x) = \inf_{x'\in X} F(x')$ and $\lim_{k\to\infty} F_{n_k}(x_{n_k}) = F(x)$.

7.2 Effective resistance and Gamma convergence

Definition 7.5. Let V_n and V be finite sets and $f_n : V_n \to V$.

1. We say that a sequence of functions $u_n \in l(V_n)$ *converges uniformly (via f_n)* to a function $u \in V$ if $\lim_{n\to\infty} \sup_{x\in V_n} |u(f_n(x)) - u_n(x)| = 0$.
2. Let $\mathcal{E}_n$ and $\mathcal{E}$ be resistance forms on V_n and V, respectively. We say $\mathcal{E}_n$ *converges to $\mathcal{E}$ in the Γ-sense* (via f_n) if for all $u \in l(V_n)$ the following hold:
   (Γ1) $\mathcal{E}(u) \le \liminf_{n\to\infty} \mathcal{E}_n(u_n)$ for all sequences $u_n \in l(V_n)$ that converge uniformly (via f_n) to u.
   (Γ2) There exist $u_n \in l(V_n)$ that converge uniformly to u such that
$$\limsup_{n\to\infty} \mathcal{E}_n(u_n) \le \mathcal{E}(u).$$

Remarks. If $V_n = V$ for all $n \in \mathbb{N}$, we set $f_n(x) = x$ for all $x \in V$.

Lemma 7.6. *Let $\mathcal{N}_n = (V, E, C_n)$ be simple networks with associated energy forms $\mathcal{E}_n := \mathcal{E}_{\mathcal{N}_n}$. If $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$ to a resistance form $\mathcal{E}$ on V such that $\mathbb{1}_x, \mathbb{1}_y \in D[\mathcal{E}]$ then*
$$\sup_{n\in\mathbb{N}} C_n(x,y) < \infty.$$

Proof. Suppose $\sup C_n(x,y) = \infty$. Then $x \ne y$, and there exists a subsequence (n_k) such that $C_{n_k}(x,y) \to \infty$. Let f_n be a recovery sequence for $\mathbb{1}_x$, i.e. $f_n \to \mathbb{1}_x$ and $\mathcal{E}_n(f_n) \to \mathcal{E}(\mathbb{1}_x)$. Since
$$\lim_{n\to\infty} (f_n(v) - f_n(w))^2 = (\mathbb{1}_x(v) - \mathbb{1}_x(w))^2 = \begin{cases} 1 & , \ v = x \ne w \text{ or } v \ne x = w \\ 0 & , \ \text{otherwise} \end{cases},$$
and $C_n(v,w)(f_n(w) - f_n(v))^2 \ge 0$ for all $n \in \mathbb{N}, v, w \in V$, we have
$$\begin{aligned}
\infty &= \lim_{k\to\infty} C_{n_k}(x,y)(f_{n_k}(y) - f_{n_k}(x))^2 \\
&\le \sum_{w\in V} \liminf_{k\to\infty} C_{n_k}(x,w)(f_{n_k}(w) - f_{n_k}(x))^2 \\
&\le \sum_{v\in V}\sum_{w\in V} \liminf_{k\to\infty} C_{n_k}(v,w)(f_{n_k}(w) - f_{n_k}(v))^2 \\
&= \sum_{e\in E} \liminf_{k\to\infty} C_{n_k}(e)(f_{n_k}(\overline{e}) - f_{n_k}(\underline{e}))^2 \\
&\le \liminf_{k\to\infty} \sum_{e\in E} C_{n_k}(e)(f_{n_k}(\overline{e}) - f_{n_k}(\underline{e}))^2 \\
&= \liminf_{k\to\infty} 2 \cdot \mathcal{E}_{n_k}(f_{n_k}) = 2 \lim_{n\to\infty} \mathcal{E}_n(f_n) = 2 \cdot \mathcal{E}(\mathbb{1}_x) < \infty.
\end{aligned}$$
This is clearly a contradiction. Hence, the statement follows. ☐

Remark. We are using the limes inferior in the last computation because it is not clear if $C_{n_k}(e)$ converges for any given $e \in E$.

Proposition 7.7 (Kasue [4]). *Let (V, E_n, C_n) be finite simple networks with the same vertex set V. Let $\mathcal{E}_n$ and R_n be the associated energy forms and effective resistances, respectively. Then $R(x,y) := \lim_{n\to\infty} R_n(x,y) \in \mathbb{R}$ exists for all $x, y \in V$, and R is a metric on V if and only if $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$, where $\mathcal{E}$ is a resistance form. In the case of convergence, there exists a simple network with vertex set V such that R and $\mathcal{E}$ are its effective resistance and energy form.*

Proof. First, we show that $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$ implies the point-wise convergence of R_n to a metric on V. Secondly, we show the converse implication.

1. Let $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$ for a resistance form $\mathcal{E}$ on $l^2(V)$. Fix $x, y \in V$ and let $\mathcal{E}_n^*(u) := \mathcal{E}_n(u)$ if $u(x) = 1, u(y) = 0$ and $\mathcal{E}_n^*(u) = +\infty$ otherwise. Then

$$\min_{\substack{u \in l^2(V) \\ u(x)=1, u(y)=0}} \mathcal{E}_n(u) = \min_{u \in l^2(V)} \mathcal{E}_n^*(u).$$

We will show that $\mathcal{E}_n^* \xrightarrow{\Gamma} \mathcal{E}^*$ (where $\mathcal{E}^*$ is defined analogously to $\mathcal{E}_n^*$):

Γ1) Let $u_n \to u$ uniformly. Since V is finite, this is equivalent to $u_n \to u$ point-wise. If $u(x) \neq 1$ or $u(y) \neq 0$, then there exists $n_0 \in \mathbb{N}$ such that $u_n(x) \neq 1$ or $u_n(y) \neq 0$ for all $n \geq n_0$. In this case, we have

$$\mathcal{E}^*(u) = \infty = \liminf_{n\to\infty} \mathcal{E}_n^*(u_n).$$

If $u(x) = 1$ and $u(y) = 0$, then

$$\mathcal{E}^*(u) = \mathcal{E}(u) \leq \liminf_{n\to\infty} \mathcal{E}_n(u_n) \leq \liminf_{n\to\infty} \mathcal{E}_n^*(u_n).$$

Γ2) Let $u \in l^2(V)$ and (u_n) be a recovery sequence for u, i.e. $\mathcal{E}_n(u_n) \to \mathcal{E}(u)$. If $u(x) \neq 1$ or $u(y) \neq 0$ as above ,we get $\mathcal{E}_n^*(u_n) = +\infty$ for all sufficiently large n. Hence,

$$\lim_{n\to\infty} \mathcal{E}_n^*(u_n) = \infty = \mathcal{E}^*(u).$$

If $u(x) = 1$ and $u(y) = 0$, let $u_n^* = \mathbb{1}_x + \mathbb{1}_{V\setminus\{x,y\}} u_n = \mathbb{1}_{\{x,y\}} u + \mathbb{1}_{V\setminus\{x,y\}} u_n$. Then $u_n^* \to u$ point-wise, and since V is finite, by Lemma 7.6, we know that

$$\sup_{n\in\mathbb{N}} C_n(v,w) < \infty \quad \forall\ v, w \in V.$$

Hence,

$$\begin{aligned}\mathcal{E}_n(u_n^*) - \mathcal{E}_n(u_n) &= \sum_{v\in V\setminus\{x,y\}} C_n(x,v)((u_n(v)-1)^2 - (u_n(v)-u_n(x))^2) \\ &+ \sum_{v\in V\setminus\{x,y\}} C_n(y,v)((u_n(v)-0)^2 - (u_n(v)-u_n(y))^2) \\ &+ C_n(x,y)(1-(u_n(x)-u_n(y))^2) \\ &\to 0\end{aligned}$$

because all sums are finite, $u_n(x) \to 1$, $u_n(y) \to 0$, $u_n(v) \to u(v)$, and $C_n(v,w)$ is bounded for all $v, w \in V$. Thus,

$$\lim_{n\to\infty} \mathcal{E}_n^*(u_n^*) = \lim_{n\to\infty} \mathcal{E}_n(u_n^*) = \lim_{n\to\infty} \mathcal{E}_n(u_n) = \mathcal{E}(u) = \mathcal{E}^*(u).$$

Hence, $\mathcal{E}_n^* \xrightarrow{\Gamma} \mathcal{E}^*$. By Proposition 5.38, we know that there exist $\phi_n \in l^2(V)$, $\phi_n(x) = 1$, $\phi_n(y) = 0$ such that

$$\mathcal{E}_n(\phi_n) = \min_{\substack{u\in l^2(V)\\ u(x)=1,u(y)=0}} \mathcal{E}_n(u)$$

or equivalently

$$\mathcal{E}_n^*(\phi_n) = \min_{u\in l^2(V)} \mathcal{E}_n^*(u).$$

Since $\mathcal{E}_n(\widetilde{\phi_n}) \leq \mathcal{E}_n(\phi_n)$, $\widetilde{\phi_n}(x) = 1$ and $\widetilde{\phi_n}(y) = 0$, we can assume $\phi_n = \widetilde{\phi_n}$, i.e. $0 \leq \phi_n \leq 1$. We see that

$$B_{0,1} := \big\{ u \in l^2(V) \mid 0 \leq u(v) \leq 1 \ \forall\ v \in V, u(x) = 1, u(y) = 0 \big\}$$

is compact because $l^2(V) \cong \mathbb{R}^{|V|}$ is finite-dimensional. Now suppose that (ϕ_{n_k}) is any subsequence of $(\phi_n) \subseteq B_{0,1}$. Then ϕ_{n_k} has a convergent subsequence $\phi_{n_{k_l}} \to \phi_0 \in B_{0,1}$. By Lemma 7.4 and its following remark, this implies

$$\mathcal{E}^*(\phi_0) = \min_{u\in l^2(V)} \mathcal{E}^*(u) = \lim_{l\to\infty} \mathcal{E}_{n_{k_l}}^*(\phi_{n_{k_l}}).$$

This means we have shown that every subsequence $(\mathcal{E}_{n_k}^*(\phi_{n_k}))$ of $(\mathcal{E}_n^*(\phi_n))$ has in turn a subsequence converging to $\min_{u\in l^2(V)} \mathcal{E}^*(u)$. Hence,

$$\lim_{n\to\infty} \mathcal{E}_n^*(\phi_n) = \min_{u\in l^2(V)} \mathcal{E}^*(u).$$

This implies

$$\begin{aligned}
\lim_{n\to\infty} R_n(x,y) &= \lim_{n\to\infty} \sup\left\{\frac{|u(x)-u(y)|^2}{\mathcal{E}_n(u)} \mid u \in l^2(V), \mathcal{E}(u)>0\right\} \\
&= \lim_{n\to\infty} \sup\left\{\frac{1}{\mathcal{E}_n(u)} \mid u \in l^2(V), u(x)=1, u(y)=0\right\} \\
&= \lim_{n\to\infty} \left(\min_{\substack{u\in l^2(V)\\ u(x)=1,u(y)=0}} \mathcal{E}_n(u)\right)^{-1} = \lim_{n\to\infty} (\mathcal{E}_n(\phi_n))^{-1} \\
&= \left(\lim_{n\to\infty} \mathcal{E}_n^*(\phi_n)\right)^{-1} = \left(\min_{u\in l^2(V)} \mathcal{E}^*(u)\right)^{-1} \\
&= \sup\left\{\frac{1}{\mathcal{E}(u)} \mid u \in l^2(V), u(x)=1, u(y)=0\right\} \\
&= \sup\left\{\frac{|u(x)-u(y)|^2}{\mathcal{E}(u)} \mid u \in l^2(V), \mathcal{E}(u)>0\right\} \\
&= R_{\mathcal{E}}(x,y) =: R(x,y).
\end{aligned}$$

Since $\mathcal{E}$ is a resistance form, $R = R_{\mathcal{E}}$ is symmetric and $R(x,y)=0$ if and only if $x=y$. R satisfies the triangle inequality because every R_n does. Hence, R is a metric on V.

2. Assume that $R(x,y) := \lim_{n\to\infty} R_n(x,y)$ is a metric on V. As shown in Theorem 5.27, $(C_n(x,y))_{x,y\in V}$ can be obtained by solving linear equation systems whose coefficient matrices only consist of linear expressions of $(R_n(x,y))_{x,y\in V}$. Hence, $C(x,y) := \lim_{n\to\infty} C_n(x,y) \in [0,\infty]$ exists for all $x,y\in V$. Suppose $C(x,y)\neq 0$. Then $C_n(x,y)>0$ for all n sufficiently large, i.e. there exists $e_{xy}\in E(x,y)$. By Lemma 5.32, we have

$$R(x,y) = \lim_{n\to\infty} R_n(x,y) \leq \lim_{n\to\infty} \frac{1}{C_n(x,y)} = \frac{1}{C(x,y)}.$$

Hence, $C(x,y) \leq 1/R(x,y) < \infty$.
Now define $\mathcal{E}(u) := \frac{1}{2}\sum_{x\in V}\sum_{y\in V} C(x,y)\cdot(u(y)-u(x))^2$. For every $u_n \to u$ uniformly and therefore point-wise, we then have

$$\mathcal{E}(u) = \lim_{n\to\infty} \mathcal{E}_n(u_n).$$

Hence, $\mathcal{E}_n \xrightarrow{\Gamma} \mathcal{E}$. What is left to show is that $\mathcal{E}$ is a resistance form on $l(X)$. We will do this by using Proposition 5.43:
(i) $\mathcal{E}(f)\geq 0$ holds by definition of $\mathcal{E}$.
(ii) Let u_0 be a non-constant function on V. Then $\mathcal{E}_n(u_0)>0$ for all $n\in\mathbb{N}$,

and there exist $x, y \in V$ such that $u_0(x) \neq u_0(y)$. Hence,

$$\begin{aligned}\lim_{n\to\infty} \frac{|u_0(x)-u_0(y)|^2}{\mathcal{E}_n(u_0)} &\leq \lim_{n\to\infty} \sup \left\{ \frac{|u(x)-u(y)|^2}{\mathcal{E}_n(u)} \mid u \in l^2(V), \mathcal{E}_n(u) > 0 \right\} \\ &= \lim_{n\to\infty} R_n(x,y) = R(x,y) < \infty.\end{aligned}$$

Thus, $\mathcal{E}(u_0) = \lim_{n\to\infty} \mathcal{E}_n(u_0) > 0$.

(iii) Let $u \in l^2(V)$. Then $\mathcal{E}(u) = \lim_{n\to\infty} \mathcal{E}_n(u) \geq \lim_{n\to\infty} \mathcal{E}_n(\widetilde{u}) = \mathcal{E}(\widetilde{u})$.

In both implications it follows that $R = R_{\mathcal{E}}$. By Proposition 5.45, there exists a simple network $\mathcal{N}$ such that $R = R_{\mathcal{E}} = R_{\mathcal{N}}$ and $\mathcal{E}_{\mathcal{N}} = \mathcal{E}$. □

Remarks.

1. The proof of Proposition 7.7 shows that for finite networks (with the same vertex set) both the effective resistance's point-wise convergence and the energy form's Gamma convergence are equivalent to the point-wise convergence of $C_n(x,y)$.
2. Unfortunately, Proposition 7.7 is not applicable to infinite networks as Example 7.8 shows. However, Example 7.9 shows that a similar result might hold if we adjust the metric framework.

Example 7.8. Let $V = \mathbb{N}$, $E = \{(n, n+1, 1), (n+1, n, 1) \mid n \in \mathbb{N}\}$ and consider

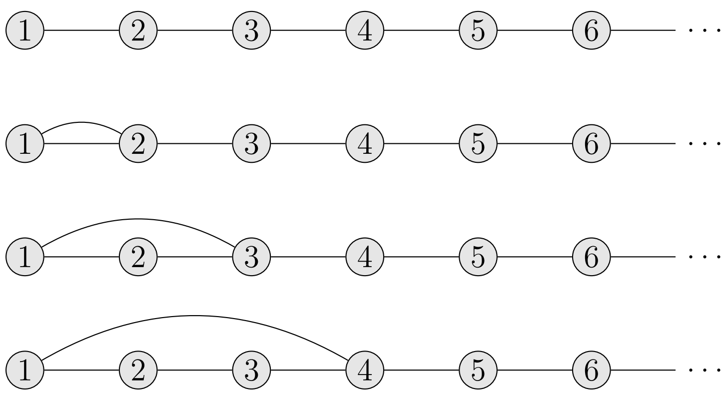

Figure 7.1: The networks $\mathcal{N}_n$, $n = 1, \ldots, 4$.

the networks $\mathcal{N}_1 := (V, E, 1)$ and $\mathcal{N}_n = (V, E \cup \{(1, n, 1), (n, 1, 1)\}, 1)$, $n \geq 2$, as shown partially in Figure 7.1. Let R_n denote the effective resistance of $\mathcal{N}_n$ and let $x, y \in \mathbb{N}$, $x < y$. Using Proposition 6.22 we can compute $R_n(x,y)$ and obtain $R_1(x,y) = y - x$ and

$$R_n(x,y) = \left(\frac{1}{y-x} + \frac{1}{x+n-y} \right)^{-1}$$

for all $n \geq y$. Hence, $\lim_{n\to\infty} R_n(x,y) = R_1(x,y)$ for all $x, y \in \mathbb{N}$.

Now let $\mathcal{E}_n$ denote the energy form of $\mathcal{N}_n$. Using the notation $\mathcal{E}_n(u) = \mathcal{E}_n(\nabla u)$ for $u \in l(V)$, we have

$$\mathcal{E}_1(u) = \sum_{x \in \mathbb{N}} (u(x+1) - u(x))^2$$

and $\mathcal{E}_n(u) = \mathcal{E}_1(u) + (u(n) - u(1))^2$. However, $\mathcal{E}_n \overset{\Gamma}{\not\to} \mathcal{E}_1$, as shown in the following. Let

$$u(x) = \sum_{k=1}^{x} \frac{1}{k} \ , \ x \in \mathbb{N}$$

and u_n be a recovery sequence for u. Then $u(x) \to \infty$ as $x \to \infty$, and

$$\mathcal{E}_1(u) = \sum_{x=1}^{\infty} \frac{1}{(x+1)^2} = \frac{\pi^2}{6} - 1 < \infty.$$

$u_n \to u$ uniformly implies $u_n(1) \to u(1) = 1$, and $\lim_{n\to\infty} |u_n(n) - u(n)| = 0$, i.e. $\lim_{n\to\infty} u_n(n) = \infty$. Hence,

$$\mathcal{E}_n(u_n) \geq (u_n(n) - u_n(1))^2 \to \infty,$$

and therefore $\mathcal{E}_n(u_n) \not\to \mathcal{E}_1(u)$.

Example 7.9. Consider the same networks as in Example 7.8. As we have shown, $\mathcal{E}_n \overset{\Gamma}{\to} \mathcal{E}_1$ does not hold. However, if we adjust our notion of Gamma convergence to better fit our underlying spaces of functions $D(\mathcal{N}_n)$, we can achieve this outcome.

First, we realize that $\mathcal{E}_n(u) = \mathcal{E}_1(u) + (u(n) - u(1))^2$ implies $D(\mathcal{N}_n) = D(\mathcal{N}_1)$. Using this we will consider Gamma convergence (as defined in Definition 7.1) on the metric space $(D(\mathcal{N}_1), \mathcal{E}_1 + o_1)$. We claim that in this sense $\mathcal{E}_n \overset{\Gamma}{\to} \mathcal{E}_1$ holds.

<u>(Γ1):</u> Let $u_n \to u$, i.e. $\mathcal{E}_1(u_n - u) + o_1(u_n - u) \to 0$. It follows that $u_n(1) \to u(1)$ and $\mathcal{E}_1(u_n - u) \to 0$. The triangle inequality in $(l^2_{\mathcal{N}_1,-}(E_1), \mathcal{E}_1)$ implies $\mathcal{E}_1(u_n) \to \mathcal{E}_1(u)$. Hence,

$$\mathcal{E}_1(u) = \liminf_{n\to\infty} \mathcal{E}_1(u_n) \leq \liminf_{n\to\infty} \mathcal{E}_1(u_n) + (u_n(n) - u_n(1))^2 = \liminf_{n\to\infty} \mathcal{E}_n(u_n).$$

<u>(Γ2):</u> Let $u \in D(\mathcal{N}_1)$. We have to find $u_n \in D(\mathcal{N}_1)$ such that $\mathcal{E}_1(u_n - u) + o_1(u_n - u) \to 0$ and $\mathcal{E}_n(u_n) \to \mathcal{E}_1(u)$. Since $\mathcal{E}_n(f + c) = \mathcal{E}_n(f)$ for all $f \in D(\mathcal{N}_1)$, $n \in \mathbb{N}$ and $o_1((u_n + c) - (u + c)) = o_1(u_n - u)$, we may assume w.l.o.g. that $u(1) = 0$. Let

$$u_n(x) := \begin{cases} u(x) - u(x)\frac{x}{n} & , \ 1 \leq x \leq n \\ u(x) - u(n) & , \ x \geq n+1 \end{cases} .$$

It follows that $o_1(u_n - u) = (u_n(1) - u(1))^2 = u(1)^2/n^2 \to 0$ and $u_n(n) = 0 = u_n(1)$. We will now show that $\mathcal{E}_1(u_n - u) \to 0$:

For $x \geq n+1$, we have $(u_n - u)(x) - (u_n - u)(x+1) = (-u(n) - (-u(n)) = 0$, and

$$(u_n - u)(n) - (u_n - u)(n+1) = 0 - u(n) - (-u(n)) = 0.$$

It follows that

$$\begin{aligned}
\mathcal{E}_1(u_n - u) &= \sum_{x=1}^{n-1} (u_n(x) - u(x) - (u_n(x+1) - u(x)))^2 \\
&= \sum_{x=1}^{n-1} \left(-u(x)\frac{x}{n} + u(x+1)\frac{x+1}{n} \right)^2 \\
&= \frac{1}{n^2} \sum_{x=1}^{n-1} (u(x+1) + x \cdot (u(x+1) - u(x)))^2 \\
&\leq \frac{2}{n^2} \left(\sum_{x=1}^{n-1} u(x+1)^2 + \sum_{x=1}^{n-1} x^2 (u(x+1) - u(x))^2 \right).
\end{aligned}$$

For convenience, let $d(x) := u(x+1) - u(x)$. Using $u(x+1) = \sum_{k=1}^{x} d(k)$ and applying Hardy's inequality[5] (see [6], p.51) we get

$$\sum_{x=1}^{\infty} \frac{u(x+1)^2}{x^2} \leq \sum_{x=1}^{\infty} \left(\frac{|d(x)| + \ldots + |d(1)|}{x} \right)^2 \leq 2^2 \sum_{x=1}^{\infty} d(x)^2 = 4 \cdot \mathcal{E}_1(u) < \infty.$$

Kronecker's Lemma[6] (see [3], p. 190 and p.194) implies

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{x=1}^{n} u(x+1)^2 = \lim_{n\to\infty} \frac{1}{n^2} \sum_{x=1}^{n} x^2 \cdot \frac{u(x+1)^2}{x^2} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{x=1}^{n} x^2 \cdot d(x)^2 = 0.$$

Hence,

$$\mathcal{E}_1(u_n - u) \leq \frac{2}{n^2} \left(\sum_{x=1}^{n-1} u(x+1)^2 + \sum_{x=1}^{n-1} x^2 \cdot d(x)^2 \right) \to 0.$$

Again it follows that $\mathcal{E}_1(u_n) \to \mathcal{E}_1(u)$ and therefore

$$\mathcal{E}_n(u_n) = \mathcal{E}_1(u_n) + \underbrace{(u_n(n) - u_n(1))^2}_{=0} = \mathcal{E}_1(u_n) \to \mathcal{E}_1(u).$$

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[5] Hardy's inequality: Let $p > 1$ and (a_k) be a sequence of non-negative real numbers. Then $\sum_{n=1}^{\infty} \left(\frac{a_1+\ldots+a_n}{n}\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p$.

[6] Kronecker's Lemma: Let $(x_k), (b_k)$ be sequences of real numbers such that $\sum_{k=1}^{\infty} x_k < \infty$, $0 \leq b_k \leq b_{k+1}$ for all $k \in \mathbb{N}$ and $b_k \to \infty$. Then $\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k x_k = 0$.

Conjectures and points of further interest

1. Does the effective resistance of an infinite, locally finite network determine the simple structure of this network, i.e. is it true that two simple, locally finite networks on the same vertex set are equal if and only if their effective resistances are equal? We believe this to be true but were not yet able to prove it.
2. Can Proposition 7.7 be extended to infinite networks if we adjust the type of convergence of functions we are using?
3. Is there an 'easy' way to compute the effective resistance of an infinite network?

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8 Selbstständigkeitserklärung

Ich versichere, dass ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Quellen und Hilfsmittel angefertigt habe, insbesondere sind wörtliche oder sinngemäße Zitate als solche gekennzeichnet. Mir ist bekannt, dass Zuwiderhandlung auch nachträglich zur Aberkennung des Abschlusses führen kann.

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