Wachspress Geometry

September 24th – 25th, 2024

Tuesday, September 24th

11:00 - 11:45
Moment Varieties of Measures on Polytopes
Bernd Sturmfels

Abstract: This lecture revisits my article with Kathlén Kohn and Boris Shapiro on the algebraic relations among moments of the uniform distribution of a polytope. In that project we had rediscovered Wachspress coordinates and adjoint polynomials. I shall start out with my old slides from 2018, and these will then guide us to the theme of this workshop.

11:45 - 12:30
Adjoints of Polytopes and Polypols
Kathlén Kohn

Abstract: Eugene Wachspress introduced polypols, which are certain nonlinear polytopes. He aimed to generalize barycentric coordinates from simplices to arbitrary polytopes and further to polypols. For this, he made use of adjoint hypersurfaces of rational polypols. Our first goal is to explain the well-definedness and geometric meaning of these adjoints for polytopes and planar polypols. Second, we encounter several appearances of these adjoints in various other applications (beyond barycentric coordinates) such as, intersection theory or statistics. In particular, in the study of scattering amplitudes in physics, positive geometries are certain real semialgebraic sets together with a rational canonical form. We explain that planar positive geometries are essentially Wachspress' rational polypols in the plane. Then, we present an explicit formula for the canonical form of a planar rational polypol in terms of defining equations of the adjoint curve and the facets of the polypol.

This talk is based on joint works with Ragni Piene, Kristian Ranestad, Felix Rydell, Boris Shapiro, Rainer Sinn, Miruna-Stefana Sorea, Bernd Sturmfels, and Simon Telen.

14:00 – 14:45
Zonotopal Wachspress coordinates
Raman Sanyal

Abstract: Every polytope is the canonical projection of a simplex. The Wachspress coordinates are particular sections of these projections that have many interesting algebraic and geometric properties. Zonotopes are Minkowski sums of line segments or, equivalently, linear projections of cubes. Points in zonotopes are thus sums of points, one from each segment. In this talk, I will discuss zonotopal Wachspress coordinates that give natural sum-representations of points in zonotopes. Zonotopal Wachspress coordinates enjoy similar nice properties that reflect the geometry of the hyperplane arrangement associated to the collection of segments. This is work in progress with Martin Winter.

14:45 - 15:30
Extensions of Maxwell-Cremona liftings to higher dimensions
Bernd Schulze

Abstract: The classical Maxwell-Cremona correspondence from the 1860s provides a one-to-one correspondence between the set of self-stresses of a bar-joint framework on a planar graph in the Euclidean plane and the set of 3-dimensional polyhedral surfaces in 3-space that project vertically onto it. This result has found numerous applications in areas such as geometric rigidity theory, polytope theory, computational geometry, and structural engineering. While there are some generalisations of this theory to liftings of d-complexes in d-space, extensions for liftings of bar-joint frameworks in d-space for d at least 3 have been missing. In this talk we introduce differential liftings on general graphs using differential forms associated with the elements of the homotopy groups of the complements to the frameworks. Such liftings play the role of integrands for the classical notion of liftings for planar frameworks. These differential liftings have a natural extension to self-stressed frameworks in higher dimensions. As a result, we generalise the notion of classical liftings to both graphs and multidimensional k-complexes in d-space ($k = 2, ..., d$).

The talk is based on joint work with Oleg Karpenkov, Fatemeh Mohammadi and Christian Mueller.

Wednesday, September 25th

10:00 - 10:45
On cosmological polytopes
Martina Juhnke

Abstract: A cosmological polytope is defined for a given Feynman diagram, and its canonical form may be used to compute the contribution of the Feynman diagram to the wavefunction of certain cosmological models. Given a subdivision of a polytope, its canonical form is obtained as a sum of the canonical forms of the facets of the subdivision. In this paper, we identify such formulas for the canonical form via algebraic techniques. It is shown that the toric ideal of every cosmological polytope admits a Gröbner basis with a squarefree initial ideal, yielding a regular unimodular triangulation of the polytope. In specific instances, including trees and cycles, we recover graphical characterizations of the facets of such triangulations that may be used to compute the desired canonical form. This is joint work with Liam Solus and Lorenzo Venturello. I will also report on recent work with Justus Bruckamp, Lina Goltermann, Eric Landin and Liam Solus, in which we study h^*-polynomials of cosmological polytopes.

11:00 - 11:45
The boundary structure and adjoint of the 2-loop Amplituhedron
Elia Mazzucchelli

Abstract: Positive loop Grassmannians are generalizations of positive Grassmannians and are motivated by physics. They are conjectured to be positive geometries, whose canonical form yields the integrand of some amplitudes in $N = 4$ super Yang-Mills theory. In this talk we will focus on the simplest nontrivial case of a two-loop Grassmannian. I will show how to understand the combinatorial structure of the complex boundary stratification and how to determine the residual arrangement. Extending a recent result for the $m = k = 2$ Amplituhedron, the adjoint geometry is unique and completely determined by requiring it to contain the residual arrangement.

11:45 - 12:30
Santaló Geometry of Convex Polytopes
Dmitrii Pavlov

Abstract: The Santaló point of a convex polytope is the interior point which leads to a polar dual of minimal volume. This minimization problem is relevant in interior point methods for convex optimization, where the logarithm of the dual volume is known as the universal barrier function. When translating the facet hyperplanes, the Santaló point traces out a semi-algebraic set. We describe and compute this geometry using algebraic and numerical techniques. We exploit connections with statistics (in particular, with Wachspress models), optimization and physics.

14:00 – 14:45
Discrete spherical Laplacian
Ivan Izmestiev

Abstract: The so-called cotangent Laplacian is a symmetric negative semidefinite matrix associated with a Delaunay triangulation of a point set in the plane. It has many applications in discrete differential geometry and is an indispensable tool in computer graphics.

In this talk I will describe a similar discretization of the spherical Laplacian. It associates a negative semidefinite self-adjoint operator to a point set in the unit sphere (or, equivalently, to a set of unit vectors). The dicrete spherical Laplacian shares many properties with its classical counterpart, namely it is related to infinitesimal isometric deformations of polyhedra inscribed into the sphere and to conformal vector fields.

This is a joint work with Wai Yeung Lam.

14:45 - 15:30
General position stresses
Louis Theran

Abstract: Let G be a $(d+1)$-connected graph with n vertices. An equilibrium stress matrix for G is an n-by-n symmetric matrix that has the all ones vector in its kernel and zeros in the off-diagonal positions corresponding to non-edges of G. The set of all equilibrium stress matrices of rank $n-d-1$ or less corresponds to the equilibrium stress matrices arising from an equilibrium stress of some framework (G, p) in dimension d. In general, this "stress variety" has a complicated structure. However, the set of equilibrium stresses for G of rank exactly $n-d-1$ that, in addition, corresponds to a general position framework in dimension d is an irreducible constructible set. Irreducibility leads to a notion of a "generic stress" G, with applications to problems in generic global and universal rigidity. I will also describe some connections to rubber band embeddings and orthogonal representations of graphs.

This talk is based on joint work with Bob Connelly and Shlomo Gortler.