Algebraische Geometrie II

Intersection Theory

Skript zur Veranstaltung
Universität Leipzig

Sommer 2023

Rainer Sinn

Fassung vom July 6, 2023

Contents

Chapter 1

Introduction

The goal of these lecture notes is to explain the necessary basics and introduce the modern tools of intersection theory as a formalism. The goal of the course is to enable students to use this formalism in practice and on examples. The focus is therefore not on proof techniques or maximal generality of the results. We will skip several of the more technical proofs and try to understand the relevant results and notions by way of examples.

1.1. Degree

The degree of a variety plays a central role in intersection theory. We will consider subvarieties of smooth projective varieties. The simplest case is $X \subset \mathbb{P}^n$. The degree has equivalent definitions in this case.

Definition. For an irreducible variety $X \subset \mathbb{P}^n$, the **degree** $\deg(X)$ (German: Grad) of X is leading coefficient of the Hilbert polynomial of X, up to a factor of $(\dim(X))!$. Concretely, if $a_d t^d + a_{d-1}t^{d-1} + \ldots + a_0$ is the Hibert polynomial of X, then $\deg(X) = d!a_d$.

The Hilbert polynomial of X is the Hilbert polynomial $P(t) \in \mathbb{Q}[t]$ of the graded algebra $K[X] = K[x_0, \ldots, x_n]/\mathcal{I}_+(X)$, the homogeneous coordinate ring of X. This means by definition that $\dim(K[X]_k) = P(k)$ for sufficiently large $k \in \mathbb{N}$. Here, $\dim(K[X]_k)$ is the dimension of the kth graded piece of $K[X]$ as a K-vector space. Equivalently, the degree is the number of points in the intersection $X \cap L$, where $L \subset \mathbb{P}^n$ is a generic linear subspace of dimension $n - \dim(X)$. In other words, L is the intersection of $d = \dim(X)$ many generic hyperplanes in $\mathbb{P}^n$ so that $X \cap L = X \cap H_1 \cap \ldots \cap H_d$ for generic hyperplanes $H_i \subset \mathbb{P}^n$. This last characterization holds for more linear spaces $L \subset \mathbb{P}^n$ if we count intersection points with multiplicity, see below.

1.2. Some basic examples

Let K be an algebraically closed field of characteristic 0 (usually the field of complex numbers). We use the standard notation $\mathbb{A}^n$ for K^n with the Zariski topology and the ring of regular functions $K[x_1, \ldots, x_n]$ as well as $\mathbb{P}^n$ for the n-dimensional projective space over K, also equipped with the Zariski topology.

Dimension 1. Consider a homogeneous polynomial $f \in K[x, y]$ of degree d and its zero set $\mathcal{V}_+(f) \subset \mathbb{P}^1$. Since f factors into precisely d linear factors $f = \prod_{i=1}^d (b_i x - a_i y)$, the set $\mathcal{V}_+(f)$

consists of d points, namely $[a_i, b_i]$. However, this statement is, of course, only true if we count each zero with the multiplicity of the corresponding linear factor of f. This is the most basic instance of counting zeros with multiplicities.

Dimension 2. In the projective plane, two curves without a common irreducible component intersect in only finitely many points. The number of points is determined by Bézout's Theorem.

1.2.1 Theorem (Bézout's Theorem). *Given two relatively prime homogeneous polynomials $f, g \in K[x, y, z]$ of degree d and e, then $\mathcal{V}_+(f) \cap \mathcal{V}_+(g) \subset \mathbb{P}^2$ is finite and consists of $d \cdot e$ points, if counted correctly with multiplicity.*

We saw one way to count the multiplicity last semester: we first compute the resultant R of of the two polynomials f and g with respect to z (at least after a generic change of coordinates). Then the multiplicity of $[a, b, c] \in \mathcal{V}_+(f, g)$ is the multiplicity of $[a, b]$ as a root of R in the sense of the previous paragraph. Geometrically, the resultant of f and g with respect to z defines the image of the projection $\pi_p \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ of $\mathcal{V}_+(f, g)$ from the point $p = [0, 0, 1]$. The above definition only makes sense if any line through p intersects $\mathcal{V}_+(f, g)$ in at most 1 point (counted here without multiplicity). This approach is computationally easy (at least using computer algebra systems), given that the genericity condition is satisfied. Bézout's Theorem then follows essentially from classical determinantal formulas to compute the resultant (via the Sylvester matrix).

However, it is technically better to have a more local definition of the intersection multiplicity. One way to achieve this in $\mathbb{P}^2$ is by looking at the **local ring**. Assuming that we are interested in $[0, 0, 1] \in \mathcal{V}_+(f, g)$, we consider the local ring $\mathcal{O}_{\mathbb{A}^2,(0,0)}$ of $\mathbb{A}^2 \cong D_+(z) \subset \mathbb{P}^2$ at $O = (0, 0)$ consisting of all rational functions a/b where $a, b \in K[x, y]$ and $b(0, 0) \neq 0$. Then $f(x, y, 1)$ and $g(x, y, 1)$ are in the maximal ideal $\mathfrak{m}_{\mathbb{A}^2,(0,0)}$ of $\mathcal{O}_{\mathbb{A}^2,(0,0)}$. The intersection multiplicity of f and g at O is defined as the dimension of the quotient $\mathcal{O}_{\mathbb{A}^2,O}/(f(x, y, 1), g(x, y, 1))$ as a K-vector space.

1.2.2 Example. Consider $f = (x + z)^2 + y^2 - z^2$ and $g = (x - z)^2 + y^2 - z^2$ intersecting at $[0, 0, 1] \in \mathbb{P}^2$. A basis of $\mathcal{O}_{\mathbb{A}^2,(0,0)}/(f(x, y, 1), g(x, y, 1))$ as a K-vector space is 1 and y. To check this claim, we can first compute $K[x, y]/((x - 1)^2 + y^2 - 1, (x + 1)^2 + y^2 - z^2)$ and the localize at the maximal ideal $\mathfrak{m} = (x, y)$. The quotient is isomorphic to $K[x, y]/(x, y^2)$ (since $f - g = -4xz$).

The intersection multiplicity of the two curves at $[0, 0, 1]$ is therefore 2.

Exercise 1.2.3.
- ⋄ Compute the resultant of the two polynomials in the previous example and verify that the multiplicity assigned to $[0, 0, 1]$ by this method is also 2.
- ⋄ Find the other intersection points of $\mathcal{V}_+(f)$ and $\mathcal{V}_+(g)$ of the curves in the previous example and determine the intersection multiplicity at those points.

A lot of work in this class will be around good notions of multiplicities and degrees. The following short excursion into projective duality gives some examples.

1.3. Projective duality

Let V be an $(n + 1)$-dimensional vector space over K so that $\mathbb{P}^n \cong \mathbb{P}(V)$. Let V^* be the dual vector space of linear forms $\ell \colon V \to K$. The dual projective space $(\mathbb{P}^n)^*$ is then $\mathbb{P}(V^*)$. In other words, a point in $(\mathbb{P}^n)^*$ corresponds to a hyperplane in $\mathbb{P}^n$, namely $[\ell] \in \mathbb{P}(V^*)$ defines $\{p \in \mathbb{P}(V)\colon \ell(p) = 0\}$. Given a hyperplane $H \subset \mathbb{P}^n$, we sometimes write $[H]$ for the corresponding point in $(\mathbb{P}^n)^*$.

Let us first look at the dual variety of a plane curve: let $C = \mathcal{V}_+(f) \subset \mathbb{P}^2$ be an irreducible plane curve (meaning that $f \in K[x, y, z]$ is irreducible). Define the dual curve C^* to be the Zariski closure of all $[L]$, where the line $L \subset \mathbb{P}^2$ is tangent to C at some point.

1.3.1 Example. Let $f = (x, y, z)A(x, y, z)^\top$ be a quadratic form represented by a symmetric matrix A of full rank 3 and $C = \mathcal{V}_+(f)$ the corresponding conic. Then the dual curve C^* is the Zariski closure of the set $\{A(x, y, z)^\top\colon [x, y, z] \in C\}$ because the gradient of f is $2A(x, y, z)^\top$ and is perpendicular to the tangent to $p = [x, y, z]$ for $p \in C$. Since $[x, y, z] \in C$ if and only if $0 = (x, y, z)A(x, y, z)^\top = (A(x, y, z)^\top)^\top A^{-1}(A(x, y, z)^\top)$, it follows that C^* is the conic defined by A^{-1}.

This fact about duality of conics (which works exactly the same for any quadratic form of full rank in any number of variables) gives a simple solution to a classical, geometric, enumerative problem: how many conics are tangent to five general lines? This means that we fix five random lines $L_1, \ldots, L_5 \subset \mathbb{P}^2$ and look for conics $C \subset \mathbb{P}^2$ such that C is tangent to L_i at some point $p_i \in L_i$.

Let's translate this problem to the dual projective plane: to each line L_i corresponds a point $[L_i] \in (\mathbb{P}^2)^*$ and the conic C is tangent to L_i (at some point) if and only if the dual curve C^* contains $[L_i]$. We have just seen that C^* is again a conic and we require it to pass through five generic points. Since the space of quadratic forms in three variables has dimension six, there is a unique such conic. The answer to the above problem is therefore 1.

The class of a curve. The **class** (German: Klasse) of a curve $C \subset \mathbb{P}^2$ is the number of tangents to C that pass through a generic point $p \in \mathbb{P}^2$. This notion was introduced and studied by Plücker.

Exercise 1.3.2. Show that the class of a curve $C \subset \mathbb{P}^2$ is equal to the degree of the dual curve $C^* \subset (\mathbb{P}^2)^*$.

One way to compute the class of $C = \mathcal{V}_+(f)$ is to consider $D_q f = \langle \nabla f, q\rangle$ the directional derivative of f in direction q. If f is homogeneous of degree d, then $D_q f$ is homogeneous of degree $d - 1$. The intersection points $\mathcal{V}_+(f, D_q f)$ are the points of C such that the corresponding tangents pass through q – if they are smooth points of C! So if f is irreducible and C is smooth, $D_q f$ cannot have a nontrivial common factor with f and Bézout's Theorem tells us that the class of C is $d(d - 1)$. The curve C_q defined by $D_q f$ is called a **polar curve** (German: Polare) of C.

However, if $p \in C$ is a singular point, $\nabla f(p) = 0$ and this point lies on every polar curve C_q but there is no tangent to C at p. The question becomes of how a singularity of C affects the above count for the class of C. If p is a node, which means that $\nabla f(p) = 0$ but the Hessian $f''(p)$ at p has rank 2, then it turns out that the intersection multiplicity of p in $\mathcal{V}_+(f, D_q f)$ is 2. So the node counts for two tangents through q and the class of a curve of degree d with δ nodes (and no other singularities) is $d(d - 1) - 2\delta$.

Exercise 1.3.3. Verify that the intersection multiplicity of a node p of a curve $\mathcal{V}_+(f)$ in the intersection with a generic polar curve $\mathcal{V}_+(D_q f)$ is 2 as claimed in the previous paragraph.

Exercise 1.3.4. What is an ordinary cusp (see the classification of singularities of plane curves) and how does it affect the class of a curve?

Degree of a dual surface. We now consider a suface $S \subset \mathbb{P}^3$ and aim to compute the degree of the dual surface S^* in some special cases. This was pioneered by Salmon in 1847. Similar to the case of plane curves, we can compute the degree of the dual surface S^* by intersecting it with a generic line $L \subset (\mathbb{P}^3)^*$. Such a line is dual to the family of planes in $\mathbb{P}^3$ containing a fixed line in $\mathbb{P}^3$, namely the intersection of all planes $\{x \in \mathbb{P}^3 \colon \ell(x) = 0\}$ with $[\ell] \in L$. So the degree of S^* is the number of points of S such that the tangent plane contains a generic, fixed line in $\mathbb{P}^3$. We can try to compute this number by the same construction as before: if $S = \mathcal{V}_+(f)$, take now two directional derivatives $D_{q_1}f$ and $D_{q_2}f$ for generic points $q_i \in \mathbb{P}^3$. Then the tangent plane T_pS to S at a nonsingular point $p \in S$ contains the line spanned by q_1 and q_2 if and only if $p \in \mathcal{V}_+(f, D_{q_1}f, D_{q_2}f)$. By this argument, we would expect S^* to have degree $d(d-1)^2$, where $d = \deg(f)$.

Again, singular points of S change this calculation, similar to plane curves. However, there is also a more serious new phenomenon in this case: **excess intersection** (German: ?überschießender Durchschnitt?). What if S is singular along a curve? Then this curve is contained in $\mathcal{V}_+(f, D_{q_1}f, D_{q_2}f)$ independent of which points $q_1, q_2 \in \mathbb{P}^3$ we choose. But dimension 1 is not the expected dimension of this intersection (hence the word excess intersection). Let us only consider the case that S is singular along a line $C \subset S$. Salmon then argues that the presence of singularities along a line should decrease the degree of S^* by $3d-4$ as follows (heuristically from a modern point of view!): the answer to this question should not really depend on the surface S too much, so let's take our surface S to be the union of a plane H containing C and a "general" surface S' of degree $d-1$. By Bézout's Theorem in $\mathbb{P}^3$, the surface S' intersects $\mathcal{V}_+(D_{q_1}f, D_{q_2}f)$ in $(d-1)^3$ points. Of these, $d-1$ lie on the line C (those are the points in $C \cap S'$). The plane H intersects $\mathcal{V}(D_{q_i}f)$ in a (plane!) curve of degree $d-1$. However, the line C is an irreducible component of each ($i = 1, 2$) of these curves. So H intersects $\mathcal{V}(D_{q_1}f, D_{q_2}f)$ in $(d-2)^2$ points outside of C (by Bézout's Theorem in the plane H). In total we have counted $(d-1)^3 - (d-1) + (d-2)^2$ intersection points that do not lie on the line C. Compared to the expected count $d(d-1)^2$, the difference is $3d-4$.

We will later see a systematic way of dealing with excess intersection and do such computations rigorously.

1.4. Tangent conics

A classical problem in intersection theory with a long history is to determine the number of conics that are tangent to five given generic conics in $\mathbb{P}^2$. Again, the condition to be tangent to a given curve at some point is a codimension 1 condition. The space of conics in $\mathbb{P}^2$ is a 5-dimensional projective space. So we expect a 0-dimensional intersection and would like to compute the number of points. The degree of the hypersurface of conics that are tangent to a given conic is 6. So a naive application of Bézout's Theorem suggests that there should be $6^5 = 7776$ conics tangent to five conics. However, this is incorrect due to excess intersection, which consists of "double lines". Let ℓ be a linear form in $K[x, y, z]$. Then $\mathcal{V}_+(\ell^2)$ is tangent to any conic, because it intersects the conic in two points and this intersection appears to be tangent. It has multiplicity 2 coming from ℓ^2. More precisely, the variety of squares, which is isomorphic to a Veronese surface $\nu_2(\mathbb{P}^2)$, is contained in any hypersurface of conics tangent to a fixed conic. So in the intersection of five "tangency hypersurfaces" of degree 6 as above, the

Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$ is always an irreducible component. The question is, how to count this in analogy to a line of singularities in the problem of computing the degree of a dual surface. We will eventually see, that in this case, the Veronese surface counts for 4512 points (and is the only excess intersection) so that the correct answer to this enumerative problem is: $6^5 - 4512 = 7776 - 4512 = 3264$.

Let us verify that the hypersurface Z of conics in $\mathbb{P}^5 = \mathbb{P}(K[x,y,z]_2)$ that are tangent to a given conic has degree 6 using another kind of tool, namely the **Riemann-Hurwitz formula**. This tool works particularly well in the context of curves. It says the following: for a non-constant morphism $f\colon C \longrightarrow C'$ of degree d of curves C, C' (irreducible and smooth), the number of **ramification points** (German: Verzweigungspunkte) is given by the formula

$$2g(C) - 2 = d(2g(C') - 2) + \sum_{P \in C} (e_P - 1).$$

This formula involves the genus $g(C)$ of the curves and the ramification index e_P. We need this formula only for $C \cong C' \cong \mathbb{P}^1$, in which case the genus is 0. The ramification index will correspond to an intersection point of higher multiplicity in our case.

Exercise 1.4.1.
- ◇ Research the definition of the Euler characteristic of a polyhedron.
- ◇ What is a triangulation of a Riemann surface?
- ◇ Look up the combinatorial proof of the Hurwitz formula (as explained, for instance, in Rick Miranda's book “Algebraic Curves and Riemann Surfaces”).

To construct the map f, we first choose coordinates on $\mathbb{P}^2$ such that the fixed conic is $\nu_2(\mathbb{P}^1) = \{[s^2, st, t^2]\colon [s,t] \in \mathbb{P}^1\} = \mathcal{V}_+(xz - y^2) = C$. Next, we choose two generic conics $\mathcal{V}_+(g)$ and $\mathcal{V}_+(h)$ that intersect C in four distinct points. Then the map φ from C to $\mathbb{P}^1$ given by $p \mapsto (g(p), h(p))$ is a nonconstant morphism given by the pencil of conics $L = \{ag + bh\colon a, b \in \mathbb{C}\}$. The degree of the hypersurface Z of conics tangent to C is the number of intersection points of Z with L. But this we can compute with the Riemann-Hurzwitz formula applied to $f = \varphi \circ \nu_2\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1$. The degree of this map is 4 and so the formula predicts 6 ramification points. Those correspond to conics in the pencil L that do not intersect C in four distinct points but rather with a point of higher multiplicity, which is a point of tangency.

1.5. The Chow Ring

The main tool to formalize and compute degrees in this class is the **intersection ring** or **Chow ring** (German: Schnitt- oder Chow-Ring). We will always study intersection problem in a complete and smooth variety X (usually curves, $\mathbb{P}^n$, Segre varieties, Grassmannians, or bundles over such varieties). To this variety X, we aim to associate a ring $A^*(X)$ generated by classes $[Y]$ of irreducible subvarieties $Y \subset X$ in such a way that the class of the intersection $[Y_1 \cap Y_2]$ of two subvarieties of X is equal to the product $[Y_1][Y_2]$ of the classes. Moreover, we want the ring $A^*(X)$ to be graded and generated in degree 1. The degree of a homogeneous element gives the codimension of the varieties it represents. Let us first discuss this formalism in some examples (that for now remain absolutely formal).

1.5.1 Example. Consider $X = \mathbb{P}^m \subset \mathbb{P}^n$ the Segre variety embedded in $\mathbb{P}^{mn+m+n}$ with the usual

Segre embedding. We will compute the degree of this variety as a subvariety of $\mathbb{P}^{mn+m+n}$. Let π_1 be the projection $\mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^m$, $(x, y) \mapsto x$, and π_2 the projection to the second factor. The Chow ring of X is $A^*(X) = \mathbb{Z}[s,t]/(s^{m+1}, t^{n+1})$, where s represents the class of $\pi_1^{-1}(H)$ for a hyperplane $H \subset \mathbb{P}^m$ and t analogously represents the class of $\pi_2^{-1}(H)$ for a hyperplane $H \subset \mathbb{P}^n$. In terms of bihomogeneous polynomials, s represents the class of a variety defined by a $(1,0)$ form. So $s + t$ represents the class of a variety defined by a $(1,1)$ form. This is exactly what we need to compute the degree of X: by definition, we want to count the number of intersection points of $X \subset \mathbb{P}^{mn+m+n}$ with $\dim(X) = m + n$ generic hyperplanes. A hyperplane in the ambient space gives a $(1,1)$ form on X. Intersecting these hyperplanes corresponds to multiplication. So we compute $(s + t)^{m+n}$ in $A^*(X)$ which gives

$$\sum_{j=0}^{m+n} \binom{m+n}{j} s^j t^{m+n-j} = \binom{m+n}{m} s^m t^n.$$

Here, we used that $s^{m+1} = 0$ and t^{n+1} hold in $A^*(X)$.

Exercise 1.5.2. Compute the degree of a threefold Segre product $X = \mathbb{P}^m \times \mathbb{P}^n \times \mathbb{P}^r$ using that $A^*(X) = \mathbb{Z}[s,t,u]/(s^{m+1}, t^{n+1}, u^{r+1})$.

Chapter 2

Divisors, Vector Bundles and Sheaves

In this chapter, we will introduce the important notions of (Weil and Cartier) divisors, vector bundles and (locally free) sheaves. On a sufficiently nice projective variety, these objects correspond to each other. This is the case on every smooth variety, which is mostly our setup here. It is, however, convenient, to have all three equivalent points of view. We will see later that this part becomes $A^1(X)$, the first graded part of the intersection ring of X (with appropriate hypotheses on X).

2.1. Sheaves

We have already seen the **structure sheaf** (German: Strukturgarbe) of an algebraic variety (without calling it this name) last term.

Definition. Let X be a topological space. A **presheaf** $\mathcal{F}$ (German: Prägarbe) of abelian groups on X assigns to every open subset $U \subset X$ an abelian group $\mathcal{F}(U)$ and to every inclusion $V \subset U$ of open sets a morphism $\rho_{UV}\colon \mathcal{F}(U) \to \mathcal{F}(V)$ of groups such that the following properties hold.

- $\mathcal{F}(\emptyset) = 0$,
- ρ_{UU} is the identity for every open subset U of X, and
- $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ for all chains of open subsets $W \subset V \subset U$.

For an open subset U of X, every element $s \in \mathcal{F}(U)$ is called a **section** (German: Schnitt) of $\mathcal{F}$ (on U).

These properties are modelled on spaces of functions for which ρ_{UV} simply corresponds to restriction of a function on U to the subset $V \subset U$.

2.1.1 Examples. (1) The assignment $\mathcal{F}(U) = C(U, \mathbb{R})$ is a sheaf of abelian groups on $\mathbb{R}^n$ associating to every open subset $U \subset \mathbb{R}^n$ the group of continuous functions from U to $\mathbb{R}$. The morphisms ρ_{UV} are restriction. The same holds for spaces of differentiable and analytic functions like C^k or C^∞ because the properties of being differentiable or analytic are **local**.

(2) The most important presheaf for us is the presheaf $\mathcal{O}$ of **regular functions** (German: reguläre Funktion) on an affine algebraic variety. Let V be an affine algebraic variety with

coordinate ring $K[V]$. To a principal open subset $D(s) = V \setminus \mathcal{V}(s)$ with $s \in K[V]$, $s \neq 0$, the ring of regular functions on $D(s)$ is the localization of $K[V]$ at $S = \{1, s, s^2, \ldots\}$. So we have $\mathcal{F}(D(s)) = K[V]_s = \mathcal{O}(D(s))$. Since the definition of a regular function on an algebraic variety is local, we have $\mathcal{O}(U)$ for every open subset $U \subset V$. The connecting morphisms ρ_{UV} are given by restriction of functions as before.

Presheaves of functions usually have more properties making them sheaves.

Definition. A presheaf $\mathcal{F}$ on a topological space X is a **sheaf** (German: Garbe) if it has the following properties.

(1) For every open cover $\{V_i\}$ of any open subset $U \subset X$ and every $s \in \mathcal{F}(U)$ such that $\rho_{UV_i}(s) = 0$ for every i it follows that $s = 0$.
(2) For every open cover $\{V_i\}$ of any open subset $U \subset X$ and all elements $s_i \in \mathcal{F}(V_i)$ with the property that $\rho_{V_iV_j}(s_i) = \rho_{V_jV_i}(s_j)$ for all i, j there is an element $s \in \mathcal{F}(U)$ such that $\rho_{UV_i}(s) = s_i$.

The second property in the above definition says that sections glue together and the first property says that they do so in a unique way.

Definition. The **stalk** $\mathcal{F}_P$ (German: Halm) of a sheaf $\mathcal{F}$ at a point $P \in X$ is the group of **germs** (German: Keime) at X. These are equivalence classes of pairs (U, s) of an open subset $U \subset X$ containing P and an element $s \in \mathcal{F}(U)$ with the equivalence relation $(U, s) \sim (V, t)$ if there is an open subset $W \subset U \cap V$ such that $\rho_{UW}(s) = \rho_{VW}(t) \in \mathcal{F}(W)$.

2.1.2 Example. The stalk of the structure sheaf $\mathcal{O}_X$ of an algebraic variety X at a point P is the local ring $\mathcal{O}_{X,P}$ with maximal ideal $\mathfrak{m}_{X,P}$ of all germs of regular functions that are 0 at P.

Exercise 2.1.3. Show that a sheaf is uniquely determined by its stalks.

The sheaf of regular functions on an algebraic variety is not only a sheaf of abelian groups but also a sheaf of rings. This simply means that $\mathcal{F}(U)$ is not only a group but also has a multiplication that makes it a ring and the restriction morhpisms are ring homomorphisms (for every open subset $U \subset X$). Over rings, we can also look at modules.

Definition. Let X be a topological space with a sheaf $\mathcal{O}_X$ of rings. A **sheaf of $\mathcal{O}_X$-modules**, also simply called an **$\mathcal{O}_X$-module** (German: Modulgarbe) is a sheaf $\mathcal{F}$ on X such that for each open subset $U \subset X$, the group $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and for each inclusion $V \subset U$ of open subsets, the restriction morphism $\mathcal{F}(U) \to \mathcal{F}(V)$ is compatible with the module structures via $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

2.1.4 Example. An ideal of a ring is a module. Sheaves of ideals are special $\mathcal{O}_X$-modules and called **ideal sheaves** (German: Idealgarbe). They correspond to **closed subschemes**.

- Let X be an affine variety with coordinate ring $A = K[X]$ and $Y \subset X$ be a closed subset. The vanishing ideal of Y defines the ideal sheaf of Y directly: for any open subset $U \subset X$ we have the ideal $\mathcal{I}_Y(U) = \{s \in \mathcal{O}_X(U) \colon \forall\, p \in Y \cap U\ s(p) = 0\} \subset \mathcal{O}_X(U)$.
- More generally, this construction works on any algebraic variety X and closed subset $Y \subset X$. On an open affine subset $U \subset X$, we have the construction above and this unique determines the ideal sheaf $\mathcal{I}_Y$ since any sheaf is uniquely determined on an open cover.

- In general, we have to keep track of multiplicities and also consider **nonreduced** structures encoded in the ideal sheaf. We do this locally: Assume that X is an affine algebraic variety with coordinate ring $A = K[X]$. In scheme language, this is written as $X = \mathrm{Spec}(A)$. Consider any ideal $I \subset A$ (not necessarily radical) and define an ideal sheaf $\mathcal{I}_Y$ by setting $\mathcal{I}_Y(U)$ to be the ideal generated by $\rho_{XU}(I)$ in $\mathcal{O}_X(U)$. The ring homomorphism $A \to A/I$ corresponds to a morphism $i\colon Y = \mathrm{Spec}(A/I) \to X = \mathrm{Spec}(A)$ of affine schemes (essentially by definition of a morphism of schemes). This is a **closed immersion** (German: abgeschlossene Immersion).

  In the general, not necessarily affine case, the above is the local construction. In the language of sheaves (and schemes), the ideal sheaf of a subscheme $Y \subset X$ is defined as the kernel of the map $i^{\#}\colon \mathcal{O}_X \to i_*\mathcal{O}_Y$ of $\mathcal{O}_X$-modules. Explicitly, for an open affine subset $U \subset X$, it is the above construction because if $U = \mathrm{Spec}(A)$ and $Y \cap U = \mathrm{Spec}(A/I)$, then the map of $\mathcal{O}_X$-modules $i^{\#}(X)\colon \mathcal{O}_X(U) = A \to A/I = (i_*\mathcal{O}_Y)(U)$ is simply the quotient map in this case.

Definition. An $\mathcal{O}_X$-module $\mathcal{F}$ is **locally free** (German: lokal frei) if there is an open cover $\{U_i\}$ of X such that $\mathcal{F}(U_i)$ is a free $\mathcal{O}_X(U_i)$-module. The **rank** (German: Rang) of a locally free $\mathcal{O}_X$-module $\mathcal{F}$ (at $P \in X$) is the rank of the module $\mathcal{F}_P$ as a $\mathcal{O}_{X,P}$ module. Locally free $\mathcal{O}_X$-modules of rank 1 are also called **invertible sheaves** (German: invertierbare Garbe).

If the algebraic variety X is connected (which is the case of irreducible varieties over $\mathbb{C}$, for example), then the rank of every locally free $\mathcal{O}_X$-module is the same at every point.

2.1.5 Example. The simplest example is the structure sheaf itself: $\mathcal{O}_X$ is a **free** $\mathcal{O}_X$-module of rank 1. Ideal sheaves tend to not be free. A notable exception are principal ideals which are equivalent to line bundles and divisors.

2.1.6 Example. Let X be an irreducible algebraic variety of dimension d. The **tangent bundle** $\mathcal{T}_X$ (German: Tangentialbündel) of X is a locally free $\mathcal{O}_X$-module of rank d. The stalk at x is $\mathcal{O}_{X,x}^d$ that can be described as a sheaf in terms of derivations on the ring (or rather their dual).

2.2. *Vector bundles*

Definition. A **(geometric) vector bundle** (German: Vektorbündel, genauer: Vektorraumbündel) of rank r on X is a variety E with a morphism $\pi\colon E \to X$ that satisfies the following properties.

- The fiber $\pi^{-1}(x)$ over a point $x \in X$ is a r-dimensional K-vector space.
- There is an open cover $\{U_i\}$ of X and isomorphisms φ_i of $U_i \times \mathbb{A}^r$ and $\pi^{-1}(U_i)$ satisfying two conditions: $(\pi \circ \varphi_i)(x, v) = x$ for all $v \in \mathbb{A}^r$ and $v \mapsto \varphi(x, v)$ is a linear isomorphism of $\mathbb{A}^r$ with $\pi^{-1}(x)$.

The variety E is called the **total space** (German: Totalraum), the variety X the **base** (German: Basis) of the bundle. An isomorphism $\varphi\colon U \times \mathbb{A}^r \to \pi^{-1}(U)$ for an open subset U of X is called a **local trivialization** (German: lokale Trivialisierung). We will often simply refer to a vector bundle by its total space (even though the data of the morphism π is crucial). A vector bundle of rank 1 is usually called a **line bundle** (German: Geradenbündel).

2.2.1 Example. Projective space $\mathbb{P}^n$ carries a tautological line bundle. To every point $[x] \in \mathbb{P}^n$, we associate the line $K \cdot x \subset \mathbb{A}^{n+1}$ that is represented by the point $[x]$ in $\mathbb{P}^n$. The total space E of this bundle is a subset of $\mathbb{P}^n \times \mathbb{A}^{n+1}$ An open cover is given by $D_+(x_i)$ for $i = 0, 1, \ldots, n$ with respect to homogeneous coordinates $[x_0, \ldots, x_n]$ on $\mathbb{P}^n$. On $D_+(x_i)$, a point has homogeneous coordinates $[x_0/x_i, \ldots, 1, \ldots, x_n/x_i]$ with a 1 in the ith position. We define $\varphi_i(x, \lambda)$ to be $([x], (\lambda x_0/x_i, \ldots, \lambda, \ldots, \lambda x_n/x_i)) \in \mathbb{P}^n \times \mathbb{A}^{n+1}$. If x is in $D_+(x_0) \cap D_+(x_1)$, then φ_0 maps $([x], \lambda)$ to $([x], (\lambda, \lambda x_1/x_0, \ldots, \lambda x_n/x_0))$ and φ_1 maps it to $([x], (\lambda x_0/x_1, \lambda, \lambda x_2/x_1, \ldots, \lambda x_n/x_1))$. The **transition function** $T_{01} \colon D_+(x_0) \cap D_+(x_1) \times \mathbb{A}^1 \to D_+(x_0) \cap D_+(x_1) \times \mathbb{A}^1$ maps $([x], \lambda)$ to $([x], \lambda x_0/x_1)$ so that $\varphi_1 \circ T_{01} = \varphi_0$. For fixed $[x]$, this is a linear automorphism of $\mathbb{A}^1$ and its inverse is $T_{10} = T_{01}^{-1}$ given by $([x], \lambda) \mapsto ([x], \lambda x_1/x_0)$.

2.2.2 Example. Let $X \subset \mathbb{A}^n$ be an irreducible and smooth affine variety of dimension d with vanishing ideal $\mathcal{I}(X) = (f_1, \ldots, f_r)$. Then the Jacobian matrix $J = (\partial_j f_i)_{i,j}$ has rank $n - d$ at every point $p \in X$.

(1) Let I be a subset of $\{1, \ldots, r\}$ of size $n - d$ and $U_I \subset X$ be the open subset of X where the matrix J_I obtained from J by deleting every row with index j not in I has rank $n - d$. Then the maps $U_I \times \mathbb{A}^{n-d} \to X \times \mathbb{A}^n$, $(x, v) \mapsto (x, v^\top J_I)$ defines a vector bundle on X of rank $n - d$, the (relative) **normal bundle** (German: Normalenbündel) of $X \subset \mathbb{A}^n$. This bundle is usually denoted by $\mathcal{N}_{X/\mathbb{A}^n}$.

(2) From the normal bundle $\mathcal{N}_{X/\mathbb{A}^n}$, we get the tangent bundle $\mathcal{T}_X$ by taking the kernel of the Jacobian instead of its row space. We will see below how this is related to the tangent bundle in the language of sheaves in Example 2.1.6.

Instead of giving a vector bundle in terms of its total space E with its projection $\pi \colon E \to X$, it can also be defined locally in terms of their transition functions. They have to satisfy the following properties. Let U_i be open subsets of X with local trivializations $\varphi_i \colon U_i \times \mathbb{A}^r \to \pi^{-1}(U_i)$ (for i in some index set I). Then $U_i \times \mathbb{A}^r$ and $U_j \times \mathbb{A}^r$ must be related by **transition functions** (German: Übergangsfunktionen) $T_{ij} \colon (U_i \cap U_j) \times \mathbb{A}^r \to (U_i \cap U_j) \times \mathbb{A}^r$ which are regular functions, the identity on $U_i \cap U_j$, and give a linear isomorphism $T_{ij}(x) \in \mathrm{GL}_r$ for every $x \in U_i \cap U_j$. Moreover, they must satisfy the **cocycle condition** (German: Kozykelbedingung)

$$T_{jk}|_{U_i \cap U_j \cap U_k} \circ T_{ij}|_{U_i \cap U_j \cap U_k} = T_{ik}|_{U_i \cap U_j \cap U_k}.$$

Exercise 2.2.3. What are the transition functions of the normal bundle $\mathcal{N}_{X/\mathbb{A}^n}$ for a smooth affine variety $X \subset \mathbb{A}^n$ (see Example 2.2.2)? What are the transition functions of the tangent bundle $\mathcal{T}_X$ as in Example 2.2.2?

Definition. Let $\pi \colon E \to X$ be a vector bundle of rank r on a smooth algebraic variety X. A **section** of E is a morphism $s \colon X \to E$ such that $\pi \circ s$ is the identity on X. For an open subset U of X, a section over U is a map $s \colon U \to E$ with $\pi \circ s = \mathrm{id}_U$.

2.2.4 Remark. In a local trivialization $U \times \mathbb{A}^r$ of E, a section is therefore a map that sends $x \in U$ to a point (x, v), which means that a section assigns a vector in $\pi^{-1}(x)$ to every $x \in X$. The trivial example is the **zero section** that assigns the zero vector to every point $x \in X$.

2.3. Sheaves and Vector Bundles

2.3.1 Construction. Let $\pi\colon E \to X$ be a vector bundle of rank r on a smooth algebraic variety X. For an open subset $U \subset X$, we assign the set of sections $\mathcal{F}_E(U)$ of E over U. Since $\pi^{-1}(x)$ is a vector space for every $x \in X$, the set $\mathcal{F}_E(U)$ is an abelian group with pointwise addition. The assignment $U \mapsto \mathcal{F}_E(U)$ is in fact the **sheaf of sections** of E.

Exercise 2.3.2. Check the defining properties of a sheaf for the sheaf of sections $\mathcal{F}_E$ of a vector bundle E.

2.3.3 Proposition. *Let E be a vector bundle of rank r on a smooth algebraic variety X. The associated sheaf $\mathcal{F}_E$ of sections is a locally free $\mathcal{O}_X$-module of rank r.*

Proof. It suffices to prove the claim on a local trivialization $U \times \mathbb{A}^r$ of the bundle E. A section is then essentially a morphism to $\mathbb{A}^r$ and therefore given as an r-tuple of regular functions on U. The $\mathcal{O}_X(U)$-module structure comes from pointwise multiplication of this r-tuple of regular functions by an element of $\mathcal{O}_X(U)$. As an $\mathcal{O}_X(U)$-module, $\mathcal{F}_E(U)$ is isomorphic to $\mathcal{O}_X(U)^r$. ■

In this way, every vector bundle E of rank r on a smooth variety X gives rise to a locally free sheaf $\mathcal{F}_E$ of rank r. This is in fact reversible.

2.3.4 Construction. Let $\mathcal{F}$ be a locally free sheaf of rank r on a smooth variety X. Then there is an open cover $\{U_i\}$ of X such that $\mathcal{F}(U_i) = \mathcal{O}_X(U_i)^r$ for each U_i. With this open cover come transition functions $T_{ij} \in \mathrm{GL}_r(\mathcal{O}_X(U_i \cap U_j))$ that satisfy the cocycle conditions (by the defining properties of sheaves: sections are uniquely determined by their restriction, here to $U_i \cap U_j \cap U_k$).

Exercise 2.3.5. Check the cocycle condition for the transition functions $T_{ij} \in \mathrm{GL}_r(\mathcal{O}_X(U_i \cap U_j))$.

We can also write the total space of the vector bundle constructed in Construction 2.3.4 by giving its coordinate ring in the following sense. Let $S(\mathcal{F})$ be the symmetric algebra on $\mathcal{F}$. This is a construction as a sheaf: locally, $\mathcal{F}(U) = \mathcal{O}_X(U)^r$ is free of rank r and $S(\mathcal{F})(U)$ is the symmetric algebra of the $\mathcal{O}_X(U)$-module $\mathcal{O}_X(U)^r$. This symmetric algebra is isomorphic to $\mathcal{O}_X(U)[x_1, \dots, x_r]$. Overall, the result is the $\mathcal{O}_X$-module $S(\mathcal{F})$. The total space of the bundle is $E = \mathrm{Spec}(S(\mathcal{F}))$. The ring homomorphisms $\mathcal{O}_X(U) \to \mathcal{O}_X(U)[x_1, \dots, x_r]$ locally define the bundle projection $\pi\colon \mathrm{Spec}(S(\mathcal{F})) \to X$.

We have not discussed all necessary technical details to understand this construction. The point is that the language of sheaves is convenient here, because takes care of the necessary gluing arguments.

2.4. Divisors

Divisors form a building block for the intersection ring $A^*(X)$, namely $A^1(X)$. Weil divisors are more elementary, Cartier divisors better suited for singular varieties. They are equivalent notions on smooth varieties. We will discuss here only Weil divisors. Before we get started, we need a little bit of commutative algebra.

2.4.1 Detour: Discrete Valuation Rings

Definition. Let K be a field. A **(discrete) valuation** (German: diskrete Bewertung) of K is a map $\nu\colon K \to \mathbb{Z} \cup \{\infty\}$ satisfying the following conditions.

- $\nu(a) = \infty$ if and only if $a = 0$
- $\nu(ab) = \nu(a) + \nu(b)$ for all $a, b \in A \setminus \{0\}$
- $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$

An integral domain A is called a **discrete valuation ring** (German: diskreter Bewertungsring) if there exists a discrete valuation on on its quotient field $\mathrm{Quot}(A)$ with discrete valuation ν such that $A = \{x \in K^*\colon \nu(x) \geq 0\} \cup \{0\}$.

2.4.1 Remark. If K carries a disrete valuation $\nu\colon K \to \mathbb{Z} \cup \{\infty\}$, then the subset $\{x \in K\colon \nu(x) \geq 0\} \cup \{0\}$ is always a local subring with maximal ideal $\mathfrak{m} = \{x \in K\colon \nu(x) > 0\}$. This follows from from the last two properties of valuations. Since ν is a homomorphism of semigroups $(A, \cdot)$ and $(\mathbb{Z}, +)$, we have $\nu(1) = 0$ and $\nu(x) = -\nu(x^{-1})$ for all $x \in K^*$.

2.4.2 Example. Let $p \in \mathbb{Z}$ be a prime and $A = \mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ with respect to the multiplicative set $S = \{m \in \mathbb{Z}\colon \gcd(m, p) = 1\} = \mathbb{Z} \setminus (\mathbb{Z}p)$. Then A is a discrete valuation ring for the p-**adic valuation** ν_p on $\mathbb{Q}$. Write a fraction $a/b = p^k c/d$ such that c and d are not divisible by p, then $\nu(a/b) = k$. The valuation ring is the set of all fractions a/b such that $\nu(a/b) \geq 0$.

2.4.3 Proposition. *A discrete valuation ring A is a local ring of dimension* 1 *and a principal ideal domain. Its maximal ideal is generated by one element t so that any element of* $\mathrm{Quot}(A)$ *can be written as ut^k for a unit $u \in A$ and a $k \in \mathbb{Z}$. The valuation of ut^k is k.*

A generator of the maximal ideal of a discrete valuation ring is called a **uniformizing parameter**.

Proof. Every unit of A has valuation 0. Let t be an element with smallest valuation. Since the image of the valuation is a subgroup of $\mathbb{Z}$ generated by one element m. By replacing ν by $1/m\nu$, we can assume $\nu(t) = 1$. For any $t' \in A$ with $\nu(t')$, it follows that $\nu(t'/t) = 0$ so that $t' = ut$ for a unit $t \in A$. Moreover, if $a \in A$ has $\nu(a) = k$ for $k \in \mathbb{N}$, then $u = a/t^k$ is a unit so that $a = ut^k$ is in the ideal generated by t. Since any element $a \in A$ that is not a unit has a positive valuation, it is contained in (t) so that A is a local ring with maximal ideal which is principal. Hence A is local of dimension 1. If $I \subset A$ is an ideal of A, $I \neq (0)$, pick an element $a \in I$ of minimial valuation, say $\nu(a) = k$. Then we conclude that $a = ut^k$ as above which implies $I = (t^k)$. ∎

There is one central example in the context of divisors, namely discrete valuation rings corresponding to irreducible polynomials in coordinate rings of smooth irreducible varieties.

2.4.4 Example. Let $X \subset \mathbb{A}^n$ be a smooth irreducible variety and $K[X] = K[x_1, \ldots, x_n]/\mathcal{I}(X)$ its coordinate ring. Let $f \in K[X]$ be an irreducible polynomial. Then the ring $A = K[X]_{(f)}$ obtained by localization of $K[X]$ at the prime ideal (f) is a discrete valuation ring of the function field $K(X)$ and its maximal ideal is generated by f.

2.4.5 Remark. A valuation ring of a field K is a subring B such that for all $x \in K^*$ we have $x \in B$ or $x^{-1} \in B$. A discrete valuation ring is a valuation ring of (Krull-)dimension 1.

2.4.2 Weil divisors

To see the point and introduce some notions, we informally discuss the case of plane curves.

Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d > 1$. For each line $L \subset \mathbb{P}^2$, we consider $L \cap C$, which is a set of d points, if counted with multiplicity. Let us write this formally as $L \cap C = \sum m_i P_i$, where m_i is the multiplicity of P_i in $L \cap C$. Such a formal sum is a **divisor** (German: Divisor) on C. As we vary L, this gives a family of divisors parametrized by the lines L, which are points in $(\mathbb{P}^2)^*$. Such a family is called a **linear system** (German: Linearsystem) of divisors on C. The embedding of C to $\mathbb{P}^2$ can be recovered from the linear system in the following way. If p is a point on C, we can consider all divisors in the linear system that contain p. These divisors correspond to lines $L \in (\mathbb{P}^2)^*$ through p. These lines form a line in $(\mathbb{P}^2)^*$ dual to p and so uniquely determine p.

For two different divisors $D = L \cap C$ and $D' = L' \cap C$ corresponding to two lines $L, L' \subset \mathbb{P}^2$, we get the rational function f/f' for defining linear forms $L = \mathcal{V}_+(f)$, $L' = \mathcal{V}_+(f')$. The restriction g of the rational function f/f' to C has zeros in the points of D and poles in the points of D' (of order equal to the multiplicity of the point in the divisor). The divisors D and D' are called **linearly equivalent** (German: linear äquivalent).

Definition. Let X be a smooth algebraic variety. A **prime divisor** (Deutsch: Primdivisor) is an irreducible closed subvariety $Y \subset X$ of codimension 1. A **Weil divisor** (German: Weil-Divisor) is an element of the free abelian group $\operatorname{Div}(X)$ generated by the prime divisors. So a divisor can be written as a finite sum $D = \sum n_i Y_i$ where $n_i \in \mathbb{Z}$ and $Y_i \subset X$ are prime divisors. A divisor is called **effective** (German: effektiv) if all weights n_i are nonnegative.

Let $X \subset \mathbb{A}^n$ be a smooth irreducible affine variety and $Y \subset X$ be a prime divisor that is given by one equation $Y = \mathcal{V}_+(f)$ for an irreducible function $f \in K[X]$. Then the localization $K[X]_{(f)}$ of the coordinate ring of X at the prime ideal generated by f is a discrete valuation ring, see Example 2.4.4. The corresponding valuation ring of the function field $K(X)$ of X is usually written as $\mathcal{O}_{X,Y}$. We write ν_Y for the discrete valuation of $K(X)$. In particular, any non-zero rational function g on X has a valuation $\nu_Y(g)$. If $\nu_Y(g)$ is positive, we say that g has a **zero** along Y of order $\nu_Y(g)$. If it is negative, we say that g has a **pole** along Y of order $|\nu_Y(g)|$.

2.4.6 Lemma. *For any nonzero rational function $g \in K(X)$ on a smooth irreducible variety X there are only finitely many prime divisors $Y \subset X$ with $\nu_Y(g) \neq 0$.*

Proof. We can assume that X is affine by restricting to an affine open subset (since the complement of this open is closed and therefore only has finitely many irreducible components). Moreover, we can also assume that our function g is regular on X so that $\nu_Y(g) \geq 0$ for all prime divisors $Y \subset X$. So we have to show that $\nu_Y(g)$ is positive only for finitely many Y. But those Y are exactly the minimal prime ideals in a primary decomposition of $(g) \subset K[X]$. ∎

Definition. Let X be a smooth irreducible algebraic variety and let $g \in K(X)^*$. The **divisor** of g, denoted by (g), is

$$(g) = \sum \nu_Y(g) Y,$$

where the sum is taken over all prime divisors of X. Any divisor equal to the divisor of a function is called a **principal divisor** (German: Hauptdivisor).

This is a finite sum (and hence well defined) by the previous Lemma 2.4.6.

Exercise 2.4.7. Show that the map sending a function $f \in K(X)^*$ to its divisor (f) gives a homomorphism of the multiplicative group $K(X)^*$ and the additive group $\mathrm{Div}(X)$. So its image is a normal subgroup of $\mathrm{Div}(X)$.

Definition. Two divisors D and D' are **linearly equivalent** (German: linear äquivalent), written $D \sim D'$, if $D - D'$ is a principal divisor. The group of divisors modulo the normal subgroup of principal divisors is called the **divisor class group** $\mathrm{Cl}(X)$ of X (German: Divisorklassengruppe).

2.4.8 Proposition. *Let A be a finitely generated K-algebra without zero divisors. (So A is the coordinate ring of an irreducible affine variety $X = \mathrm{Spec}(A)$ over K.) Then A is a unique factorization domain (German: faktorieller Ring) if and only if A is integrally closed and $\mathrm{Cl}(\mathrm{Spec}(A)) = 0$.*

Proof. The main thing to prove here is that in an integrally closed domain A every prime ideal of height 1 is principal if and only if $\mathrm{Cl}(\mathrm{Spec}(A)) = 0$. For details and a proof of this fact, see for example Hartshorne, Part II, Proposition 6.2. ■

This fact is useful for computing the first examples of divisor class groups.

2.4.9 Example. The divisor class group of $\mathbb{A}^n$ is trivial because $K[x_1, \dots, x_n]$ is a unique factorization domain.

2.4.10 Example. If you know some algebraic number theory: If A is a Dedekind domain, then the ideal class group from algebraic number theory is just the divisor class group of $\mathrm{Spec}(A)$. So Proposition 2.4.8 is a generalization of the fact that a Dedekind domain is a unique factorization domain if and only if the ideal class group is trivial.

2.4.11 Proposition. *Let $X = \mathbb{P}^n$ be n-dimensional projective space over K. For any divisor $D = \sum n_i Y_i$, define the degree of D as $\deg(D) = \sum n_i \deg(Y_i)$, where $\deg(Y_i)$ is the degree of the hypersurface $Y_i \subset \mathbb{P}^n$. Let $H = \mathcal{V}_+(x_0)$ be a hyperplane.*

(1) If D is any divisor of degree d, then $D \sim dH$.
(2) For any $f \in K(\mathbb{P}^n)^*$, $\deg(f) = 0$.
(3) The degree function gives an isomorphism $\deg\colon \mathrm{Cl}(\mathbb{P}^n) \to \mathbb{Z}$.

Proof. First, let g be a homogeneous polynomial of degree d and factor it into irreducible factors $g = g_1^{e_1} \cdot \ldots \cdot g_r^{e_r}$. Each factor g_i defines a prime divisor $Y_i \subset \mathbb{P}^n$ of degree $d_i = \deg(g_i)$. Checking the affine charts, it makes sense to define define the divisor of g to be $(g) = \sum_{i=1}^r e_i Y_i$ of degree $d = \sum e_i d_i$ on $\mathbb{P}^n$. A rational function f on $\mathbb{P}^n$ is the quotient g/h of two homogeneous polynomials g and h of the same degree. With the above definition, we have $(f) = (g) - (h)$ so that $\deg(f) = 0$.

Now let D be a divisor of degree d. We can write it as a difference $D = D_1 - D_2$ of effective divisors D_1 and D_2 of degrees d_i with $d = d_1 - d_2$. Since each D_i is effective, we can write it as $D_i = (g_i)$, the divisor of a homogeneous polynomial g_i of degree d_i. This is possible because every prime ideal of the homogeneous coordinate ring of $\mathbb{P}^n$ of height 1 is principal. Then $D - dH = (f)$ for the rational function $f = g_1/x_0^d g_2$ showing (1). (3) now follows from (1) and (2) because $\deg(H) = 1$. ■

We will later see more divisor class groups. Feel free to browse the literature for more examples. For instance, Hartshorne shows in Section 6 of Part II that $\mathrm{Cl}(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z}^2$ and discusses the case of curves. This includes the example that the divisor class group of a smooth plane cubic decomposes as a continuous group $\mathrm{Cl}^o(X)$ and $\mathbb{Z}$. In this case, the genus is 1 and the continuous part is isomorphic to the curve X itself. More generally, this part is an abelian variety of dimension equal to the genus of the curve.

2.4.3 Divisors and line bundles

We will see that divisors and line bundles are the same on sufficiently nice algebraic varieties. We will restrict our discussion here to smooth and irreducible varieties.

Definition. An **invertible sheaf** is a locally free $\mathcal{O}_X$-module of rank 1.

2.4.12 Proposition. *If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on X, then so is $\mathcal{L} \otimes \mathcal{M}$ (on any ringed space X). If $\mathcal{L}$ is invertible, then $\mathcal{L} \otimes \mathcal{L}^\vee \cong \mathcal{O}_X$.*

Proof. The first part is straightforward since $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$. For the second, the dual sheaf $\mathcal{L}^\vee$ is $\mathcal{H}\mathrm{om}(\mathcal{L}, \mathcal{O}_X)$ so that $\mathcal{L}^\vee \otimes \mathcal{L} \cong \mathcal{H}\mathrm{om}(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_X$. ■

Definition. For any ringed space, we define the **Picard group** $\mathrm{Pic}(X)$ of X to be the group of isomorphism classes of invertible sheaves on X with multiplication $\otimes$.

The previous proposition shows that this is in fact a group with $\mathcal{L}^{-1} = \mathcal{L}^\vee$.

The point is that the Picard group of a smooth variety is isomorphic to its divisor class group. We will show this via line bundles with fixed sections, which turn out to be the same as Weil divisors.

Definition. A **rational section** of a line bundle $\mathcal{L}$ on an irreducible variety X is a section of $\mathcal{L}$ on some open and dense subset $U \subset X$. (A rational section on a reducible variety is a rational section that does not vanish identically on any irreducible component.) To a rational section s of $\mathcal{L}$, associate the Weil divisor

$$\mathrm{div}(s) = \sum_Y \nu_Y(s) Y.$$

Here, the valuation $\nu_Y(s)$ makes sense as follows: on any local trivialization of $\mathcal{L}$ on an open subset $U \subset X$ such that $U \cap Y$ is Zariski-dense in Y, the section s is a rational function. Since any two trivializations differ by an invertible function (namely a transition function), the valuation does not depend on the choice of local trivialization. Also, $\nu_Y(s)$ is 0 for all but finitely many prime divisors Y.

Consider the set of pairs $\{(\mathcal{L}, s)\}$ of line bundles with a fixed nonzero rational section of $\mathcal{L}$ up to isomorphism. A pair $(\mathcal{L}, s)$ is isomorphic to $(\mathcal{L}', s')$ if there is an isomorphism $\varphi \colon \mathcal{L} \to \mathcal{L}'$ of line bundles and a dense open subset $U \subset X$ on which the sections s' and $\varphi \circ s$ of $\mathcal{L}'$ differ by a regular function on U. This set forms an abelian group under tensor product with identity $(\mathcal{O}_X, 1)$. The map div gives a homomorphism from the set of isomorphism classes of pairs $(\mathcal{L}, s)$ and the group of divisors $\mathrm{Div}(X)$. We will use that line bundles and invertible sheaves can be identified.

2.4.13 Proposition. *If X is normal, then the map* div *is injective. (A variety X is normal if every local ring $\mathcal{O}_{X,x}$ is an integrally closed domain.)*

Proof. Since we already know that div is a group homomorphism, it suffices to show that any pair $(\mathcal{L}, s)$ that gets mapped to $0 \in \mathrm{Div}(X)$ is isomorphic to $(\mathcal{O}, 1)$. We consider the map of invertible sheaves $\times s \colon \mathcal{O} \to \mathcal{L}$ given by multiplication of s. So on a local trivilization of $\mathcal{L}$ over U, this maps an element $t \in \mathcal{O}_X(U)$ to $st \in \mathcal{L}(U)$. Here, st is a section of the line bundle $\mathcal{L}$ over U because the divisor of s is 0 so that s has no poles on X and in particular not on U. Now choose a local isomorphism $\iota \colon \mathcal{L}|_U \to \mathcal{O}_U$ on $U \subset X$. The composition with $\times s$ gives a map $\times s' \colon \mathcal{O}_U \to \mathcal{O}_U$ that is multiplication by $s' = \iota(s)$. Since $\mathrm{div}(s) = 0$, this function s' has no poles and is therefore regular on U. The same holds for $1/s'$ because s has no zeroes either. Therefore, $\times s'$ is an isomorphism, which implies that $\times s$ is an isomorphism over U. Since we can cover X by local trivializations of $\mathcal{L}$, this shows that $(\mathcal{L}, s)$ is isomorphic to $(\mathcal{O}, 1)$. ■

Definition. Let X be a smooth irreducible algebraic variety and D a Weil divisor on X. Define the sheaf $\mathcal{O}_X(D)$ on open subsets $U \subset X$ by

$$\{t \in K(X)^* \colon \mathrm{div}|_U t + D|_U \geq 0\} \cup \{0\}.$$

Here, $\mathrm{div}|_U t$ means that we take the divisor of t considered as a rational function on U (so we only consider prime divisors of U) and $D|_U$ is $\sum_{U \cap Y \neq \emptyset} n_Y (Y \cap U)$ if $D = \sum_Y n_Y Y$.

Here, $\mathcal{O}_X(D)(U)$ is an abelian group with respect to addition of rational functions. This is well defined because $\nu_Y(t_1 + t_2) \geq \min\{\nu_Y(t_1), \nu_Y(t_2)\}$ for every prime divisor Y so that $\mathrm{div}|_U(t_1 + t_2) \geq -D|_U$ whenever the same holds for each $\mathrm{div}|_U(t_i)$.

We now have to show that this sheaf is invertible. We will use that every local ring $\mathcal{O}_{X,x}$ of a smooth variety X is a unique factorization domain.

2.4.14 Proposition. *Let X be a smooth irreducible algebraic variety and $Y \subset X$ a prime divisor. Then there is an open cover $\{U_i\}$ of X and sections $f_i \in K(X)$ such that $Y \cap U_i$ is defined by f_i. In particular, if $D = \sum_Y n_Y Y$ is a Weil divisor, then there is an open cover $\{U_i\}$ of X such that $D|_{U_i} = \mathrm{div}(f_i)$ for suitable $f_i \in K(X)$.*

Proof. Let $x \in X$ be a point. If $x \notin Y$, then there is an neighborhood U of X such that $Y \cap U = \mathcal{V}(1)$ and $1 \in \mathcal{O}_X(U)$. If $x \in Y$, then the vanishing ideal of Y in $\mathcal{O}_{X,x}$ is a prime ideal of height 1. Since $\mathcal{O}_{X,x}$ is a unique factorization domain, this ideal is principal so that locally around x, Y is defined by one equation $f_x \in K(X)$. Since any algebraic variety is a noetherian topological space, this proves the first claim. To define any Weil divisor $D = \sum_Y n_Y Y$, we refine take a common refinement for all finitely many prime divisors Y in D with non-zero coefficient n_Y. If $Y \cap U$ is locally defined by a function $f \in K(X)$, then $n_Y Y$ is defined by f^{n_Y}. Since taking divisors is a group homomorphism $\mathrm{div} \colon K(X)^* \to \mathrm{Div}(X)$, this proves the second claim. ■

2.4.15 Proposition. *Let X be a smooth algebraic variety and D be a Weil divisor on X. The sheaf $\mathcal{O}_X(D)$ is an invertible sheaf.*

Proof. Cover X by open subsets U such that $D|_U = \mathrm{div}(f)$ as in Proposition 2.4.14. Then locally, we get a map $\mathcal{O}|_U \to \mathcal{O}(D)|_U$, $t \mapsto tf^{-1}$. Indeed, this is well defined because $\mathrm{div}|_U(tf^{-1}) = \mathrm{div}|_U(t) - D|_U + D|_U = \mathrm{div}|_U(t) \geq 0$ since t is a regular function on U. The above morphism of (local!) invertible sheaves is an isomorphism whose inverse is given by multiplication with f, which shows that $\mathcal{O}(D)$ is locally free of rank 1. ■

2.4.16 Proposition. *Let X be a smooth algebraic variety and $(\mathcal{L}, s)$ be a pair of a line bundle and a rational section. Then the pair $(\mathcal{L}, s)$ is isomorphic to $(\mathcal{O}(\mathrm{div}(s)), 1)$, where 1 is a rational section of $\mathcal{O}(\mathrm{div}(s))$ defined away from the zeroes and poles of s locally by the constant function 1.*

Proof. Write $D = \mathrm{div}(s)$ throughout this proof. Let us first show that $\mathcal{O}(D)$ and $\mathcal{L}$ are isomorphic invertible sheaves. We know that locally, $\mathcal{O}|_U$ is isomorphic to $\mathcal{O}(D)|_U$ by a defining equation f of D on U via $t \mapsto tf^{-1}$ (see proof of Proposition 2.4.15). Define the map $\varphi_U : \mathcal{O}(D)(U) \to \mathcal{L}(U)$, $t \mapsto st$ in the following sense. Fix an isomorphism $\sigma : \mathcal{L}|_U \to \mathcal{O}|_U$ as $\mathcal{O}$-modules. The rational function $st \in K(X)$ is a regular function on U, because $\mathrm{div}|_U(st) = \mathrm{div}|_U(s) + \mathrm{div}|_U(t) = D|_U + \mathrm{div}|_U(t) \geq 0$ by definition of $\mathcal{O}_X(D)$. So st is in $\mathcal{O}(U)$ and $\sigma^{-1}(st)$ is a section in $\mathcal{L}(U)$. Conversely, for every regular function f on U, the element $f/s \in K(X)$ is in $\mathcal{O}(D)(U)$ and maps to f under multiplication by s. So the above map is an isomorphism of $\mathcal{O}_X(U)$-modules over U. To see that this gives an isomorphism φ of $\mathcal{O}(D)$ and $\mathcal{L}$ as sheaves (globally, so to speak), we have to check compatibility of these isomorphisms for an open cover of X with the restriction maps. This is fine because we multiply locally by s everywhere and we get the desired isomorphism.

Finally, we check that the above morphism can be chosen to take the rational section 1 of $\mathcal{O}(D)$ to s. Let us make concrete how 1 as a rational section of $\mathcal{O}(D)$ first. Write $D = \sum_{i=1}^k n_i Y_i$ with prime divisors $Y_i \subset X$ and $n_i \neq 0$. Set $\mathrm{supp}(D) = \bigcup_{i=1}^k Y_i$. Then $1 \in K(X)$ is a rational function on $U = X \setminus \mathrm{supp}(D)$ such that $\mathrm{div}|_U(1) + D|_U \geq 0$. This is the rational section 1 of $\mathcal{O}(D)$. On this set U, we above isomorphism maps the rational section 1 of $\mathcal{O}(D)$ to the rational section s of $\mathcal{L}$. ∎

So overall, on a smooth, irreducible algebraic variety X, we have an isomorphism between the group of Weil divisors $\mathrm{Div}(X)$ and the group of isomorphism classes of pairs $(\mathcal{L}, s)$ of line bundles with rational sections. The divisor class group $\mathrm{Cl}(X)$ is isomorphic to the group of isomorphism classes of line bundles. This follows from the above discussion because linear equivalence of divisors corresponds to forgetting the chosen rational section of the line bundle.

Indeed, if s and s' are rational sections of a line bundle $\mathcal{L}$, then $\mathrm{div}(s) - \mathrm{div}(s')$ is principal so that the Weil divisors associated to $(\mathcal{L}, s)$ and $(\mathcal{L}, s')$ are linearly equivalent. Conversely, if $D - D' = \mathrm{div}(s/s')$, then $\mathcal{O}(D)$ is isomorphic to $\mathcal{O}(D')$ via $t \mapsto st/s'$ so that D corresponds to $(\mathcal{O}(D), 1)$ and D' to $(\mathcal{O}(D'), 1) \cong (\mathcal{O}(D), s'/s)$.

We will later see how vector bundles of higher rank give rise to classes in the Chow ring (rather than divisors) via the construction of **Chern classes**.

2.4.4 Line bundles on projective space

2.4.17 Example. Remember how a homogeneous polynomial $f \in K[x_0, \ldots, x_n]$ is not a function on $\mathbb{P}^n$? Rather, it is a global section of a line bundle on $\mathbb{P}^n$ usually called $\mathcal{O}(d)$ (or more precisely $\mathcal{O}_{\mathbb{P}^n}(d)$) which belongs to the linear system of divisors generated by the hypersurface $\mathcal{V}_+(f)$. Clearly, a homogeneous polynomial $f \in K[x_0, \ldots, x_n]$ of degree d is a function on every open subset $D_+(x_i)$ for $i = 0, 1, \ldots, n$ given by its dehomogenization. We describe the line bundle on the local trivializations $D_+(x_i) \times \mathbb{A}^1$ via the transition functions. On $D_+(x_0) \cap D_+(x_1)$, the transition function T_{01} is given by $(x_1/x_0)^d$ because

$$f(1, x_1/x_0, \ldots, x_n/x_0) = (x_1/x_0)^d f(x_0/x_1, 1, \ldots, x_n/x_1).$$

This last condition says that f takes the same value as a section on $D_+(x_0)$ and $D_+(x_1)$ on their intersection. Since the divisor associated to a line bundle is the zero-set of a section, the line bundle corresponds to the linear system defined by $\mathcal{V}_+(f)$.

Exercise 2.4.18. Show that $\mathcal{O}(d)$ on $\mathbb{P}^1$ has a $(d+1)$-dimensional space of global sections. In particular, the line bundles are not isomorphic!

2.4.19 Example. We computed the divisor class group of $\mathbb{P}^n$ in Proposition 2.4.11, which we now know is isomorphic to the Picard group of $\mathbb{P}^n$. This means: $\mathrm{Pic}(\mathbb{P}^n)$ is the abelian group generated by $\mathcal{O}(1)$.

Exercise 2.4.20. Verify that $\mathcal{O}(m+n) = \mathcal{O}(m) \otimes \mathcal{O}(n)$.

2.4.21 Construction. Every vector bundle E has a **dual bundle** $E^\vee$: In terms of a local trivialization $U \times \mathbb{A}^r$, it simply means that we take $U \times (\mathbb{A}^r)^*$. The transition functions then are different: If T_{ij} is the transition function for local trivializations $\varphi_i \colon U_i \times \mathbb{A}^r$, then for the dual bundle we get $(T_{ij}^{-1})^\top$. Indeed, for fixed $x \in U_i \cap U_j$, the transition function $T_{ij}^\vee$ of the dual bundle is defined by $\left(T_{ij}^\vee(\ell)\right)(T_{ij}v) = \ell(v)$ for all $v \in \mathbb{A}^r$ and all linear functions $\ell \colon \mathbb{A}^r \to \mathbb{A}^1$. This implies $T_{ij}^\vee = (T_{ij}^{-1})^\top$.

2.4.22 Example. The line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ is dual to the tautological bundle in Example 2.2.1 which means that the tautological bundle is the bundle $\mathcal{O}(1)^\vee$ which is also written as $\mathcal{O}(-1)$. We computed the transition function $T_{01}^\vee$ of the tautological bundle to be x_0/x_1. For $\mathcal{O}(1)$, we computed it to be x_1/x_0. So we have $T_{01}^\vee = \left(T_{01}^{-1}\right)^\top$ showing that $\mathcal{O}(-1)$ is dual to $\mathcal{O}(1)$.

Exercise 2.4.23. Show that $\mathcal{O}(d)$ is the dual bundle of $\mathcal{O}(-1)^{\otimes d}$, the d-th tensor power of the tautological bundle. In a local trivialization $U \times \mathbb{A}^r$, the tensor product of a bundle with itself is $U \times (\mathbb{A}^r \otimes \mathbb{A}^r)$.

In general, vector bundles of higher rank can be complicated. For general education, here is a result for $\mathbb{P}^1$ classifying all vector bundles in terms of line bundles.

2.4.24 Theorem (Grothendieck). *Any vector bundle E on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles. Any line bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(d)$ for some $d \in \mathbb{Z}$.*

For more general varieties, like $\mathbb{P}^2$, there are vector bundles of higher rank that are not decomposable into line bundles (on $\mathbb{P}^2$, the tangent bundle does not decompose, for example).

Chapter 3

Chow Ring

We discuss the definition of Chow groups for smooth algebraic varieties and sketch some properties and methods to determine the Chow ring. We will not go into the details of the proof of the intersection product, at least for now. We might return to the technical aspects (based on Fulton's book on Intersection Theory).

3.1. Cycles

Throughout, X will mostly be a smooth irreducible algebraic variety. For the basic definitions, this is usually not important. The most important reducible case for us to keep in mind is the reduced, 0-dimensional case.

Definition. Let X be an algebraic variety. The **group of cycles** on X, denoted by $Z(X)$, is the free abelian group generated by the set of irreducible subvarieties of X. Write $Z_k(X)$ for the (sub)group of cycles that are formal linear combinations of irreducible subvarieties of dimension k. The elements of $Z_k(X)$ are called k-**cycles**. A cycle $Z = \sum n_i Y_i$ is **effective** if the coefficients n_i are all nonnegative.

The group of cycles is graded by dimension (as a $\mathbb{Z}$-module), that is $Z(X) = \bigoplus_k Z_k(X)$. If X has dimension n, then $Z_{n-1}(X) = \mathrm{Div}(X)$ is the group of Weil divisors on X. Recall that two divisors D and D' in $\mathrm{Div}(X)$ are linearly equivalent if there is a rational function $f \in K(X)$ on X such that $D - D' = (f)$. Another way to view this, is to consider f as a map φ_f from X to $\mathbb{P}^1$: if f is defined in $x \in X$, map it to $[f(x), 1] \in \mathbb{P}^1$; if f is not defined in $x \in X$, map it to $[1, 0] \in \mathbb{P}^1$. In this way, we can think of (f) as $\varphi_f^{-1}([0,1]) - \varphi_f^{-1}([1,0])$. (More precisely, we should not take the preimage but the pull back.) We generalize this point of view to k-cycles in X for any k.

Exercise 3.1.1. Why is φ_f as defined in the previous paragraph a morphism of algebraic varieties? How is it defined locally where $f = g/h$ for $g, h \in \mathcal{O}_X(U)$?

For the following definition of equivalence, we need to properly take multiplicity into account. If X is a smooth, irreducible affine variety with coordinate ring $K[X]$ and $Y \subset X$ is defined by an ideal $I \subset K[X]$, take a primary decomposition of I and let $P_1, \ldots, P_s$ be the minimal primes of I. We associate a multiplicity ℓ_i to each irreducible component $Y_i \subset Y$ corresponding to the prime P_i as follows in terms of the length of a module. Localize $K[X]/I$ at the prime P_i and consider it as a module over the localization of $K[X]$ at P_i. As such it has a **length**, namely the maximal length of a chain of submodules (which is well-defined by the Jordan-Hölder Theorem). The cycle associated to the subscheme Y is then $\sum \ell_i Y_i$ which we write as $\langle Y \rangle$.

Definition. Let $\mathrm{Rat}(X) \subset Z(X)$ be the subgroup generated by differences of the form

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle ,$$

where t_0, t_1 are in $\mathbb{P}^1$ and Φ is an irreducible subvariety of $\mathbb{P}^1 \times X$ not contained in any fiber $\{t\} \times X$ of the projection $\mathbb{P}^1 \times X \to \mathbb{P}^1$ to the first factor. We say that two cycles are **rationally equivalent** if their difference is in $\mathrm{Rat}(X)$.

Locally on $\mathbb{A}^1 \times U$, the subscheme $\Phi \cap (\{t_0\} \times X)$ is given by the ideal $I \subset K[x] \otimes K[U]$ generated by $x - t_0$ and the ideal defining Φ and we mean the cycle associated to this subscheme with multiplicities as above.

Exercise 3.1.2. How does the rational equivalence of $(n-1)$-cycles on an n-dimensional variety translate to linear equivalence concretely? Consider the graph of the above map φ_f.

Definition. The **Chow group** of X is the quotient

$$A(X) = Z(X)/\mathrm{Rat}(X),$$

the **group of rational equivalence classes of cycles** on X. If $Y \in Z(X)$ is a cycle, we write $[Y] \in A(X)$ for its equivalence class.

3.1.3 Proposition. *The Chow group is also graded by dimension*

$$A(X) = \bigoplus_k A_k(X),$$

where $A_k(X)$ is the group of rational equivalence classes of k-cycles.

Proof. Let $\Phi \subset \mathbb{P}^1 \times X$ be irreducible of dimension $k+1$. The ideal $I \subset K[x] \otimes K[U]$ generated by $x - t_i$ and the ideal of Φ on an affine chart $\mathbb{A}^1 \times U$ has dimension k because $x - t_i$ is not a zero divisor. This shows that $\mathrm{Rat}(X)$ is graded by dimension. The grading of $Z(X)$ therefore decends to the quotient $Z(X)/\mathrm{Rat}(X) = A(X)$. ∎

We will write $A^c(X)$ for $A_{\dim(X)-c}(X)$. In other words, we think of the Chow group as graded by codimension. (For singular varieties, this notation would conflict established conventions.)

We will not define the intersection product making the Chow group into a commutative, graded ring for now. We assume its existence and might come back to the proof in Fulton's book [FIT]. For its defining propert, we need the notion of transverse intersection.

Definition. Two irreducible subvarieties A and B of an irreducible variety X intersect **transversely** at a point $p \in X$ if A, B, and X are smooth at p and $T_pA + T_pB = T_pX$. They are **generically transverse** if they intersect transversely at a general point of each irreducible component C of $A \cap B$.

Two cycles $A = \sum m_i A_i$ and $B = \sum n_j B_j$ are **generically transverse** if each A_i is generically transverse to each B_j.

The condition $T_pA + T_pB = T_pX$ is equivalent to $\operatorname{codim}(T_pA \cap T_pB) = \operatorname{codim}(T_pA) + \operatorname{codim}(T_pB)$ and is an open condition on p.

3.1.4 Theorem. *If X is a smooth quasi-projective variety, then there is a unique product structure on $A(X)$ satisfying the following conditions.*

If two subvarieties A and B of X are generically transverse, then $[A] \cdot [B] = [A \cap B]$. This structure makes $A(X)$ into an associative, commutative ring graded by codimension $\bigoplus_{c=0}^{\dim(X)} A^c(X)$.

Definition. The ring $(A(X), +, \cdot)$ on the Chow group with the intersection product $\cdot$ in Theorem 3.1.4 is called the **Chow ring** of X. The degree map $\deg\colon A(X) \to \mathbb{Z}$ is the homomorphism of abelian groups defined by taking any class of a closed point to 1 and $A_k(X)$ to 0 for every $k > 0$.

Chapter 1 contains some examples and in particular a discussion of how the Chow ring relates to Bézout's Theorem.

3.2. Computing the Chow Ring

We will take a look at a few methods that can be used to determine the Chow ring of some examples. Rational equivalence can be hard to understand...

Let's go by codimension first.

3.2.1 Proposition.
(1) *If X is an irreducible algebraic variety, then $A^0(X) \cong \mathbb{Z}$ and generated by the **fundamental class** $[X] \in A^0(X)$.*
(2) *If X has irreducible components $X_1, \dots, X_r$, then the classes $[X_i]$ generate a free abelian subgroup of $A(X)$ of rank r.*

Proof. The first claim is clear. The second follows from the fact that $\mathbb{P}^1 \times X$ has irreducible components $\mathbb{P}^1 \times X_i$ so that any irreducible variety $\Phi \subset \mathbb{P}^1 \times X$ is contained in one of the varieties $\mathbb{P}^1 \times X_i$. ∎

3.2.2 Example. The Chow group of a set of points $X = \{p_1, \dots, p_r\}$ is the free abelian group on its irreducible components, so $\mathbb{Z}^r = \mathbb{Z}p_1 \oplus \dots \oplus \mathbb{Z}p_r$. (More generally, the Chow group of a 0-dimensional scheme is the same as the Chow group of the underlying variety.)

Going up in codimension by 1, we find the divisor class group, which is the same as the Picard group on a smooth irreducible variety. See the discussion above, and in particular Exercise 3.1.2, for an argument.

3.2.3 Proposition. *If X is irreducible of dimension n, then $A_{n-1}(X) = A^1(X)$ is equal to the divisor class group of X.* ∎

3.2.1 Mayer-Vietoris and excision

3.2.4 Proposition. *Let X be an algebraic variety.*

(1) *(Mayer-Vietoris) If X_1 and X_2 are closed subvarieties of X, then there is an exact sequence of $\mathbb{Z}$-modules*

$$A(X_1 \cap X_2) \to A(X_1) \oplus A(X_2) \to A(X_1 \cup X_2) \to 0.$$

(2) *(Excision) If $Y \subset X$ is a closed subvariety and $U = X \setminus Y$ its complement, then the inclusion and restriction of maps of cycles give an exact sequence of $\mathbb{Z}$-modules*

$$A(Y) \to A(X) \to A(U) \to 0.$$

If X is smooth, then the map $A(X) \to A(U)$ is a ring homomorphism.

Let us clarify the maps. If $Y \subset X$ is a closed subvariety, then cycles on $\mathbb{P}^1 \times Y$ are also cycles on $\mathbb{P}^1 \times X$, which induces a map $\mathrm{Rat}(Y) \to \mathrm{Rat}(X)$. So we get a well defined map $A(Y) \to A(X)$ induced by the inclusion $Y \to X$. The intersection of a subvariety of X with the open subset $U = X \setminus Y$ is a (possibly empty) subvariety of U, which gives a restriction map $Z(X) \to Z(U)$. Since we can also restrict from $\mathbb{P}^1 \times X$ to $\mathbb{P}^1 \times U$, this map sends $\mathrm{Rat}(X)$ to 0 in $A(U)$ and we get a group homomorphism $A(X) \to A(U)$.

Proof of Proposition 3.2.4. For any variety X, there is an exact sequence of $\mathbb{Z}$-modules

$$Z(\mathbb{P}^1 \times X) \to Z(X) \to A(X) \to 0$$

by definition of the Chow group $A(X)$. Here, the left-hand map, which we denote by ∂_X, takes an irreducible variety $\Phi \subset \mathbb{P}^1 \times X$ to 0 if Φ is contained in a fiber; otherwise, it takes Φ to $\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle$ for $t_0 = [0,1]$ and $t_1 = [1,0]$ in $\mathbb{P}^1$. We begin with claim (2), excision. Consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & Z(\mathbb{P}^1 \times Y) \arrow[r] \arrow[d, "\partial_Y"'] & Z(\mathbb{P}^1 \times X) \arrow[r] \arrow[d, "\partial_X"'] & Z(\mathbb{P}^1 \times U) \arrow[r] \arrow[d, "\partial_U"'] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & Z(U) \arrow[r] \arrow[d] & 0 \\
 & A(Y) \arrow[r] \arrow[d] & A(X) \arrow[r] \arrow[d] & A(U) \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The two rows in the middle are evidently exact because a cycle $[A]$ for an irreducible variety $A \subset X$ gets mapped to $[A \cap U] = 0$ in $Z(U)$ if and only if $A \subset Y$. The three columns are exact by definition of the Chow group. A diagram chase shows that the map $A(X) \to A(U)$ is surjective with kernel equal to $A(Y)$ (rather the image of $A(Y)$ in $A(X)$). If X is smooth, the restriction $[A] \mapsto [A \cap U]$ from $A(X) \to A(U)$ is a ring homomorphism by Theorem 3.1.4.

To prove the claim (1), set $Y = X_1 \cap X_2$ and assume that $X = X_1 \cup X_2$. We then have a map $Z(Y) \to Z(X_1) \oplus Z(X_2)$ taking $[A]$ to $([A], -[A])$ where we think of an irreducible subvariety $A \subset Y$ as a subvariety of X_i by inclusion $Y \to X_i$. Taking the sum as a map from $Z(X_1) \oplus Z(X_2) \to Z(X)$ sending (A, B) to $A + B$ with respect to the inclusions $X_i \to X$, we get a right exact sequence $Z(Y) \to Z(X_1) \oplus Z(X_2) \to Z(X) \to 0$. We take the analogous maps $Z(\mathbb{P}^1 \times Y) \to Z(\mathbb{P}^1 \times X_1) \oplus Z(\mathbb{P}^1 \times X_2)$ and $Z(\mathbb{P}^1 \times X_1) \oplus Z(\mathbb{P}^1 \times X_2) \to Z(\mathbb{P}^1 \times X)$. These fit together to make the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & Z(\mathbb{P}^1 \times Y) \arrow[r] \arrow[d, "\partial_Y"'] & Z(\mathbb{P}^1 \times X_1) \oplus Z(\mathbb{P}^1 \times X_2) \arrow[r] \arrow[d, "\partial_{X_1} \oplus \partial_{X_2}"'] & Z(\mathbb{P}^1 \times X) \arrow[r] \arrow[d, "\partial_X"'] & 0 \\
0 \arrow[r] & Z(Y) \arrow[r] \arrow[d] & Z(X_1) \oplus Z(X_2) \arrow[r] \arrow[d] & Z(X) \arrow[r] \arrow[d] & 0 \\
& A(Y) \arrow[r] & A(X_1) \oplus A(X_2) \arrow[r] & A(X) &
\end{tikzcd}$$

Again, a diagram chase shows that the bottom row is exact and the right-hand map is surjective, giving the exact Mayer-Vietoris sequence in claim (1). ■

3.2.2 Affine space

We can stratify nice varieties into affine varieties and use the exact sequences in Proposition 3.2.4 to describe their Chow group. To that end, let us take a look at affine space.

3.2.5 Proposition. *The Chow group of $\mathbb{A}^n$ is isomorphic to $\mathbb{Z}$ and generated by the fundamental class meaning $A(\mathbb{A}^n) = \mathbb{Z}[\mathbb{A}^n]$.*

Proof. We already know that $A_0(\mathbb{A}^n) = \mathbb{Z}[\mathbb{A}^n]$. We show that every irreducible proper subvariety $Y \subset \mathbb{A}^n$ is rationally equivalent to 0. Choose coordinates $x_1, \ldots, x_n$ on $\mathbb{A}^n$ such that the origin does not lie in Y. Define

$$W^o = \{(t, tx) \in \left(\mathbb{A}^1 \setminus \{0\}\right) \times \mathbb{A}^n : x \in Y\}.$$

The fiber of W^o over a point $t \in \mathbb{A}^1 \setminus \{0\}$ under the projection to the first factor is $t \cdot Y$. Let $W \subset \mathbb{P}^1 \times \mathbb{A}^n$ be the closure of W^o in $\mathbb{P}^1 \times \mathbb{A}^n$, which is irreducible since it is the image of the irreducible variety $(\mathbb{A}^1 \setminus \{0\}) \times Y$ under the morphism $(t, x) \mapsto (t, tx)$. The fiber of W over $t_1 = 1 \in \mathbb{A}^1 \setminus \{0\}$ (with respect to the projection to $\mathbb{P}^1$) is Y. To show that the fiber of W over $t_0 = \infty \in \mathbb{P}^1$ is empty, pick a polynomial g in the vanishing ideal of Y that has a nonzero constant coefficient c. The rational function $G(t, x) = g(x/t)$ on $(\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^n$ extends to a regular function on $(\mathbb{P}^1 \setminus \{0\}) \times \mathbb{A}^n$ with constant value c on the fiber $\{\infty\} \times \mathbb{A}^n$. So the fiber of W over $\{\infty\}$ is empty and Y is rationally equivalent to 0 as claimed. ■

Exercise 3.2.6. Show that the fiber of W as in the proof of the previous proposition over 0 is the cone with vertex $0 \in \mathbb{A}^n$ over the intersection $\overline{Y} \cap H_\infty$, where $\overline{Y}$ is the projective closure of Y in $\mathbb{P}^n$ and $H_\infty = \mathbb{P}^n \setminus \mathbb{A}^n$ the hyperplane at infinity.

Using excision, this result immediately gives the following statement.

3.2.7 Corollary. *If $U \subset \mathbb{A}^n$ is a nonempty open set, then $A(U) = A^0(U) = \mathbb{Z}[U]$.*

3.2.3 Projective Space

Using the Chow group of $\mathbb{A}^n$ as a building block, we can compute the Chow group of $\mathbb{P}^n$ by its affine stratification built inductively from $\mathbb{P}^n = \mathbb{A}^n \cup H_\infty$ using that H_∞ is $\mathbb{P}^{n-1}$.

3.2.8 Theorem. *The Chow group of $\mathbb{P}^n$ is isomorphic to $\mathbb{Z}[s]/(s^{n+1})$ as a graded $\mathbb{Z}$-module (where the polynomial ring is graded by degree as usual). The intersection product is given by $s^i \cdot s^j = s^{i+j}$ for all $i, j \geq 0$.*

Proof. Write $\mathbb{P}^n = D_+(x_0) \cup \mathcal{V}_+(x_0)$, which is the beginning of the stratification $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$ into affine pieces. Setting $Y = \mathcal{V}_+(x_0)$ and $U = D_+(x_0)$, excision in Proposition 3.2.4(2) tells us that $A(\mathbb{P}^n)$ surjects onto $A(U) = \mathbb{Z}[\mathbb{A}^n]$ with kernel $A(\mathbb{P}^{n-1})$. By induction and using that the Chow group $A(\mathbb{P}^n)$ is graded by dimension, it follows that $A(\mathbb{P}^n)$ is a free abelian group on $n+1$ generators $[\overline{\mathbb{A}^0}], \ldots, [\overline{\mathbb{A}^n}]$.

Assuming the existence of the intersection product, it is determined by basic linear algebra. Let ζ be the class of a hyperplane, which generates $A^1(\mathbb{P}^n)$. Since any k generic hyperplanes in $\mathbb{P}^n$ intersect transversely in a linear space of codimension k, the class ζ^k is represented by $[\mathbb{P}^{n-k}]$, a general $(n-k)$-plane in $\mathbb{P}^n$. Sending ζ to s gives the isomorphism of the intersection ring $A(\mathbb{P}^n)$ with $\mathbb{Z}[s]/(s^{n+1})$. ∎

3.2.9 Corollary. *If $X \subset \mathbb{P}^n$ is an irreducible variety of dimension m and degree d, then the Chow groups in dimension at least m are $A_m(\mathbb{P}^n \setminus X) \cong \mathbb{Z}/(d)$ and $A_{m'}(\mathbb{P}^n \setminus X) = \mathbb{Z}$ for $m' > m$.*

Proof. By excision Proposition 3.2.4(2), there is an exact sequence $A(X) \longrightarrow A(\mathbb{P}^n) \longrightarrow A(\mathbb{P}^n \setminus X) \longrightarrow 0$. Since $A_m(X) \cong \mathbb{Z}$ is generated by the fundamental class $[X]$ of X, which is mapped to $d[\mathbb{P}^m]$ in $\mathbb{A}(\mathbb{P}^n)$, we get the first claim $A_m(\mathbb{P}^n \setminus X) \cong \mathbb{Z}/(d)$. The second claim follows by the same exact sequence using the fact that $A_{m'}(X) = 0$ for $m' > \dim(X) = m$. ∎

3.2.10 Corollary (Bézout's Theorem). *Let $X_1, \ldots, X_k$ be irreducible subvarieties of $\mathbb{P}^n$ of codimension c_i with $\sum_{i=1}^k c_i \leq n$ and assume that the X_i intersect generically transversely. Then*

$$\deg(X_1 \cap X_2 \cap \ldots \cap X_k) = \prod_{i=1}^{k} \deg(X_i).$$

Here, we define the degree of a class $[A] \in A_k(X)$ to be $\deg([A] \cdot [\mathbb{P}^{n-k}])$ for any $k > 0$.

Exercise 3.2.11. Use excision to compute the Chow groups of Segre varieties $\mathbb{P}^m \times \mathbb{P}^n$ (or also with arbitrary number of factors, not just two). Can you explain the intersection product (again assuming its existence) by fnding transverse intersections of cycles? Use the intersection ring to compute the degree of the Segre variety $\mathbb{P}^m \times \mathbb{P}^n$ in its Segre embedding.

3.2.4 Affine stratifications

The point of this section is to formalize the idea that we used to compute the Chow group of $\mathbb{P}^n$ in Theorem 3.2.8.

Definition. We say that X is **stratified** by subvarieties U_i if there are finitely many U_i, each U_i is irreducible and locally closed in X, X is a disjoint union of the U_i, and the closure of any U_i is a union of U_j. The sets U_i are called the **(open) strata** of the stratification, while their closures $\overline{U_i}$ are called the **closed strata**. A stratification with strata U_i is **affine** if each open stratum is isomorphic to some $\mathbb{A}^k$. It is called **quasi-affine** if each U_i is isomorphic to an open subset of some $\mathbb{A}^k$.

3.2.12 Remark. The open strata can be recovered from the closed strata since

$$U_i = Y_i \setminus \left(\bigcup_{Y_j \subsetneq Y_i} Y_j \right).$$

3.2.13 Proposition. *Let $\{U_i\}$ be a quasiaffine stratification of an algebraic variety X. Then $A(X)$ is generated by the classes of the closed strata.*

Proof. We use induction on the number of strata. If there is only one stratum, then the claim is exactly Corollary 3.2.7.

Now pick a minimal stratum U_0 (with respect to in dimension). Since the closure of U_0 is a union of strata, it must already be closed. This implies that $U = X \setminus U_0$ is stratified by the strata other than U_0. By induction, $A(U)$ is generated by the closures of the strata other than U_0. Using again Corollary 3.2.7 for U_0 we get $A(U_0) = \mathbb{Z}[U_0]$. We use the exact sequence

$$A(U_0) \to A(X) \to A(U) \to 0$$

given by excision Proposition 3.2.4(2). The Chow group $A(U)$ is generated by the closed strata in U. These generators come from (the same) closed strata in X. So the exact sequence shows that $A(X)$ is also generated by the closed strata of the stratification. ■

It can happen that the classes of closed strata in a quasi-affine stratification of a variety X can be 0 in $A(X)$.

3.2.14 Example. We can stratify $\mathbb{A}^1 = (\mathbb{A}^1 \setminus \{p\}) \cup \{p\}$ for any point $p \in \mathbb{A}^1$. This is a quasiaffine stratification but the class of p in $A(X) = \mathbb{Z}[\mathbb{A}^1]$ is 0.

For affine stratifications, this cannot happen by the following general, recent result.

3.2.15 Theorem (Totaro, 2014). *The classes of the closed strata in an affine stratification of a variety X form a basis of $A(X)$.*

Chapter 4

Grassmannians

Let V be a finite-dimensional vector space over an algebraically closed field K. We write $\mathrm{Gr}(k, V)$ for the **Grassmannian** of k-dimensional linear subspaces of V. If $V = K^n$, then we simply write $\mathrm{Gr}(k, n)$. In the projective situation, we use black board notation: $\mathbb{G}(k, n)$ is the Grassmannian of k-dimensional projective subspaces of an n-dimensional projective space. The translation to the linear setup is simply $\mathbb{G}(k, n) = \mathrm{Gr}(k+1, n+1)$. The Grassmannians are projective varieties via the **Plücker embedding**: we map a linear subspace of V spanned by a basis $v_1, \ldots, v_k$ to $v_1 \wedge v_2 \wedge \ldots \wedge v_k \in \Lambda^k V$, which is, up to scaling, independent of the choice of the basis. The image of this map is exactly the variety of **decomposable elements** in $\Lambda^k V$, which is an algebraic variety invariant under scaling. This is the affine cone over the Grassmannian in the Plücker embedding. If $V = K^n$ and we write the vectors $v_1, \ldots, v_k$ as rows of a matrix A, then $v_1 \wedge \ldots \wedge v_k$ can be expanded in the basis $e_{i_1} \wedge \ldots \wedge e_{i_k}$ of $\Lambda^k K^n$ for $i_1 < i_2 < \ldots < i_k$ in terms of the $k \times k$-minors of A. The coefficient of $e_{i_1} \wedge \ldots \wedge e_{i_k}$ is the $k \times k$ minor of A with columns $i_1, \ldots, i_k$.

4.1. Bundles on the Grassmannian

Similar to the tautological bundle on projective space, there is a similar bundle on a Grassmannian called the **universal subbundle**. This can be used to construct the **universal quotient bundle** and to describe the tangent bundle of Grassmannians. This is our first goal.

4.1.1 Universal subbundle

Definition. Let $\pi \colon E \to B$ be a vector bundle of rank r. A **subbundle** of E is a closed set $E' \subset E$ such that $\pi^{-1}(x) \cap E'$ is a vector subspace of $\pi^{-1}(x)$ for every $x \in B$ and $\pi|_{E'} \colon E' \to B$ is a vector bundle of some rank k.

We fix a K-vector space V of dimension $n \in \mathbb{N}$ and a $k \in \mathbb{N}_0$. Let $X = \mathrm{Gr}(k, V) \times V$ be the trivial bundle of rank n over $\mathrm{Gr}(k, V)$ whose fiber over every point is V. The total space of the **universal subbundle** $\mathcal{S}$ is, as a subset of X, the incidence correspondence

$$\mathcal{S} = \{([\Lambda], x) \in \mathrm{Gr}(k, V) \times V \colon x \in \Lambda\}$$

consisting of pairs of k-dimensional subspaces $\Lambda \subset V$ and points x in Λ. The bundle morphism of $\mathcal{S}$ is, of course, the projection to the first factor.

Exercise 4.1.1. Let V be an $(n+1)$-dimensional K-vector space. Show that the **universal k-plane** $\Phi = \{([\Lambda], x) \colon x \in \Lambda\} \subset \mathbb{G}(k, \mathbb{P}(V)) \times \mathbb{P}(V)$ is a closed subvariety of $\mathbb{G}(k, \mathbb{P}(V)) \times \mathbb{P}(V)$ of dimension $k + (k+1)(n-k)$ that is defined by bilinear forms on $\mathbb{P}(\Lambda^{k+1} V) \times \mathbb{P}(V)$.

4.1.2 Example. For $k = 1$, we have $\mathrm{Gr}(1, V) = \mathbb{P}(V)$ and the universal subbundle is the tautological bundle $\mathcal{O}_{\mathbb{P}(V)}(-1)$ on $\mathbb{P}(V)$ as discussed in Example 2.2.1.

The universal subbundle is indeed a vector bundle of rank k on $\mathrm{Gr}(k, V)$. To show this, we will locally trivialize the bundle. We recall affine charts on the Grassmannian in a more abstract setup than last term.

4.1.3 Construction. For any subspace $\Gamma \subset V$ of dimension $n-k$ there is an affine chart $U_\Gamma \subset \mathrm{Gr}(k, V)$ of all k-dimensional subspaces that are complementary to Γ; that is $U_\Gamma = \{[\Lambda] \in \mathrm{Gr}(k, V) \colon \Lambda \cap \Gamma = \{0\}\}$. This is an open subset of $\mathrm{Gr}(k, V)$. Indeed, set $\eta = w_1 \wedge w_2 \wedge \ldots \wedge w_{n-k}$ for a basis $w_1, \ldots, w_{n-k}$ of Γ so that $U_\Gamma = \{[\omega] \in \mathrm{Gr}(k, V) \colon \omega \wedge \eta \neq 0\}$ is the complement of a hyperplane section of $\mathrm{Gr}(k, V)$. To see abstractly that U_Γ is isomorphic to an affine space $\mathbb{A}^{k(n-k)}$ of dimension $k(n-k)$, fix $[\Omega] \in U_\Gamma$ so that $V = \Gamma \oplus \Omega$. Let $\pi_\Gamma \colon V \longrightarrow \Gamma$ and $\pi_\Omega \colon V \longrightarrow \Omega$ be the quotient maps. A k-dimensional subspace $\Lambda \in V$ that is complementary to Γ is isomorphic to Ω via π_Ω. Write $\pi_\Omega^{-1} \colon \Omega \longrightarrow \Lambda \subset V$ for the inverse and define the linear map $\varphi \colon \Omega \longrightarrow \Gamma$ by the composition $\pi_\Gamma \circ \pi_\Omega^{-1}$. The subspace Λ is then the graph of φ in $V = \Omega \oplus \Gamma$. Conversely every linear map from Ω to Γ gives rise to a k-dimensional subspace of V, namely its graph. This establishes a bijection of U_Γ with $\mathrm{Hom}(\Omega, \Gamma) \cong \mathbb{A}^{k(n-k)}$. To see that this is an isomorphism of algebraic varieties, we have to choose coordinates, see Exercise 4.1.4. This exercise shows that the entry $a_{i,j}$ of the matrix A is a regular function on U_Γ making the bijection $U_\Gamma \cong \mathbb{A}^{k(n-k)}$ an isomorphism of affine algebraic varieties.

Exercise 4.1.4. Explicitly show that $U_\Gamma \cong \mathbb{A}^{k(n-k)}$ for the choices of subspaces $\Gamma = \mathrm{span}\{e_{k+1}, \ldots, e_n\}$ and $\Omega = \mathrm{span}\{e_1, \ldots, e_k\}$ in K^n. The concrete version of the abstract construction above is that U_Γ is the set of k-dimensional subspaces of K^n that are the row spaces of matrices of the form

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & a_{1,1} & a_{1,2} & \ldots & a_{1,n-k} \\ 0 & 1 & \ldots & 0 & a_{2,1} & a_{2,2} & \ldots & a_{2,n-k} \\ \vdots & & \ddots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \ldots & 1 & a_{k,1} & a_{k,2} & \ldots & a_{k,n-k} \end{pmatrix}$$

because the rowspan of B corresponds to the linear map $\varphi \colon \Omega \longrightarrow \Gamma$ given by the transpose of the matrix $A = (a_{ij})$.

4.1.5 Proposition. *The subset $\mathcal{S}$ of X whose fiber over a point $[U] \in \mathrm{Gr}(k, V)$ is the subspace $U \subset V$ is a vector bundle of rank k over $\mathrm{Gr}(k, V)$.*

Proof. We first show that $\mathcal{S}$ is an algebraic subset of the trivial bundle $X = \mathrm{Gr}(k, V) \times V$ and then give local trivializations of $\mathcal{S}$.

In the Plücker embedding, $[U]$ is given by $\eta = v_1 \wedge v_2 \wedge \ldots \wedge v_k \in \Lambda^k V$ for a basis $v_1, \ldots, v_k$ of U. So an element $x \in V$ is in U if and only if $\eta \wedge x \in \Lambda^{k+1} V$ is 0. This is an algebraic condition. Explicitly, let A be the $(k+1) \times n$ matrix whose rows are $v_1, \ldots, v_k$ and x (in some basis of V). Then $\eta \wedge x = 0$ is equivalent to the vanishing of all $(k+1) \times (k+1)$-minors of A.

On $X = \mathrm{Gr}(k, V) \times V$, these minors are bilinear equations in the Plücker coordinates of U and the coordinates of x. Hence they are well-defined and $\mathcal{S}$ is a closed subset of X.

On the usual affine charts as in Construction 4.1.3, the local trivialization of $\mathcal{S}$ is simply given by the projection π_Ω, namely as $\pi_1^{-1}(U_\Gamma) \to U_\Gamma \times \Omega$, $([\Lambda], x) \mapsto ([\Lambda], \pi_\Omega(x))$. ■

Exercise 4.1.6. What are the transition functions of the universal subbundle of $\mathrm{Gr}(k, V)$?

4.1.2 Universal quotient bundle

The universal subbundle $\mathcal{S}$ on $\mathrm{Gr}(k, V)$ is a subbundle of the trivial bundle $\mathrm{Gr}(k, V) \times V$ on $\mathrm{Gr}(k, V)$ and we can construct the quotient bundle $\mathcal{Q}$ of the trivial one modulo the universal subbundle, which gives the **universal quotient bundle** $\mathcal{Q}$ on $\mathrm{Gr}(k, V)$. The fiber over $[\Lambda] \in \mathrm{Gr}(k, V)$ is simply V/Λ, which is a $(n-k)$-dimensional vector space so that $\mathcal{Q}$ is a vector bundle of rank $n-k$ on $\mathrm{Gr}(k, V)$.

As a locally free sheaf of $\mathcal{O}_{\mathrm{Gr}(k,V)}$-modules, the quotient bundle $\mathcal{Q}$ is given by the exact sequence

$$0 \to \mathcal{S} \to \mathcal{O}^{\dim(V)} \cong \mathcal{O} \otimes_K V \to \mathcal{Q} \to 0,$$

where we write $\mathcal{O}$ for the structure sheaf $\mathcal{O}_{\mathrm{Gr}(k,V)}$ of the Grassmannian. For the tensor product $\mathcal{O} \otimes_K V$, we abused notation and wrote V for the constant sheaf with stalks V (as K-vector spaces) so that $\mathcal{O} \otimes_K V$ is the sheaf of sections of the trivial vector bundle. The point is that the sheaf defined by this exact sequence is locally free of rank $\dim(V) - k$ and hence the sheaf of sections of a vector bundle on $\mathrm{Gr}(k, V)$. There is an open cover of $\{U_\Gamma\}$ of $\mathrm{Gr}(k, V)$ so that $\mathcal{S}$ is locally free using the local trivialization $U_\Gamma \times \Omega$. A section $s\colon U_\Gamma \to \Omega$ of $\mathcal{S}$ on this local trivialization maps to $\Lambda \subset V$ by $\pi_\Omega^{-1} \circ s$ with the isomorphism $\pi_\Omega|_\Lambda\colon \Lambda \to \Omega$. The composition of $\pi_\Omega^{-1} \circ s$ with the projection map $V \to V/\Lambda$ is a section of $\mathcal{Q}$ over U_Γ. A section $s\colon U_\Gamma \to V$ of the trivial bundle $\mathcal{O} \otimes V$ maps to 0 in $\mathcal{Q}$ if it satisfies $s([\Lambda]) \in \Lambda$ for all $[\Lambda] \in U_\Gamma$, which is the condition for being a section of $\mathcal{S}$. We give local trivializations of $\mathcal{Q}$ below in Proposition 4.1.8.

4.1.7 Example. If $k = \dim(V) - 1$ so that $\mathrm{Gr}(k, V) = \mathbb{P}(V^*)$, the universal quotient bundle $\mathcal{Q}$ is isomorphic to $\mathcal{O}_{\mathbb{P}(V^*)}(1)$. The above exact sequence is

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathbb{P}(V^*)} \otimes V \to \mathcal{Q} \to 0$$

and $\mathcal{Q}$ is a line bundle so that $\mathcal{Q} = \mathcal{O}(d)$ for some $d \in \mathbb{Z}$ (see Example 2.4.19). Let's determine d.

Let $V = K^{n+1}$ and choose local trivializations of $\mathcal{S}$ with $\Gamma_i = \mathrm{span}\{e_i\}$ for $i = 0, 1$ and $\Omega_0 = \mathrm{span}\{e_1, e_2, \ldots, e_n\}$ as well as $\Omega_1 = \mathrm{span}\{e_0, e_2, \ldots, e_n\}$. A subspace $[\Lambda] \in U_{\Gamma_0} \cap U_{\Gamma_1}$ is represented by two maps $\varphi_i\colon \Omega_i \to \Gamma_i$. The graph of φ_0 representing a linear subspace Λ in $\Gamma_0 \oplus \Omega_0$ is the row span of a $n \times (n+1)$ matrix of the form

$$A = \begin{pmatrix} x_1 & -1 & 0 & \ldots & 0 \\ x_2 & 0 & -1 & & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ x_n & 0 & 0 & & -1 \end{pmatrix}$$

meaning that $\varphi_0(-e_j) = x_j e_0$ for $j = 1, \ldots, n$. A normal vector of this hyperplane is the vector $(1, x_1, x_2, \ldots, x_n) \in (K^{n+1})^*$ in the dual basis. In the affine chart U_{Γ_1}, the subspace is represented by $\varphi_1\colon \Omega_1 \to \Gamma_1$, say $\varphi_1(-e_j) = y_j e_1$ for $j = 0, 2, \ldots, n$, which corresponds to the matrix

$$B = \begin{pmatrix} -1 & y_0 & 0 & \ldots & 0 \\ 0 & y_2 & -1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & y_n & 0 & & -1 \end{pmatrix}.$$

Here, a normal vector is $(y_0, 1, y_2, \ldots, y_n)$. We know that $y_0 \neq 0$ because $\Lambda \cap \Gamma_0 = \{0\}$. Since they represent the same hyperplane, the two normal vectors are equal up to scaling so that $(1, x_1, x_2, \ldots, x_n) = x_1/y_0(y_0, 1, y_2, , \ldots, y_n)$. Since this is the transition function of $\mathcal{O}_{\mathbb{P}(V^*)}(1)$, the bundle $\mathcal{Q}$ is isomorphic to $\mathcal{O}_{\mathbb{P}(V^*)}(1)$ as claimed.

4.1.8 Proposition. *The universal quotient bundle $\mathcal{Q}$ is a vector bundle of rank* $\dim(V) - k$ *on* $\mathrm{Gr}(k, V)$.

Proof. We locally trivialize $\mathcal{Q}$ over the open sets $U_\Gamma \subset \mathrm{Gr}(k, V)$ as $U_\Gamma \times \Gamma$. For $[\Lambda] \in U_\Gamma$ represented by $\varphi\colon \Omega \to \Gamma$ (for fixed $\Omega \subset V$ with $\Omega \cap \Gamma = \{0\}$), the isomorphism $V/\Lambda \to \Gamma$ is given by mapping $(x, y) \in \Omega \oplus \Gamma$ to $y - \varphi(x) \in \Gamma$; this linear map induces a linear isomorphism $V/\Lambda \to \Gamma$ because Λ is the kernel of that linear map. Now let Γ_1 and Γ_2 be two subspaces of V of dimension $n-k$ and pick a subspace $\Omega \subset V$ that is complementary to both (so that $\Omega \cap \Gamma_i = \{0\}$ for $i = 1, 2$). The transition function of $\mathcal{Q}$ from $U_{\Gamma_1} \times \Gamma_1 \to U_{\Gamma_2} \times \Gamma_2$ then comes from the isomorphisms $\alpha_i\colon V/\Omega \to \Gamma_i$ ($i = 1, 2$) by mapping $([\Lambda], v) \in U_{\Gamma_1} \times \Gamma_1$ to $([\Lambda], \alpha_2 \circ \alpha_1^{-1}(v))$, which indeed lands in $U_{\Gamma_2} \times \Gamma_2$. ∎

Exercise 4.1.9. What are the transition functions of the bundle $\mathcal{Q}$ in coordinates given by choosing bases of Ω and Γ on $\mathrm{Hom}(\Omega, \Gamma) \cong \mathbb{A}^{k(n-k)}$?

4.1.3 Tangent bundle

In terms of these two bundles $\mathcal{S}$ and $\mathcal{Q}$ on a Grassmannian $\mathrm{Gr}(k, V)$, we can describe the tangent bundle. Let us first look at a point. In a local chart U_Γ of $\mathrm{Gr}(k, V)$, the point Ω corresponds to $0 \in \mathrm{Hom}(\Omega, \Gamma)$ (see Construction 4.1.3 for notation). The vector space $\mathrm{Hom}(\Omega, \Gamma)$ is $\mathbb{A}^{k(n-k)}$ with $n = \dim(V)$. The tangent space to 0 in $\mathbb{A}^m$ is isomorphic to $\mathbb{A}^m$. So point-wise, using $\Gamma = V/\Omega$, we have $T_{[\Lambda]} \mathrm{Gr}(k, V) = \mathrm{Hom}(\Lambda, V/\Lambda)$. Globally, this suggests that $\mathcal{T}_{\mathrm{Gr}(k,V)} = \mathcal{H}\mathrm{om}(\mathcal{S}, \mathcal{Q})$ as locally free $\mathcal{O}_{\mathrm{Gr}(k,V)}$-modules.

4.1.10 Theorem. *The tangent bundle $\mathcal{T}_{\mathrm{Gr}(k,V)}$ of the Grassmannian is isomorphic to the vector bundle $\mathcal{H}\mathrm{om}(\mathcal{S}, \mathcal{Q})$, where $\mathcal{S}$ is the universal subbundle and $\mathcal{Q}$ the universal quotient bundle of* $\mathrm{Gr}(k, V)$.

Proof. We have just seen that we have this isomorphism on every fiber so that we have to check that these local isomorphisms extend to an isomorphism of bundles. Let $U = U_\Gamma \cap U_{\Gamma'}$ be an open subset of the Grassmannian with the notation in Construction 4.1.3 and pick a point $[\Omega] \in U$. Then U is the open subset of linear maps in $\mathrm{Hom}(\Omega, \Gamma)$ whose graphs meet Γ' trivially. As a subset of $\mathrm{Hom}(\Omega, \Gamma')$, this is the set of linear maps whose graphs meet Γ trivially. The two representations of a subspace in U are related via the isomorphisms $\alpha\colon \Gamma \to V/\Omega \leftarrow \Gamma'\colon \beta$ by mapping $\varphi \in \mathrm{Hom}(\Omega, \Gamma)$ to $\beta^{-1} \circ \alpha \circ \varphi \in \mathrm{Hom}(\Omega, \Gamma')$. The vector space $\mathrm{Hom}(\Omega, \Gamma)$ is the tangent space to U_Γ at Ω and the same for Γ'. We now argue that the transition functions of the tangent bundle and the bundle $\mathcal{H}\mathrm{om}(\mathcal{S}, \mathcal{Q})$ are the same showing that $\mathcal{T}_{\mathrm{Gr}(k,V)} \cong \mathcal{H}\mathrm{om}(\mathcal{S}, \mathcal{Q})$.

For the tangent bundle, the transition function from $(\mathcal{T}_{\mathrm{Gr}(k,V)}|_{U_\Gamma})|_{U_{\Gamma'}}$ to $(\mathcal{T}_{\mathrm{Gr}(k,V)}|_{U_{\Gamma'}})|_{U_\Gamma}$ is the derivative of the transition function of the Grassmannian. The charts U_Γ and $U_{\Gamma'}$ are isomorphic to $\mathrm{Hom}(\Omega,\Gamma)$ and $\mathrm{Hom}(\Omega,\Gamma')$, respectively. The transition is given by $\beta^{-1}\circ\alpha$ with the maps above. Since this is linear, the differential of this map is again $\beta^{-1}\circ\alpha$ as a map from the tangent space $\mathrm{Hom}(\Omega,\Gamma) = T_{[\Omega]}U_\Gamma$ to $\mathrm{Hom}(\Omega,\Gamma') = T_{[\Omega]}U_{\Gamma'}$.

The bundle $\mathcal{H}\mathrm{om}(\mathcal{S},\mathcal{Q})$ has the same transition functions completing the proof. ∎

4.1.11 Example. For $\mathbb{P}^n = \mathrm{Gr}(1,K^{n+1})$, this theorem is essentially the **Euler sequence**

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)\otimes V \to \mathcal{T}_{\mathbb{P}^n} \to 0.$$

We have seen in Example 4.1.2 that the universal subbundle $\mathcal{S}$ on $\mathrm{Gr}(1,K^{n+1})$ is $\mathcal{O}_{\mathbb{P}^n}(-1)$ so that $\mathcal{T}_{\mathbb{P}^n} \cong \mathcal{H}\mathrm{om}(\mathcal{O}_{\mathbb{P}^n}(-1),\mathcal{Q}) \cong \mathcal{O}_{\mathbb{P}^n}(1)\otimes\mathcal{Q}$ by Theorem 4.1.10. We get the Euler sequence from

$$0 \to \mathcal{S} \to \mathcal{O}_{\mathbb{P}^n}\otimes V \to \mathcal{Q} \to 0$$

by tensoring this sequence with the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ using $\mathcal{O}(d)\otimes\mathcal{O}(1) = \mathcal{O}(d+1)$.

Locally, at a point, the tangent space $T_p\mathbb{P}^n$ can be identified with $(K^{n+1})/p$ by choosing a basis $p = K\cdot v$ of the line p. This isomorphism is the quotient map induced on $(K^{n+1})/p$ by the differential $dq_v\colon T_vU \to T_{[v]}\mathbb{P}^n$ of the quotient map $q\colon U = V\setminus\{0\} \to \mathbb{P}^n$, $x \mapsto [x]$. Since the quotient map is linear, the differential dq_v is also simply the quotient map. But how is can we identify T_vU with K^{n+1}? The Zariski tangent space to U at v is $(\mathfrak{m}_v/\mathfrak{m}_v^2)^*$, where $\mathfrak{m} = (x_0 - v_0,\dots,x_n - v_m)$ is the maximal ideal generated by the linear forms given by the coordinates of v. An element $w\in K^{n+1}$ is a linear form on $\mathfrak{m}_v$ by sending a polynomial f to $(D_wf)(v)$, the directional derivative of f in direction w at v. This isomorphism α_v of K^{n+1} with the Zariski tangent space $(\mathfrak{m}_v/\mathfrak{m}_v^2)^*$ depends on the choice of v: the isomorphism $\alpha_{\lambda v}$ of K^{n+1} with $T_{\lambda v}U$ is $w \mapsto (f \mapsto (D_wf)(\lambda v))$. So we get two surjections $\varphi_v = dq_v\circ\alpha_v\colon K^{n+1} \to K^{n+1}/K\cdot v$ as well as $\varphi_{\lambda v} = dq_{\lambda v}\circ\alpha_{\lambda v}\colon K^{n+1} \to K^{n+1}/K\cdot v$. We have $\lambda\varphi_v(u) = \varphi_{\lambda v}(u)$ and this map depends on the scaling of v even though $[v] = [\lambda v] \in \mathbb{P}^n$ for any non-zero λ. To get a well-defined map to $T_p\mathbb{P}^n$, independent of the scaling, we simply tensor with a 1-dimensional vector space: there is a natural isomorphism $\mathrm{span}\{v\}^*\otimes V/\mathrm{span}\{v\} \to T_{[v]}\mathbb{P}^n$ sending (ℓ,u) to $\ell(v)\varphi_v(u)$. A rescaling of v therefore does not affect this map. This isomorphism is the local version of the isomorphism of $\mathcal{O}_{\mathbb{P}^n}(1)\otimes\mathcal{Q}$ with $\mathcal{T}_{\mathbb{P}^n}$.

4.2. Lines in Projective 3-space

Our next goal is to find an affine stratification of $\mathrm{Gr}(k,V)$, which we will achieve in terms of Schubert varieties. We start with the case $k=2$ and $\dim(V)=4$ as a warm up. The construction relies on the choice of a complete flag, which is a nested and maximal sequence of vector subspaces. Since the general linear group of V acts transitively on the set of complete flags, it turns out that the rational equivalence classes of the Schubert varieties do not depend on the choice of the flag. We will give a reason for this below, see Theorem 4.3.1. The Schubert varieties themselves do depend on the flag.

4.2.1 Construction. Fix a **complete flag** $\mathcal{V}$ in $\mathbb{P}^3$, that is a point $p\in\mathbb{P}^3$, a line $L\subset\mathbb{P}^3$

containing p and a plane $H \subset \mathbb{P}^3$ containing L. Then we have the following **Schubert varieties**

$$\begin{aligned}
\Sigma_{0,0} &= \mathbb{G}(1,3) \\
\Sigma_{1,0} &= \{\Lambda \colon \Lambda \cap L \neq \emptyset\} \\
\Sigma_{2,0} &= \{\Lambda \colon p \in \Lambda\} \\
\Sigma_{1,1} &= \{\Lambda \colon \Lambda \subset H\} \\
\Sigma_{2,1} &= \{\Lambda \colon p \in \Lambda \subset H\} \\
\Sigma_{2,2} &= \{\Lambda \colon L = \Lambda\}
\end{aligned}$$

The indexing is chosen such that $\Sigma_{a,b}$ is the set of lines in $\mathbb{P}^3$ that intersect the $(2-a)$-dimensional projective space of the flag $\mathcal{V}$ in a point and the $(3-b)$-dimensional projective space in a line. With this indexing, $\Sigma_{a,b}$ has codimension $a+b$ in $\mathbb{G}(1,3)$. For simplicity, we will usually drop trailing zeroes in the index (e.g. write Σ_1 for $\Sigma_{1,0}$). If we want to refer to the fixed flag, we write $\Sigma_{a,b}(\mathcal{V})$. If the Schubert variety depends only on one subspace of the flag, we also sometimes write the subspace instead of the flag, e.g. $\Sigma_2(p)$ instead of $\Sigma_{2,0}(\mathcal{V})$.

4.2.2 Proposition. *The Schubert varieties $\Sigma_{a,b} \subset \mathbb{G}(1,3)$ defined above are irreducible.*

Proof. The variety $\Sigma_{1,0}$ is irreducible because it is the projection of the irreducible incidence correspondence $\{(L', p) \in \mathbb{G}(1,3) \times L \colon p \in L'\}$. This incidence correspondence remembers the intersection point of a line $L' \in \Sigma_{1,0}$ with L. Its fibers over $p \in L$ are isomorphic to $\mathbb{P}^2$ (thought of as the image of $\mathbb{P}^3$ under projection away from $p \in L$). The same argument works for $\Sigma_{2,1}$.

The variety $\Sigma_{2,0}$ is isomorphic to $\mathbb{P}^2$ by projection away from p. The variety $\Sigma_{1,1}$ is the dual projective plane of H. The last one, $\Sigma_{2,2}$ is a point. ∎

Exercise 4.2.3. Show that $\Sigma_{2,1}$ is irreducible by setting up the appropriate incidence correspondence.

Definition. A **Schubert cell** in $\mathbb{G}(1,3)$ is the quasi projective variety $\Sigma^o_{a,b}$ obtained from $\Sigma_{a,b}$ by removing all intersections of $\Sigma_{a,b}$ with the other Schubert varieties $\Sigma_{a',b'}$ in $\mathbb{G}(1,3)$ that are strictly contained in $\Sigma_{a,b}$.

The poset of Schubert varieties in $\mathbb{G}(1,3)$ is the following.

$$\begin{tikzcd}
 & & \Sigma_2 \arrow[dr] & & \\
\{L\} = \Sigma_{2,2} \arrow[r] & \Sigma_{2,1} \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[r] & \mathbb{G}(1,3) \\
 & & \Sigma_{1,1} \arrow[ur] & &
\end{tikzcd}$$

4.2.4 Example. The Schubert cell Σ^o_1 is $\Sigma_1 \setminus \left(\Sigma_2 \cup \Sigma_{1,1}\right)$. Concretely, Σ^o_1 is the following set of lines in $\mathbb{P}^3$.

$$\Sigma^o_1 = \{\Lambda \subset \mathbb{P}^3 \colon \Lambda \cap L \neq \emptyset \text{ but } p \notin \Lambda \text{ and } \Lambda \not\subset H\}$$

4.2.5 Proposition. *The Schubert cell Σ^o_1 is isomorphic to an affine space.*

Proof. Let H' be a plane in $\mathbb{P}^3$ containing p but not containing L so that $H' \cap L = \{p\}$. Any line $\Lambda \in \Sigma_1^o$ intersects L in a point that is not p and so it intersects H' in a unique point. Since any such line Λ is not contained in H, the intersection point of Λ with H does not lie in $H' \cap H$, which is a line in H. So we get two maps $\Sigma_1^o \to (L \setminus \{p\}) \cong \mathbb{A}^1, \Lambda \mapsto \Lambda \cap L$, and $\Sigma_1^o \to (H' \setminus (H' \cap H)) \cong \mathbb{A}^2$, $\Lambda \mapsto \Lambda \cap H'$. The line Λ is uniquely determined by these two intersection points because $\Lambda \cap L \subset H$, whereas $\Lambda \cap H' \not\subset H$. So the product of these two maps gives an isomorphism $\Sigma_1^o \to \mathbb{A}^1 \times \mathbb{A}^2 \cong \mathbb{A}^3$. ■

Exercise 4.2.6. Show that the map $\Sigma_1^o \to L \setminus \{p\}$ sending Λ to the intersection point $\Lambda \cap L$ is indeed a morphism of algebraic varieties.

Exercise 4.2.7. Show that $\Sigma_{2,1}^o$ is isomorphic to $\mathbb{A}^1$. Then show that the Schubert cells Σ_2^o and $\Sigma_{1,1}$ are isomorphic to $\mathbb{A}^2$ by a similar argument as above (for Σ_1^o).

So the Schubert varieties are the closed strata of an affine stratification of $\mathbb{G}(1,3)$. Proposition 3.2.13 shows that the Chow group of $\mathbb{G}(1,3)$ is generated by the classes of the Schubert varieties. Write $\sigma_{a,b}$ for the class of $\Sigma_{a,b}$ in $A(\mathbb{G}(1,3))$.

4.2.8 Remark. Since any two complete flags $\mathcal{V}$ and $\mathcal{V}'$ in $\mathbb{P}^n$ are related by a change of coordinates, the induced action of PGL_{n+1} on $\mathbb{G}(k,n)$ maps the Schubert cells and Schubert varieties with respect to the flags to each other. So the rational equivalence class of a Schubert variety does not depend on the choice of a complete flag in $\mathbb{P}^n$, see Theorem 4.3.1.

Now that we have generators of the Chow group of $\mathbb{G}(1,3)$ as a free abelian group, we can determine the intersection product, assuming its existence, because the Schubert varieties intersect transversely.

4.2.9 Theorem. *The six Schubert classes $\sigma_{a,b} \in A^{a+b}(\mathbb{G}(1,3))$ for $0 \le b \le a \le 2$ freely generate $A(\mathbb{G}(1,3))$ as a graded abelian group. The intersection product is given by*

$$\sigma_1^2 = \sigma_{1,1} + \sigma_2 \quad (A^1 \times A^1 \to A^2) \tag{4.2.10}$$

$$\sigma_1 \cdot \sigma_{1,1} = \sigma_1 \cdot \sigma_2 = \sigma_{2,1} \quad (A^1 \times A^2 \to A^3) \tag{4.2.11}$$

$$\sigma_1 \cdot \sigma_{2,1} = \sigma_{2,2} \quad (A^1 \times A^3 \to A^4) \tag{4.2.12}$$

$$\sigma_{1,1}^2 = \sigma_2^2 = \sigma_{2,2},\ \sigma_{1,1} \cdot \sigma_2 = 0 \quad (A^2 \times A^2 \to A^4). \tag{4.2.13}$$

Proof. Proposition 3.2.13 shows that the Schubert classes generate $A(\mathbb{G}(1,3))$. The formulae for the intersection show that the two generators of $A^2(\mathbb{G}(1,3))$ are independent so that freely generate $A(\mathbb{G}(1,3))$.

To prove the formulae for intersection, we will use that two Schubert varieties $\Sigma_{a,b}(\mathcal{V})$ and $\Sigma_{a',b'}(\mathcal{V}')$ for two generic complete flags $\mathcal{V}$ and $\mathcal{V}'$ intersect generically transversely (which we will prove later, at least in special cases). This is implied by general results, for example Kleiman's Theorem (see below, Theorem 4.3.1). For simplicity, write $\Sigma_{a,b}$ and $\Sigma'_{a',b'}$.

Let's start with intersections of Schubert cycles of complementary dimension, in which case we simply have to count the number of intersection points by generic transversality. To compute σ_2^2, we look at

$$\Sigma_2 \cap \Sigma_2' = \{\Lambda \in \mathbb{G}(1,3) \colon p \in \Lambda \text{ and } p' \in \Lambda\},$$

which consists only of the unique line spanned by p and p'. This means that $\sigma_2^2 = \sigma_{2,2}$ as claimed.

The intersection of $\Sigma_{1,1}$ and $\Sigma'_{1,1}$ consists of all lines Λ such that $\Lambda \subset H$ and $\Lambda \subset H'$, which means that $\Lambda = H \cap H'$ and $\sigma_{1,1}^2 = \sigma_{2,2}$.

The Schubert varieties Σ_2 and $\Sigma'_{1,1}$ are disjoint because $p \notin H'$. The intersection product of σ_2 and $\sigma_{1,1}$ is therefore 0.

Finally, $\Sigma_1 \cap \Sigma'_{2,1}$ contains all lines Λ such that $\Lambda \cap L \neq \emptyset$ and $p' \in \Lambda \subset H'$. Since L intersects H' in a unique point which is not p', there is a unique such line. So we have $\sigma_1 \cdot \sigma_{2,1} = \sigma_{2,2}$.

Next, we discuss the intersections that have codimension 1. The intersection $\Sigma_1 \cap \Sigma'_2$ is the set of lines Λ intersecting L and containing the point p', which is the set of lines spanned by p' and a point in L. With respect to a flag starting with p' and containing the plane spanned by p' and L, this is the Schubert variety $\Sigma_{2,1}$. So we get $\sigma_1 \cdot \sigma_2 = \sigma_{2,1}$. The intersection of Σ_1 with $\Sigma'_{1,1}$ is also $\Sigma_{2,1}$ with respect to a flag containing the point $L \cap H'$ and the plane H' so that $\sigma_1 \cdot \sigma_{1,1} = \sigma_{2,1}$.

The most interesting case is $\Sigma_1 \cap \Sigma'_1$ because this is not a Schubert variety. We know that σ_1^2 has codimension 2 so that it is unique a $\mathbb{Z}$-linear combination

$$\sigma_1^2 = \alpha\sigma_{1,1} + \beta\sigma_2.$$

We determine these coefficients by intersecting with classes of complementary dimension. (This is known as the method of undetermined coefficients.)

Assuming the existence of the intersection product, we have associativity so that

$$(\alpha\sigma_{1,1} + \beta\sigma_2)\sigma_2 = \sigma_1^2\sigma_2 = \sigma_1(\sigma_1\sigma_2) = \sigma_1\sigma_{2,1} = \sigma_{2,2}.$$

Multiplying out the left hand side gives $\alpha 0 + \beta\sigma_{2,2} = \beta\sigma_{2,2}$ (using $\sigma_{1,1} \cdot \sigma_2 = 0$ and $\sigma_2^2 = \sigma_{2,2}$). By comparing coefficients with the right-hand side of the above equation, this implies $\beta = 1$. A similar computation for $\sigma_1^2 \cdot \sigma_{1,1}$ gives $\alpha = 1$. In sum, we get $\sigma_1^2 = \sigma_{1,1} + \sigma_2$ as claimed.

This completes the description of the intersection product on $A(\mathbb{G}(1,3))$. ■

Assuming the existence of the intersection product, this result gives the following description of $A(\mathbb{G}(1,3))$.

4.2.14 Corollary. *The Chow ring $A(G(1,3))$ is isomorphic as a graded $\mathbb{Z}$-algebra to*

$$\mathbb{Z}[\sigma_1, \sigma_2]/(\sigma_1^3 - 2\sigma_1\sigma_2, \sigma_1^2\sigma_2 - \sigma_2^2).$$

Proof. The grading of $A = \mathbb{Z}[\sigma_1, \sigma_2]/(\sigma_1^3 - 2\sigma_1\sigma_2, \sigma_1^2\sigma_2 - \sigma_2^2)$ is given by assigning codimension 1 to σ_1 and codimension 2 to σ_2. With this grading, the two relations are homogeneous of degree 3 and 4, respectively. In the Chow ring, the relations $\sigma_1^2 = \sigma_{1,1} + \sigma_2$, $\sigma_1\sigma_2 = \sigma_{2,1}$, and $\sigma_1^2\sigma_2 = \sigma_{2,2}$ hold. So $\sigma_{1,1}$ is $\sigma_1^2 - \sigma_2$ in A and so forth. The two ways of writing $\sigma_{2,1}$ as $\sigma_1\sigma_2$ and $\sigma_1\sigma_{1,1}$ give rise to the first relation $\sigma_1^3 - 2\sigma_1\sigma_2 = 0$. The three ways of writing $\sigma_{2,2}$ as $\sigma_{1,1}^2$, $\sigma_1\sigma_{2,1}$, and σ_2^2 give rise to the second relation. ■

The quotient ring on the right-hand side of the above equation is a 1-dimensional $\mathbb{Z}$-algebra (Krull dimension). In this sense, the Chow ring of $A(\mathbb{G}(1,3))$ is a complete intersection. This fact generalizes to Chow rings of any Grassmannian.

Let us check that Schubert cycles indeed intersect transversely by determining the tangent space to Σ_2. The tangent spaces to other Schubert cycles $\Sigma_{a,b}$ have a similar description and imply transversality of the intersections in the above proof (Exercise 4.2.16).

4.2.15 Proposition. *Let $\Sigma = \Sigma_2(p)$ be the Schubert cycle of lines in $\mathbb{P}^3 = \mathbb{P}(V)$ containing p. For $[L] \in \Sigma$, write $\widehat{L} \subset V$ for the 2-dimensional subspace corresponding to $L \subset \mathbb{P}(V)$. Using the identification of $T_{[L]}\mathbb{G}(1,3)$ with $\mathrm{Hom}(\widehat{L} \to V/\widehat{L})$, we have*

$$T_{[L]}\Sigma = \{\varphi \in \mathrm{Hom}(\widehat{L}, V/\widehat{L}) \colon \varphi(p) = 0\}.$$

This proposition implies in particular that two Schubert cycles $\Sigma_2(p)$ and $\Sigma_2(p')$ for two distinct points $p, p' \in \mathbb{P}^3$ intersect transversely at the line $L = \langle p, p' \rangle$, which is their unique intersection point. Indeed,

$$T_{[L]}\Sigma(p) \cap T_{[L]}\Sigma(p') = \{\varphi \in \mathrm{Hom}(\widehat{L}, V/\widehat{L}) \colon \varphi(p) = 0 \text{ and } \varphi(p') = 0\} = \{0\}$$

because p and p' span L.

Proof of Proposition 4.2.15. Choose a vector space $\Gamma \subset V$ complementary to $\widehat{L}$ so that the open subset U_Γ of the Grassmannian can be identified with the vector space $\mathrm{Hom}(\widehat{L}, \Gamma)$ (see Construction 4.1.3). Then $U_\Gamma \cap \Sigma_2(p)$ is the linear subspace of $\mathrm{Hom}(\widehat{L}, \Gamma)$ containing those φ with $\varphi(p) = 0$. Since this is a linear condition on φ, the tangent space has the same description, as claimed. ∎

Exercise 4.2.16. Compute the tangent spaces of the Schubert varieties $\Sigma_{a,b}$ for $0 \leq b \leq a \leq 2$ at a general point $[\Lambda] \in \Sigma_{a,b}$ as above. Use your description to show that the intersections in the proof of Theorem 4.2.9 are generically transverse.

Exercise 4.2.17. Which of the Schubert varieties $\Sigma_{a,b}$ with $0 \leq b \leq a \leq 2$ are smooth varieties? If they are singular, what is their singular locus?

4.2.1 Applications

A straightforward application is the following result.

4.2.18 Corollary. *Given four general lines $L_i \subset \mathbb{P}^3$, there are 2 lines that intersect all of them.*

Proof. The Schubert variety of lines meeting a fixed line L_i is $\Sigma_1(L_i)$. For four general lines, these Schubert varieties meet transversely so that the number of points in their intersection is given by the degree of σ_1^4, which is equal to

$$\sigma_1^4 = (\sigma_{1,1} + \sigma_2)^2 = \sigma_{1,1}^2 + 2\sigma_{1,1}\sigma_2 + \sigma_2^2 = \sigma_{2,2} + 0 + \sigma_{2,2} = 2\sigma_{2,2},$$

which has degree 2. ∎

Let us try one application to **secant varieties**. Let $C \subset \mathbb{P}^3$ be a smooth curve of degree d and genus g. Consider the map $\tau \colon C \times C \dashrightarrow \mathbb{G}(1,3)$ mapping two points $p, q \in C$ to the line $L = \langle p, q \rangle$ spanned by these two points. This is, as such, not defined for $p = q$ so that we only get a rational map from the product $C \times C$ to the Grassmannian of lines in $\mathbb{P}^3$. We want to determine the class of the (closure of the) image of this map, which we expect to be a surface having codimension 2 in $\mathbb{G}(1,3)$.

4.2.19 Proposition. *Let $C \subset \mathbb{P}^3$ be a smooth irreducible curve of degree d and genus g that is not contained in any hyperplane in $\mathbb{P}^3$. The closure of the image of the map $\tau\colon C \to C \dashrightarrow \mathbb{G}(1,3)$ mapping (p,q) to the line $L = \langle p,q\rangle \subset \mathbb{P}^3$ has class*

$$\left(\binom{d-1}{2} - g\right)\sigma_2 + \binom{d}{2}\sigma_{1,1} \in A^2(\mathbb{G}(1,3)).$$

Proof. We compute the class of $\tau(C \times C)$ by specialization. The image has dimension 2 for general reasons (since a general fiber of τ can only contain finitely many points). So we know that the class $[\tau(C \times C)]$ is a $\mathbb{Z}$-linear combination $\alpha\sigma_2 + \beta\sigma_{1,1}$. Since $\sigma_2\sigma_{1,1} = 0$, we get β by couting the lines in $[\tau(C \times C)]\sigma_{1,1}$. Those lines intersect the curve C in two points and they are contained in the plane H of a flag so that $[\Sigma_{1,1}(H)] = \sigma_{1,1}$. This plane H intersects C in d distinct points so that it contains $\binom{d}{2}$ lines through two points of C. This shows that $\beta = \binom{d}{2}$.

Similarly, intersecting with σ_2 gives α. Choosing a flag starting with $p \in \mathbb{P}^3$, we have $\Sigma_2(p) = \{[L] \in \mathbb{G}(1,3) \colon p \in L\}$. The secants of C through p make the intersection $\Sigma_2(p) \cap \tau(C \times C)$ and we count these lines as follows. Consider the projection $\pi_p\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ away from p. The restriction of this map to C gives a birational morphism to its image, which is therefore a plane curve of degree d and genus g. A secant to C through p gives a point in the image that has two preimages. This will be a **node** of the image (because our projection center p is general). The number of nodes is given by the degree-genus formula as $\binom{d-1}{2} - g$. This is the number of secants through a general point p and we get $\alpha = \binom{d-1}{2} - g$ as claimed. ∎

4.2.20 Corollary. *If C and C' are two rational normal curves in $\mathbb{P}^3$ (so both are obtained by a generic change of coordinates from the twisted cubics), then there are 10 lines that are secants to both at the same time.*

Proof. For this, we intersect the classes of $\tau(C \times C)$ and $\tau(C' \times C')$, which are both rationally equivalent to $\sigma_2 + 3\sigma_{1,1}$ to get

$$(\sigma_2 + 3\sigma_{1,1})^2 = \sigma_2^2 + 6\sigma_2\sigma_{1,1} + 9\sigma_{1,1}^2 = 10\sigma_{2,2}.$$

This class has degree 10 and we only need to verify that the varieties intersect generically transversely. This can be done by hand (exercise) or by invoking Kleinman's transversality theorem (at least in charactersitic 0). ∎

4.2.2 Specialization – static

Static specialization is a method used to intersect Schubert cycles. We will do this for the example $\sigma_1^2 = \sigma_{1,1} + \sigma_2$ below. The basic idea is to intersect Schubert varieties corresponding to flags that are not generic so that determining the subspaces in their intersection becomes easier; but still generic enough so that the intersection remains transversal. Concretely, for σ_1^2 in $A(\mathbb{G}(1,3))$, this looks as follows.

4.2.21 Example. Pick two compelete flags $\mathcal{V}$ and $\mathcal{V}'$ in $\mathbb{P}^3$ such that the line L in $\mathcal{V}$ intersects the line L' in $\mathcal{V}'$ in a point p. Then L and L' span a plane H in $\mathbb{P}^3$ (since they intersect). So as sets, we get

$$\Sigma_1(L) \cap \Sigma_1(L') = \{[\Lambda] \in \mathbb{G}(1,3) \colon \Lambda \cap L \neq \emptyset \neq \Lambda \cap L'\}$$

Since the span of L and L' is the plane H, this means that a line $[\Lambda]$ in $\Sigma_1(L) \cap \Sigma_1(L')$ is contained in H or contains p so that

$$\Sigma_1(L) \cap \Sigma_1(L') = \Sigma_2(p) \cup \Sigma_{1,1}(H).$$

This equality holds on the level of sets. To show that this is also what we expect in the Chow ring, we have to show that the intersection is generically transverse, which then implies $\sigma_1^2 = \sigma_1 + \sigma_{1,1}$ as desired. We do this for both irreducible components of the intersection by the description of the tangent space to Schubert varieties.

First, pick a general point $[\Lambda]$ on $\Sigma_2(p) \subset \Sigma_1(L) \cap \Sigma_1(L')$ so that $p \in \Lambda$. Set K to be the span of Λ and L and K' to be the span of Λ and L'. By Exercise 4.2.22, the tangent spaces are

$$T_{[\Lambda]}\Sigma_1(L) = \{\varphi\colon \varphi(\widehat{p}) \subset \widehat{K}/\widehat{\Lambda}\} \text{ and } T_{[\Lambda]}\Sigma_1(L') = \{\varphi\colon \varphi(\widehat{p}) \subset \widehat{K'}/\widehat{\Lambda}\}.$$

Since the planes K and K' are distinct, they intersect in the line Λ, which shows that the intersection of the above tangent spaces consists of those linear maps $\varphi\colon \widehat{\Lambda} \to V/\widehat{\Lambda}$ that have p in the kernel. This vector space has dimension 2 so that the intersection is transversal at $[\Lambda]$.

Secondly, pick again a general point $[\Lambda]$ on $\Sigma_{1,1}(H) \subset \Sigma_1(L) \cap \Sigma_1(L')$. Let q be the intersection point of Λ and L and q' be the intersection point of Λ and L'. Then Proposition 4.2.15 shows

$$T_{[\Lambda]}\Sigma_1(L) = \{\varphi\colon \varphi(q) \in \widehat{H}/\widehat{\Lambda}\} \text{ and } T_{[\Lambda]}\Sigma_1(L') = \{\varphi\colon \varphi(q') \in \widehat{H}/\widehat{\Lambda}\},$$

so that $T_{[\Lambda]}\Sigma_1(L) \cap T_{[\Lambda]}\Sigma_1(L') = \{\varphi \in \operatorname{Hom}(\widehat{\Lambda}, V/\widehat{\Lambda})\colon \varphi(\widehat{\Lambda}) \subset \widehat{H}/\widehat{\Lambda}\}$. Again, this is a 2-dimensional vector space so that the intersection is generically transversal along $\Sigma_{1,1}(H)$ as well.

Exercise 4.2.22. Fix a line $L \subset \mathbb{P}^3$ and consider the Schubert variety

$$\Sigma_1(L) = \{[\Lambda] \in \mathbb{G}(1,3)\colon \Lambda \cap L \neq \emptyset\}.$$

For a line $[\Lambda] \in \Sigma_1(L)$ that is not equal to L, let q be the intersection point of Λ and L. Let K be the span of $\Lambda \cup L$, which is a plane. Show that $[\Lambda]$ is a smooth point of $\Sigma_1(L)$ with tangent space

$$T_{[\Lambda]}\Sigma_1(L) = \left\{\varphi \in \operatorname{Hom}(\widehat{\Lambda}, V/\widehat{\Lambda})\colon \varphi(\widehat{q}) \subset \widehat{K}/\widehat{\Lambda}\right\}.$$

4.3. General Grassmannians

The goal of this section is to give a description of the Chow ring of $\operatorname{Gr}(k, V)$ in terms of Schubert cycles. Transversality of intersections of such cycles follows, in characteristic 0, from Kleiman's Theorem that we have referred to before. Let's finally see a sketch of the proof.

4.3.1 Kleiman's Theorem

4.3.1 Theorem (Kleiman's Theorem). *Let K be an algebraically closed field of characteristic 0. Let G be an algebraic group (e.g. $G = \mathrm{GL}_n(K)$) that acts transitively on an algebraic variety X (e.g. $X = \operatorname{Gr}(k, K^n)$). Let $A \subset X$ be an irreducible closed subset.*

(a) *If $B \subset X$ is an irreducible closed set, then there is an open dense subset $U \subset G$ such that gA is generically transverse to B for all $g \in U$.*
(b) *If $\varphi\colon Y \to X$ is a morphism of varieties and Y irreducible, then the preimage $\varphi^{-1}(gA)$ is generically reduced and of the same codimension as $A \subset X$ for all g in an open subset $U \subset G$ or it is generically empty.*
(c) *If G is affine (like $G = \mathrm{GL}_n(K)$), then $[gA] = [A]$ in $A(X)$ for any $g \in G$.*

Sketch of proof. The first claim (a) follows from the second (b) for $Y = B$. We sketch a proof of part (b). The map $G \to X$, $g \mapsto gx$ is surjective for any $x \in X$ since G acts transitively on X. The fibers of this map are the cosets of the stabilizer of x; concretely, the fiber of gx is $g\,\mathrm{stab}(x)$, where $\mathrm{stab}(x)$ is the stabilizer of x. In particular, all fibers of this map have the same dimension, which is then $\dim(G) - \dim(X)$. Set

$$\Gamma = \{(x, y, g) \in A \times Y \times G\colon gx = \varphi(y)\}$$

Again, since G acts transitively on X, the projection of Γ to $A \times Y$ is surjective and its fibers are the stabilizer cosets. This shows that Γ has dimension $\dim(A) + \dim(Y) + (\dim(G) - \dim(X))$. The fiber over $g \in G$ of the projection $\Gamma \to G$ is isomorphic to $\{(x, y) \in A \times Y\colon gx = \varphi(y)\}$, which is $\varphi^{-1}(gA)$. If this projection is dominant, then for general $g \in G$ the fiber $\varphi^{-1}(gA)$ has dimension $\dim(A) + \dim(Y) - \dim(X) = \dim(Y) - \mathrm{codim}_X(A)$ as claimed in part (b). If the projection $\Gamma \to G$ is not dominant, then a general fiber $\varphi^{-1}(gA)$ is empty.

We can from now on assume that $\varphi^{-1}(gA)$ is non-empty for general $g \in G$ so that the projection $\Gamma \to G$ is dominant. We will show that $\varphi^{-1}(gA)$ is smooth at a general point, which implies that it is generically reduced. We essentially follow Harthorne's proof (Chapter III, Section 10 – which also contains the context on smooth morphisms).

Let $h\colon A_{sm} \times G \to X$ be the morphism mapping (x, g) to gx where A_{sm} is the (non-empty, open) set of smooth points of A. This morphism is dominant because G acts transitively on X. By generic smoothness (characteristic 0!), there is an open subset $U \subset X$ such that $h\colon h^{-1}(U) \to U$ is smooth (which essentially means that the differential of h is surjective). Since G acts on $A_{sm} \times G$ by left multiplication on G and this action commutes with the morphism h, the morphism $h\colon h^{-1}(gU) \to gU$ is also smooth (for any $g \in G$). The group G acts transitively on X so that the translates gU of U cover X. This shows that the morphism h is smooth everywhere.

Let us consider the subset $\Gamma' = \{(x, y, g) \in A_{sm} \times Y_{sm} \times G\colon gx = \varphi(y)\}$ where the points $x \in A$ and $y \in Y$ are smooth. Then Γ' is smooth for general reasons by the smoothness of h. In fact, Γ' is the fiber product

$$\begin{tikzcd}
\Gamma' = (A_{sm} \times G) \times_X Y_{sm} \arrow[r, "\pi_Y"] \arrow[d, "\pi_{A\times G}"'] & Y_{sm} \arrow[d, "\varphi"] \\
A_{sm} \times G \arrow[r, "h"'] & X
\end{tikzcd}$$

in which the map π_Y is smooth by base extension of h and Γ' is smooth (as a K-variety) because both π_Y is smooth and Y_{sm} is smooth over K. We now look at the projection $\pi_G\colon \Gamma' \to G$, $(x, y, g) \mapsto g$. Again, by generic smoothness, there is an open subset V of G such that $\pi_G\colon \pi_G^{-1}(V) \to V$ is smooth. This implies that $\pi_G^{-1}(g)$ will be smooth for every $g \in V$ and in particular, this will be generically reduced. Moreover, smoothness implies that each connected component of $\pi_G^{-1}(g)$ in Γ' has the same codimension equal to $\mathrm{codim}_X(A)$ as expected.

So now we have to take care of $\Gamma_s = \Gamma \setminus \Gamma'$. If $\pi_G\colon \Gamma_s \to G$ is not dominant, then the proof is done with the above argument. So we assume now that this map is dominant. Then a general fiber $\pi_G^{-1}(g) \cap \Gamma_s$ has dimension at most $\dim(\Gamma') - \dim(G)$ which is strictly smaller than $\dim(\Gamma) - \dim(G)$. The latter is the dimension of each connected component of the complete fiber $\pi_G^{-1}(g)$ for general $g \in G$ so that overall, the fiber $\pi_G^{-1}(g)$ is generically reduced.

The last claim (c) is easier in case that the algebraic group G is a direct product of general linear groups because G is then an open subset of the vector space of matrices of some size over K containing the identity matrix. Indeed, if $G = \mathrm{GL}_{k_1} \times \ldots \times \mathrm{GL}_{k_r}$, then its natural embedding into $\mathrm{Mat}_{k_1 \times k_1}(K) \times \ldots \times \mathrm{Mat}_{k_r \times k_r}(K)$ has the desired properties. To see that $[gA] = [A] \in A(X)$, let L be the projective line spanned by the identity and g and set

$$\Phi = \{(g, x) \in (G \cap L) \times X \colon g^{-1}x \in A\} \subset \mathbb{P}^1 \times X.$$

The fiber over the identity in L is A and the fiber over g is gA showing that $[A]$ and $[gA]$ are rationally equivalent as claimed. ∎

Exercise 4.3.2. (Easy:) Show that the general linear group $\mathrm{GL}(V)$ acts transitively on $\mathrm{Gr}(k, V)$ for any $k = 1, 2, \ldots, \dim(V)$ and more generally on complete flags in V.
(Harder:) Does $\mathrm{GL}(V)$ also act transitively on the set of pairs $([\Lambda], \varphi)$ of subspaces $[\Lambda] \in \mathrm{Gr}(k, V)$ and tangent vectors $\varphi \in T_{[\Lambda]} \mathrm{Gr}(k, V)$?

4.3.2 Schubert cells, cycles, and varieties

We first describe the Schubert varieties in $\mathrm{Gr}(k, V)$ for a fixed $k \in \{1, \ldots, n\}$.

4.3.3 Construction. Again, we choose a **complete flag** $\mathcal{V}$ in an n-dimensional K-vector space V consisting of an inclusion of subspaces

$$\{0\} \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = V$$

such that $\dim(V_i) = i$. For a weakly decreasing sequence $a = (a_1, a_2, \ldots, a_k)$ of nonnegative integers bounded by $n - k$ (that is $n - k \geq a_1 \geq a_2 \geq \ldots \geq a_k \geq 0$), we define the **Schubert variety** $\Sigma_a(\mathcal{V}) \subset \mathrm{Gr}(k, V)$ to be

$$\Sigma_a(\mathcal{V}) = \left\{[\Lambda] \in \mathrm{Gr}(k, V) \colon \ \dim(\Lambda \cap V_{n-k+i-a_i}) \geq i \text{ for all } i\right\}.$$

Since $n - k$ is the codimension of Λ, the expected dimension of the intersection $\Lambda \cap V_{n-k+i}$ is i. Since we subtract a_i in the index of the subspace V_j in the definition of the Schubert variety, we are essentially saying that the k-plane Λ intersects the flag in dimension i already a_i steps earlier than expected. To compare with the above notation, this means that $\Sigma_{1,1} \subset \mathrm{Gr}(2, V)$ is the set of planes in V that have a 1-dimensional intersection one step earlier than expected: in this case, that means that the plane in V intersects the plane of the flag in a line since the expected dimension of this intersection is 0; the second 1 says that it intersects the flag in a 2-dimensional space one step earlier than expected meaning that it is contained in the hyperplane H.

Another way to look at this is to consider the sequence of subspaces

$$\{0\} \subset V_1 \cap \Lambda \subset V_2 \cap \Lambda \subset \ldots \subset V_{n-1} \cap \Lambda \subset V_n \cap \Lambda = \Lambda.$$

In this sequence, any subspace is either equal to the one coming before it or its dimension increases by exactly 1. This happens k times in total because Λ has dimension k. In this interpretation, the Schubert variety consists of all k-planes in V for which the ith jump in this sequence occurs (at least) a_i steps early.

To simplify notation a little bit, we will drop trailing zeroes in the indexing sequences meaning that we write for example $(1, 1, 1)$ for $(1, 1, 1, 0, 0, \ldots, 0)$ in case that $k > 3$. Also, for a constant sequence $(b, b, \ldots, b)$ of length r, we will write b^r.

4.3.4 Example. (1) The Schubert variety $\Sigma_{n-k+1-\ell}(\mathcal{V})$ is the variety of k-planes in $\mathrm{Gr}(k, V)$ that intersect the ℓ-dimensional subspace V_ℓ of the flag nontrivially. In particular, the variety $\Sigma_1(\mathcal{V})$ is the variety of k-planes that meet V_{n-k} nontrivially.

(2) The Schubert variety $\Sigma_{(n-\ell)^k}(\mathcal{V})$ consists of all k-planes in V that are contained in V_ℓ. To see this, note that the index $n - k + i - a_i$ simplifies to $\ell - k + i$, which after k steps is ℓ. Similarly, the Schubert variety $\Sigma_{(n-k)^r}(\mathcal{V})$ is the set of k-planes that contain V_r (for $r \leq k$). In this case, the computation $n - k + i - a_i = r$ shows that there are r jumps in dimension of the sequence $(\Lambda \cap V_j)_j$.

Definition. The **Schubert classes** are the rational equivalence classes of the Schubert varieties in $A(\mathrm{Gr}(k, V))$. As before, we write σ_a for the class $[\Sigma_a(\mathcal{V})]$ of the Schubert variety indexed by the sequence a.

Kleiman's Theorem 4.3.1 shows that the rational equivalence class does not depend on our choice of complete flag $\mathcal{V}$ of V because the general linear group on V acts transitively on complete flags so that $[\Sigma_a(\mathcal{V})] = [\Sigma_a(\mathcal{V}')]$ and the above definition makes sense.

There are different ways to index Schubert varieties in the literature. The one that we have copied here from Eisenbud and Harris's book 3264 & all that has the following properties.

4.3.5 Proposition. *Denote by $\geq$ the termwise partial order on the indexing sequences so that $(a_1, \ldots, a_k) \geq (b_1, \ldots, b_k)$ if and only if $a_i \geq b_i$ for $i = 1, \ldots, k$. With this notation, we have $\Sigma_a \subset \Sigma_b$ if and only if $a \geq b$.*

Proof. This follows immediately from the definition (maybe most directly from the jumps in dimension for the sequence $(V_j \cap \Lambda)_j$ of vector spaces). ∎

Definition. The **Schubert cell** Σ_a^o is defined as before

$$\Sigma_a^o = \Sigma_a \setminus \left(\bigcup_{b > a} \Sigma_b \right).$$

We show now in general that the Schubert cells give an affine stratification of the Grassmannian $\mathrm{Gr}(k, V)$.

4.3.6 Theorem. *The locally closed subset Σ_a^o is isomorphic to $\mathbb{A}^{k(n-k)-|a|}$. In particular, Σ_a^o is irreducible and smooth. The Schubert varieties Σ_a are irreducible and of codimension $|a|$ in $\mathrm{Gr}(k, V)$.*

The tangent space to Σ_a^o at $[\Lambda] \in \Sigma_a^o$ is the subspace of $T_{[\Lambda]}\operatorname{Gr}(k,V) = \operatorname{Hom}(\Lambda, V/\Lambda)$ consisting of those linear maps $\varphi\colon \Lambda \to V/\Lambda$ that map

$$V_{n-k+i-a_i} \cap \Lambda \subset \Lambda$$

to

$$\left(V_{n-k+i-a_i} + \Lambda\right)/\Lambda \subset V/\Lambda$$

for all $i = 1, \dots, k$.

Proof. The proof relies on an explicit description in terms of the Plücker coordinates of k-planes $[\Lambda] \in \Sigma_a$. To simplify our life, let us choose the coordinate flag with $V_i = \operatorname{span}\{e_1, \dots, e_i\}$ in V with respect to a chosen basis $(e_1, \dots, e_n)$ of V. We next use the sequence $(V_j \cap \Lambda)_j$ to pick a suitable basis for Λ. The first subspace of this sequence that is guaranteed to be non-zero is $V_{n-k+1-a_1} \cap \Lambda$. Moreover, the one that must have dimension at least 2 will be $V_{n-k+2-a_2} \cap \Lambda$; and so on. So we can choose a basis $(v_1, \dots, v_k)$ of Λ with $v_i \in V_{n-k+i-a_i}$. This shows in particular that the coordinates of v_i corresponding to e_j with $j > n-k+i-a_i$ are all 0.

If $[\Lambda] \in \Sigma_a^o$, then the dimension jumps in the sequence $(V_j \cap \Lambda)_j$ occur exactly at the places specified by the sequence a so that we can choose a basis with $v_i \in V_{n-k+i-a_i} \setminus V_{n-k+i-1-a_{i-1}}$. Indeed, $V_{n-k+1-a_1} \cap \Lambda$ will have dimension exactly 1 and it is the first non-zero vector space of our sequence. Similarly, $V_{n-k+2-a_2} \cap \Lambda$ will have dimension exactly 2 and it is the first one of dimension 2 in the sequence; and so on. Overall, this means that the coordinate of v_i corresponding to the basis vector $e_{n-k+i-a_i}$ is non-zero. By rescaling v_i, we can assume that this coordinate is 1. Using row operations on the matrix with rows v_i representing Λ, we can also assume that the coordinate of v_j corresponding to $e_{n-k+i-a_i}$ is 0 if $j \neq i$. As an example, consider $k = 4$, $n = 9$ and $a = (3, 2, 2, 1)$ where this matrix will look as follows.

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & 1 & 0 \end{pmatrix}$$

The point is that this choice of basis of Λ makes the submatrix A_b of the matrix A with rows v_i the identity for $b = (n-k+1-a_1, n-k+2-a_2, \dots, n-a_k)$. In particular, the corresponding Plücker coordinate of Λ is 1.

It follows that the Schubert cell Σ_a^o is contained in the open subset $U_\Gamma \subset \operatorname{Gr}(k,V)$ of all k-planes that are complementary to the span Γ of all basis vectors whose indices are not in b. In this subset U_Γ, the Schubert cell Σ_a^o is the coordinate subspace of all matrices with $a_{ij} = 0$ for $j > n-k+i-a_i$, which are the zeroes of v_i according to the last sentence in the first paragraph of the proof. In particular, Σ_a^o is smooth and irreducible. Its dimension is given by the number of such zeroes. There are a_i additional zeroes in row i compared to a general element of U_Γ so that the codimension of Σ_a^o in $\operatorname{Gr}(k,V)$ is $|a|$ as claimed. (Indeed, the vector v_i has $n-k+i-a_i$ non-zero entries of which i many are 1 or 0. Those are the same for every element in U_Γ in this basis so that we see a_i additional zeroes compared to a general element.)

The Schubert variety Σ_a is the closure of Σ_a^o. To see this, pick any basis $(v_1, \dots, v_k)$ for $[\Lambda] \in \Sigma_a$. The condition on the matrix for $[\Lambda]$ to be in Σ_a is a rank condition in the following sense. For $j = 1, \dots, k$, build the matrix M_j whose first k rows are $v_1, \dots, v_k$ and then append as rows a basis of our flag, so $e_1,\dots,e_{n-k+j-a_j}$. The condition $\dim(\Lambda \cap V_{n-k+j-a_j}) \geq j$ translates to the row-rank of M_j being at most $k + (n-k+j-a_j) - j = n - a_j$. It is exactly equal to $n - a_j$ for $j = 1,\dots,k$ if and only if $[\Lambda] \in \Sigma_a^o$. For $[\Lambda] \in \Sigma_a$, it can be even smaller. However, the minors of M_j vanishing on Σ_a^o also vanish on Σ_a. In our chart U_Γ, those minors correspond to the linear equations vanishing on $\Sigma_a^o \subset U_\Gamma$. So they define Σ_a^o showing that every polynomial equation vanishing on Σ_a^o also vanishes on Σ_a.

The tangent space to Σ_a^o at a point $[\Lambda]$ is easy to deduce from this description because Σ_a^o is really a coordinate subspace of an affine space $\mathbb{A}^{k(n-k)} \cong U_\Gamma \subset \mathrm{Gr}(k,V)$. The point is to describe this coordinate subspace abstractly in terms of linear maps in $\mathrm{Hom}(\Lambda, V/\Lambda)$ as in Exercise 4.1.4. If A is the matrix representing a linear map $\varphi\colon \Lambda \to V/\Lambda$, then the row span of the matrix B obtained from the transpose of A by adding the jth coordinate vector to be column b_j is the representation of the linear space in the basis v_i as above. We get the desired description for the linear map from $V \to V/\Lambda$ represented by this matrix as follows. The span of the first j rows of this matrix B is $V_{n-k+j-a_j} \cap \Lambda$. These rows have zeroes in every column with index $\ell > n-k+j-a_j$, unless this index is in b. In other words, the image of this row span under the linear map represented by this matrix is contained in $(V_{n-k+j-a_j} + \Lambda)/\Lambda$ as claimed. So Σ_a^o corresponds to this coordinate subspace of $\mathrm{Hom}(\Lambda, V/\Lambda)$ and so then does its tangent space $T_{[\Lambda]}\Sigma_a^o$. ■

Again, excision Proposition 3.2.13 implies that we now know generators of $A(\mathrm{Gr}(k,V))$ as an abelian group.

4.3.7 Corollary. *The Chow group $A(\mathrm{Gr}(k,V))$ is generated, as an abelian group, by the Schubert classes σ_a. The group $A_0(\mathrm{Gr}(k,V))$ is free of rank 1 and generated by $\sigma_{(n-k)^k}$.*

Proof. The class of a point is $\sigma_{(n-k)^k} \in A(\mathrm{Gr}(k,V))$ by Example 4.3.4(2) with $\ell = k$ and all points are rationally equivalent (since $\mathrm{GL}(V)$ acts transitively on $\mathrm{Gr}(k,V)$). That the Schubert classes generate the Chow group is Proposition 3.2.13. ■

4.3.8 Corollary. *Both $(\sigma_{n-k})^k$ and $(\sigma_{1^k})^{n-k}$ are equal to the class $\sigma_{(n-k)^k}$ of a point in $A(\mathrm{Gr}(k,n))$ so that we have*

$$(\sigma_{n-k})^k = (\sigma_{1^k})^{n-k} = \sigma_{(n-k)^k} \in A^{k(n-k)}(\mathrm{Gr}(k,n)).$$

Proof. Let's begin with $(\sigma_{1^k})^k$: Fix a flag $\mathcal{V}$ in K^n and denote the hyperplane of the flag by H. Then Example 4.3.4(2) with $\ell = n-1$ says that $\Sigma_{1^k}(H)$ is the set of k-planes that are contained in H. The tangent space to $\Sigma_{1^k}(H)$ at a general $[\Lambda] \in \Sigma_{1^k}(H)$ is

$$T_{[\Lambda]}\Sigma_{1^k}(H) = \{\varphi \in \mathrm{Hom}(\Lambda, V/\Lambda)\colon \varphi(\Lambda) \subset H\}$$

by Theorem 4.3.6. So if we take $(n-k)$ general hyperplanes $H_1,\dots,H_{n-k} \subset K^n$, then there is a unique k-plane in $\bigcap_{i=1}^{n-k} \Sigma_{1^k}(H_i)$, namely $\Lambda = \bigcap_{i=1}^{n-k} H_i$. The description of the tangent space above implies that the intersection is transversal at Λ so that this intersection has the class of a point as claimed.

For $(\sigma_{n-k})^k$, we argue similarly. In this case, $\Sigma_{(n-k)}(\mathcal{V})$ is the set of k-planes that contain V_1 (again, see Example 4.3.4(2) for $r = 1$) so that Theorem 4.3.6 implies that

$$T_{[\Lambda]}\Sigma_{(n-k)} = \{\varphi \in \mathrm{Hom}(\Lambda, V/\Lambda)\colon V_1 \subset \ker(\varphi)\}.$$

Now take k general 1-dimensional subspaces $L_1, \dots, L_k$ and consider $\bigcap_{i=1}^k \Sigma_{(n-k)}(L_i)$. This intersection again contains a unique k-plane, namely $\Lambda = \mathrm{span}(L_1 \cup \ldots \cup L_k)$ and the interesction at this plane is transversal. ■

Exercise 4.3.9. Let $n = \dim(V)$. First, show that there is a natural isomorphism of algebraic varieties $\mathrm{Gr}(k, V)$ and $\mathrm{Gr}(n-k, V)$ taking a k-plane Λ to $\Lambda^\perp = \{\ell \in V^* : \Lambda \subset \ker(\ell)\}$. Then show that this isomorphism takes the Schubert variety $\Sigma_{1^k}(H)$ to $\Sigma_k(H^\perp)$. More generally, it takes $\Sigma_i(W)$ for a fixed subspace $W \subset V$ of dimension $n-k+1-i$ to the Schubert variety $\Sigma_{1^i}(W^\perp)$ which consists of those $(n-k)$-planes Λ' in V^* such that $\Lambda' + W^\perp \neq V^*$.

4.3.10 Example. The Schubert varieties Σ_a are not necessarily smooth (in contrast to the Schubert cells). For instance, $\Sigma_1 \subset \mathrm{Gr}(2, 4)$ is actually a singular quadric in $\mathbb{P}^4$ because it has rank 4 only.

The homogeneous vanishing ideals of the Schubert varieties are known and very nice: they are generated by the ideal of the Grassmannian and linear equations. In terms of the coordinate flag, the linear equations are just Plücker coordinates. For now, we do not need this result.

4.4. The intersection product

Assuming the existence of the intersection product, we show that the Schubert cycles freely generate the Chow ring and derive a combinatorial formula for the intersection product (involving the Littlewood-Richardson coefficients). We will see transversality of the intersections directly by the description of the tangent spaces.

Definition. Two flags $\mathcal{V}$ and $\mathcal{W}$ in an n-dimensional K-vector space V are **transverse** if $V_i \cap W_{n-i} = \{0\}$ for all $i = 0, 1, \dots, n$.

Exercise 4.4.1. Show that the two following conditions are equivalent to the flags $\mathcal{V}$ and $\mathcal{W}$ in K^n being transverse.

(a) $\dim(V_i \cap W_j) = \max\{0, i + j - n\}$ for all $i, j \in \{0, 1, \dots, n\}$

(b) There exists a basis $x_1, \dots, x_n$ of K^n such that $V_i = \mathrm{span}\{x_1, \dots, x_i\}$ for $i = 1, \dots, n$ and $W_j = \mathrm{span}\{x_n, \dots, x_{n+1-j}\}$ for $j = 1, \dots, n$.

Exercise 4.4.2. Show that the set of pairs of transverse flags are a non-empty open subset in the space of all pairs of flags. Show that the pairs of flags are a projective variety. Furthermore, show that $\mathrm{GL}(V)$ acts transitively on the set of pairs of transverse flags in V.

Definition. Fix a Schubert cell Σ_a^o with respect to the flag $\mathcal{V}$ and pick $[\Lambda] \in \Sigma_a^o$. The **induced flag** on Λ (by $\mathcal{V}$) is the complete flag

$$\{0\} \subsetneq \Lambda \cap V_{n-k+1-a_1} \subsetneq \Lambda \cap V_{n-k+2-a_2} \subsetneq \ldots \subsetneq \Lambda \cap V_{n-a_k} = \Lambda.$$

We will use the notation $\Lambda_i^{\mathcal{V}} = \Lambda \cap V_{n-k+i-a_i}$ for the i-th dimensional subspace of Λ in the induced flag.

4.4.3 Lemma. *Let $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ be Schubert varieties in $\mathrm{Gr}(k, V)$ defined relative to transverse flags $\mathcal{V}$ and $\mathcal{W}$. Let $[\Lambda]$ be a general point of their intersection.*

(1) $[\Lambda]$ *does not lie in any strictly smaller Schubert variety* $\Sigma_{a'}(\mathcal{V}) \subsetneq \Sigma_a(\mathcal{V})$.
(2) *The flags induced on* Λ *by* $\mathcal{V}$ *and* $\mathcal{W}$ *are transverse.*

Proof. Since the flags are transverse, the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ is generically transverse by Kleiman's Theorem 4.3.1. Directly, it says that there is an open subset U_1 of $\mathrm{GL}(V)$ such that $\Sigma_a(\mathcal{V})$ and $g\Sigma_b(\mathcal{V}) = \Sigma_b(g\mathcal{V})$ intersect transversally. However, there is also an open subset U_2 of $\mathrm{GL}(V)$ such that $g\mathcal{V}$ is transverse to $\mathcal{V}$. So for every $g \in U_1 \cap U_2$, both the intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_b(g\mathcal{V})$ and the flags $\mathcal{V}$ and $g\mathcal{V}$ are transverse. Since any pair of transverse flags can be moved to any other by an automorphism of V, we get that $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ is generically transverse for any transverse pair of flags $\mathcal{V}$ and $\mathcal{W}$. In particular, the intersection has the expected codimension $|a| + |b|$ in $\mathrm{Gr}(k, V)$. If $\Sigma_{a'}(\mathcal{V})$ is strictly contained in $\Sigma_a(\mathcal{V})$, then the sequence a' is strictly larger than a in at least one entry and $|a'| > |a|$. So the dimension of the intersection $\Sigma_{a'}(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ is smaller than the dimension of $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$. Therefore a general point of $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$ does not lie on $\Sigma_{a'}(\mathcal{V})$ for any $a' > a$.

To show the second claim, we will check the property in the above definition of transverse flags. So we have to show that $\Lambda_i^{\mathcal{V}} \cap \Lambda_{k-i}^{\mathcal{W}} = \{0\}$ which is equivalent to

$$\Lambda \cap V_{n-k+i-a_i} \cap W_{n-i-b_{k-i}} = \{0\}.$$

To simplify notation, let us fix i and write $d = n - k + i - a_i$ and $e = n - i - b_{k-i}$. Consider the incidence correspondence

$$\Phi = \{([\Lambda], v) \in (\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})) \times \mathbb{P}(V_d \cap W_e) \colon v \in \Lambda\}.$$

For $([\Lambda], v) \in \Phi$ we have that $v \in \Lambda \cap V_d \cap W_e$. To show the claim, it suffices to prove that $\dim(\Phi) < \dim(\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W}))$ because then the projection from Φ to the first factor cannot be dominant meaning exactly that the induced flags on a general point of that intersection is transversal as claimed.

Since the flags $\mathcal{V}$ and $\mathcal{W}$ are transverse, the intersection $V_d \cap W_e$ has the expected dimension $\max\{0, d + e - n\}$, where $d + e - n = n - k - a_i - b_{k-1}$. If $V_d \cap W_e = \{0\}$ is trivial, then there is nothing more to show. So we can assume that $d + e - n > 0$.

Now pick $[v] \in \mathbb{P}(V_d \cap W_e)$. We describe the fiber over v of the projection $\Phi \to \mathbb{P}(V_d \cap W_e)$ to the second factor using the quotient space $V/\mathrm{span}\{v\}$. The flags $\mathcal{V}$ and $\mathcal{W}$ on V induce flags on $\overline{\mathcal{V}}$ and $\overline{\mathcal{W}}$ by setting $\overline{V_j} = (V_j + \mathrm{span}\{v\})/\mathrm{span}\{v\}$ and similarly $\overline{W_j} = (W_j + \mathrm{span}\{v\})/\mathrm{span}\{v\}$ for all $j = 1, 2, \ldots, n$. Since $v \in V_d \cap W_e$, we have $\overline{V_{j-1}} = \overline{V_j}$ for some $j \leq d$ as well as $\overline{W_{j-1}} = \overline{W_j}$ for some $j \leq e$. For $([\Lambda], v) \in \Phi$ it follows that the plane $\overline{\Lambda} = \Lambda/\mathrm{span}\{v\} \subset V/\mathrm{span}\{v\}$ lies in the Schubert varieties $\Sigma_{\overline{a}}(\overline{\mathcal{V}})$ and $\Sigma_{\overline{b}}(\overline{\mathcal{W}})$ where the sequence $\overline{a}$ of length $k-1$ is obtained from a by deleting a_i and similarly $\overline{b}$ from b by deleting b_{k-i} because $\dim(\overline{\Lambda} \cap \overline{V_j}) = \dim(\Lambda \cap V_j)$ if $v \in V_j$. These Schubert varieties $\Sigma_{\overline{a}}(\overline{\mathcal{V}})$ and $\Sigma_{\overline{b}}(\overline{\mathcal{W}})$ lie in $\mathrm{Gr}(k-1, V/\mathrm{span}\{v\})$ and intersect generically transversely, which implies that the fibers over v have dimension

$$\dim(\mathrm{Gr}(k-1, V/\mathrm{span}\{v\})) - \sum_{j \neq i} a_j - \sum_{j \neq k-i} b_j = (k-1)(n-k) - (|a| - a_i) - (|b| - b_{k-i}).$$

Since v varies in $\mathbb{P}(V_d \cap W_e)$, the dimension of Φ is

$$\begin{aligned}\dim(\Phi) &= (d+e-n-1) + (k-1)(n-k) - (|a|-a_i) - (|b|-b_{k-i}) \\ &= (n-k-a_i-b_{k-1}-1) + (k-1)(n-k) - (|a|-a_i) - (|b|-b_{k-i}) \\ &= k(n-k) - |a| - |b| - 1 \\ &< \dim(\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})).\end{aligned}$$

■

4.4.4 Proposition. *Let $\mathcal{V}$ and $\mathcal{W}$ be transverse flags in K^n. Let $\Sigma_a(\mathcal{V})$ and $\Sigma_b(\mathcal{W})$ be two Schubert varieties in $\mathrm{Gr}(k,n)$ with $|a|+|b| = k(n-k)$. Then $\Sigma_a(\mathcal{V})$ intersects $\Sigma_b(\mathcal{W})$ transversely in a unique point if $a_i + b_{k+1-i} = n-k$ for each $i = 1,\dots,k$. The two Schubert varieties are disjoint otherwise. This implies the following formulae in $A(\mathrm{Gr}(k,n))$.*

$$\sigma_a\sigma_b = \left\{ \begin{array}{cl} \sigma_{(n-k)^k} & \text{if } a_i + b_{k+1-i} = n-k \text{ for all } i = 1,2,\dots,k, \\ 0 & \text{otherwise.} \end{array} \right\}$$

Proof. Since $|a|+|b| = \dim(\mathrm{Gr}(k,n))$ and the fact that the intersection is generically transvere, the intersection will be 0-dimensional and hence transverse. We have to count the number of intersection points. Fix $i = 1,2,\dots,k$ and consider the ith condition $\dim(\Lambda \cap V_{n-k+i-a_i}) \geq i$ for $[\Lambda] \in \Sigma_a(\mathcal{V})$ and the $(k+1-i)$th condition $\dim(\Lambda \cap W_{n-i+1-b_{k+1-i}}) \geq k+1-i$ for $[\Lambda] \in \Sigma_b(\mathcal{W})$. So the dimensions of the two subspaces $\Lambda \cap V_{n-k+i-a_i}$ and $\Lambda \cap W_{n-i+1-b_{k-i+1}}$ add up to at least $\dim(\Lambda) + 1$ which means that they have to intersect nontrivially. In particular, the subspaces $V_{n-k+i-a_i}$ and $W_{n-i+1-b_{k+1-i}}$ of V meet nontrivially. The flags $\mathcal{V}$ and $\mathcal{W}$ are transverse so that this implies $n-k+i-a_i+n-i+1-b_{k+1-i} \geq n+1$. This inequality simplifies to $a_i + b_{k+1-i} \leq n-k$.

This argument shows $\Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W}) = \emptyset$ if $a_i + b_{k+1-i} > n-k$ for some $i = 1,2,\dots,k$. The assumption $|a|+|b| = \sum_{i=1}^k (a_i + b_{k+1-i}) = k(n-k)$ implies that we must have $a_i + b_{k+1-i} = n-k$ if all inequalities $a_i + b_{k+1-i} \leq n-k$ $(i = 1,2,\dots,k)$ hold. In this case, the intersection $\Gamma_i = V_{n-k+i-a_i} \cap W_{n+1-i-b_{k+1-i}}$ has dimension 1. Moreover, since $[\Lambda] \in \Sigma_a(\mathcal{V}) \cap \Sigma_b(\mathcal{W})$, we have $\Gamma_i \subset \Lambda$. We get such a 1-dimensional space Γ_i for each $i = 1,2,\dots,k$ and in the notation of Exercise 4.4.1(b), $\Gamma_i = \mathrm{span}\{e_{n-k+i-a_i}\}$. So these subspaces of Λ are in general position and their union spans Λ, which is therefore unique. ■

This proposition gives a duality between $A^m(\mathrm{Gr}(k,n))$ and $A_m(\mathrm{Gr}(k,n))$. For any Schubert index $a = (a_1,\dots,a_k)$, we will write $a^* = (n-k-a_k, n-k-a_{k-1},\dots,n-k-a_1)$ for its **dual index** which then satisfies $\sigma_a\sigma_{a^*} = \sigma_{(n-k)^k}$. This duality is the reason for the following statement.

4.4.5 Corollary. *The Schubert classes form a free basis for $A(\mathrm{Gr}(k,V))$ and the intersection form $A^m(\mathrm{Gr}(k,V)) \times A_m(\mathrm{Gr}(k,V)) \to \mathbb{Z}$ given by mapping $\sigma_{(n-k)^k}$ to $\deg(\sigma_{(n-k)^k}) = 1$ have the Schubert classes as dual bases.* ■

We can use this duality to compute the coordinates of a cycle $\alpha \in A^m(\mathrm{Gr}(k,V))$ by the **method of undetermined coefficients** as before in the proof of Theorem 4.2.9.

4.4.6 Corollary. *For any $\alpha \in A^m(\mathrm{Gr}(k,V))$, we have*

$$\alpha = \sum_{|a|=m} \deg(\alpha \cdot \sigma_{a^*})\sigma_a,$$

where $\deg\colon A_0(\mathrm{Gr}(k,V)) \to \mathbb{Z}$ *is defined by* $\deg(\sigma_{(n-k)^k}) = 1$*. In particular, for any Schubert classes* σ_a *and* σ_b *in* $\mathrm{Gr}(k,V)$*, the product* $\sigma_a \cdot \sigma_b$ *is equal to*

$$\sum_{|c|=|a|+|b|} \gamma_{a,b;c}\, \sigma_c,$$

where $\gamma_{a,b;c} = \deg(\sigma_a \cdot \sigma_b \cdot \sigma_{c^*})$. ■

The coefficients $\gamma_{a,b;c}$ appearing in the previous Corollary are called **Littlewood-Richardson coefficients** and appear in many contexts in representation theory (of groups). Ways to compute them is still an active area of research. It is in general not clear, for example, which ones are 0 and which ones are 1. They can also be greater than 1 and there is no complete description of these cases. There are a few more accessible cases like **special Schubert classes** where we have Pieri's formula.

4.5. *Pieri's Formula*

Definition. A **special Schubert class** is a class $\sigma_a \in A(\mathrm{Gr}(k,V))$, where the Schubert index $a = (a_1, 0, 0, \ldots, 0)$ has only one non-zero entry.

4.5.1 Proposition (Pieri's Formula). *For any Schubert class* $\sigma_a \in A(\mathrm{Gr}(k,V))$ *and any integer b, the product of* σ_a *with the special Schubert class* σ_b *is*

$$\sigma_a \cdot \sigma_b = \sum \sigma_c,$$

where the sum is over all Schubert indices c such that $|c| = |a| + b$ *and* $c_i \geq a_i$ *for* $i = 1, 2, \ldots, k$ *as well as* $c_i \leq a_{i-1}$ *for* $i = 2, 3, \ldots, k$.

Proof. By the Littlewood-Richardson rule in Corollary 4.4.6, the claim is equivalent to

$$\sigma_a \sigma_b \sigma_{c^*} = \begin{cases} \sigma_{(n-k)^k} & \text{if } a_i \leq c_i \leq a_{i-1} \text{ for all } i \\ 0 & \text{otherwise} \end{cases}$$

To show this, pick three transverse flags $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$. By definition,

$$\Sigma_a(\mathcal{V}) = \{[\Lambda] : \Lambda \cap V_{n-k+i-a_i} \geq i \text{ for all } i\}$$

and, since $(c^*)_i = (n - k - c_{k+1-i})$,

$$\Sigma_{c^*}(\mathcal{W}) = \{[\Lambda] : \Lambda \cap W_{i+c_{k+1-i}} \geq i \text{ for all } i\}.$$

Set $A_i = V_{n-k+i-a_i} \cap W_{k+1-i+c_i}$. Since the flags $\mathcal{V}$ and $\mathcal{W}$ are transversal, either $\dim(A_i) = 0$ or $\dim(A_i) = c_i - a_i + 1 > 0$. For any $[\Lambda] \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$, we have $\Lambda \cap A_i \neq \{0\}$ (using the ith condition $\dim(\Lambda \cap V_{n-k+i-a_i}) \geq i$ from $\Sigma_a(\mathcal{V})$ and the jth condition $\dim(\Lambda \cap W_{j-c_{k+1-j}}) \geq j$ for $\Sigma_{c^*}(\mathcal{W})$ with $j = k+1-i$). So for $\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ to be non-empty, the dimension of A_i has to be positive. In particular, the intersection is empty if $c_i < a_i$. So we now assume $c_i \geq a_i$ for all $i = 1, 2, \ldots, k$. Choosing an adapted basis $(e_1, \ldots, e_n)$ of V such that

$V_i = \operatorname{span}\{e_1, \dots, e_i\}$ and $W_j = \operatorname{span}\{e_n, \dots, e_{n+1-j}\}$ as in Exercise 4.4.1(b), the intersection A_i is $\operatorname{span}\{e_{n-k+i-c_i}, \dots e_{n-k+i-a_i}\}$. Set $A = \operatorname{span}(A_1 \cup \dots \cup A_k)$. The dimension of A is bounded by

$$\dim(A) \leq \sum_{i=1}^{k} (c_i - a_i + 1) = k + b.$$

We have equality here if and only if the ranges of indices $[n-k+i-1-c_{i-1}, n-k+i-1-a_{i-1}]$ and $[n-k+i-c_i, n-k+i-a_i]$ do not intersect for any $i = 2, 3, \dots, k$, which is equivalent to $c_i \leq a_{i-1}$.

Suppose that $\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ is non-empty and that $[\Lambda] \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W})$ is general. Then $\Lambda \cap A_i$ is actually $\Lambda_i^{\mathcal{V}} \cap \Lambda_{k+1-i}^{\mathcal{W}}$ since $\Lambda_i^{\mathcal{V}} = \Lambda \cap V_{n-k+i-a_i}$ and $\Lambda_j^{\mathcal{W}} = \Lambda \cap W_{j-c_{k+1-j}}$. Lemma 4.4.3(2) implies that $\Lambda \subset A$.

We show the last desired inequality $c_i \leq a_{i-1}$ by considering the dimension of A: The Schubert variety $\Sigma_b(\mathcal{U})$ consists of the k-planes $\Lambda \subset V$ that intersect the general linear subspace $U = U_{n-k+1-b} \subset V$ of dimension $n-k+1-b$ nontrivially. So if the triple intersection $\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W}) \cap \Sigma_b(\mathcal{U})$ is nonempty, then this subspace U must intersect A nontrivially. This implies that $\dim(U) + \dim(A) \geq \dim(V) + 1$, showing $\dim(A) \geq k + b$. With the above inequality on the dimension of A, this implies $\dim(A) = k + b$, which is equivalent to $c_i \leq a_{i-1}$.

So we now know that $\sigma_a\sigma_b\sigma_{c^*} = 0$ if c_i does not satisfy the inequalities $a_i \leq c_i \leq a_{i-1}$. In case the inequalities hold, we can explicitly find the unique intersection point in $\Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W}) \cap \Sigma_b(\mathcal{U})$ in terms of A: Since $U = U_{n-k+1-b}$ is general of codimension $k + b - 1$, it will intersect A in a 1-dimensional space, say $U \cap A = \operatorname{span}\{v\}$. Write $v = v_1 + v_2 + \dots + v_k$ with $v_i \in A_i$, which is a unique decomposition in this case (because $\dim(A) = k + b$ implies $A = \bigoplus A_i$). For an intersection point $[\Lambda] \in \Sigma_a(\mathcal{V}) \cap \Sigma_{c^*}(\mathcal{W}) \cap \Sigma_b(\mathcal{U})$, we have $\Lambda \subset A$ and therefore $\Lambda \cap U \neq \{0\}$ as well as $\Lambda = \Sigma(\Lambda \cap A_i)$. It follows that $v \in \Lambda$, which implies $v_i \in \Lambda$ for all $i = 1, 2, \dots, k$, which finally shows $\Lambda = \operatorname{span}\{v_1, v_2, \dots, v_k\}$ so that $\sigma_a\sigma_b\sigma_{c^*}$ is the class of a single point (using transversality of the intersection by Kleiman's Theorem 4.3.1). ∎

This formula implies a relation among the Schubert classes, that will reappear later in the context of Chern classes.

4.5.2 Corollary. *In the Chow ring $A(\mathrm{Gr}(k, n))$, the following relation holds.*

$$(1 + \sigma_1 + \sigma_2 + \dots + \sigma_{n-k}) \cdot (1 - \sigma_1 + \sigma_{1,1} - \sigma_{1,1,1} + \dots + (-1)^k \sigma_{1^k}) = 1,$$

where 1 *is the fundamental class* $[\mathrm{Gr}(k, n)] \in A^0(\mathrm{Gr}(k, n))$.

Proof. By Pieri's formula Proposition 4.5.1, we have

$$\sigma_\ell \sigma_{1^m} = \sigma_{l,1^m} + \sigma_{l+1,1^{m-1}}$$

for any $\ell \leq n-k$ and $m \leq k$. Now consider the terms of the product $\sum_{i=0}^{n-k} \sigma_i \cdot \sum_{j=0}^{k} (-1)^j \sigma_{1^j}$ in the claim by codimension. The homogeneous part of this product in $A^d(\mathrm{Gr}(k, n))$ is $\sum_{i=0}^{d} (-1)^i \sigma_{d-i}\sigma_{1^i}$ for all $d \geq 1$. Using the above equation, this product simplifies to

$$\sum_{i=0}^{d} (-1)^i \sigma_{d-i}\sigma_{1^i} = \sigma_d - (\sigma_{d-1,1} + \sigma_d) + (\sigma_{d-2,1,1} + \sigma_{d-1,1}) - + \dots + (-1)^{d-1}(\sigma_{1^d} + \sigma_{2,1^{d-2}}) + (-1)^d \sigma_{1^d} = 0$$

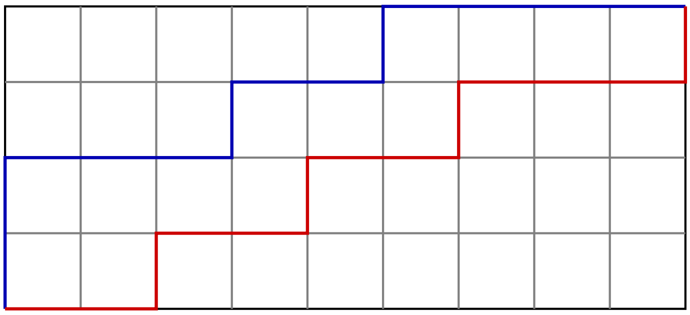

Figure 4.1: Young diagrams of Schubert indices $(5, 3, 0, 0)$ in blue and $(9, 6, 4, 2)$ in red.

because the sum telescopes. So the only term that remains is the fundamental class $1 \cdot 1$ as claimed. ■

We can use Young diagrams to count the number of Schubert classes in $\mathrm{Gr}(k, n)$: The Young diagram of the Schubert index $(a_1, \dots, a_k)$ of any Schubert class fits into a $k \times (n-k)$ grid. In the ith row of this grid, it contains the left most a_i boxes. So it is uniquely determined by its bottom right boundary. This boundary inside the box consists of edges going to the right (east) or up (north). Traveling from the southwest corner to the north east corner, every Young diagram of a Schubert index is bounded by k edges going north and $(n-k)$ edges going east. If we list them in order, we get a sequence of length n with k letters E and $(n-k)$ letters N. These sequences are in one-to-one correspondence with Young diagrams of Schubert indices of Schubert classes.

4.5.3 Example. The bottom right boundaries of the Young diagrams of the Schubert indeces $(5, 3, 0, 0)$ and $(9, 6, 4, 2)$ in a 4×9 grid are shown in Figure 4.1 in blue and in red, respectively. The corresponding sequences of edge directions are NNEEENEENEEEE and EENEENEENEEEN.

4.5.4 Corollary. *As a $\mathbb{Z}$-module, the Chow group $A(\mathrm{Gr}(k, n))$ is isomorphic to $\mathbb{Z}^{\binom{n}{k}}$.* ■

Chapter 5

Chern classes

The goal of this chapter is to introduce Chern classes and apply them to counting problems. We aim to describe Chern classes for vector bundles on smooth varieties (mostly projective space or Grassmannians) that are **globally generated** which means that they have plenty of global sections. The general construction of Chern classes is more technical. We will not go through the general construction here (at least not yet).

We write $H^0(X,E)$ (or simply $H^0(E)$) for the K-vector space of global sections $s\colon X \to E$ for any vector bundle E on X.

Definition. A vector bundle E of rank r on an irreducible variety X is **globally generated** if there are global sections $s_1,\dots,s_k$ of E such that the vector space $E_p = \pi^{-1}(p)$ attached to p is the linear span of $s_1(p),\dots,s_k(p)$.

Exercise 5.0.1. Show that a line bundle $\mathcal{L}$ on X is globally generated if and only if there are global sections $s_1,\dots,s_k$ of $\mathcal{L}$ that do not have any common zeroes.

Exercise 5.0.2. Prove that a vector bundle E of rank r is globally generated if and only if there is a surjective morphism $O_X^k \to \mathcal{E}$ of sheaves for some $k \in \mathbb{N}$, where $\mathcal{E}$ is the locally free sheaf of sections of the vector bundle E.

5.1. Line bundles

The Chern class of a line bundle on a smooth variety X is essentially the associated Weil divisor (as an element of $A^1(X)$).

5.1.1 Construction. Let X be a smooth and irreducible variety and $\mathcal{L}$ a line bundle on X. A rational section s of $\mathcal{L}$ defines the Weil divisor $\operatorname{div}(s)$ (compare Section 2.4.3). The **first Chern class** of $\mathcal{L}$, denoted by $c_1(\mathcal{L})$, is defined as the rational equivalence class of $\operatorname{div}(s)$ for any rational section s of $\mathcal{L}$.

This is well defined as we saw in Section 2.4.3. In fact, we can summarize the discussion as follows.

5.1.2 Proposition. *If X is an irreducible and smooth variety of dimension n, then c_1 is a group isomorphism*

$$c_1\colon \operatorname{Pic}(X) \to A_{n-1}(X).$$

5.1.3 Example. On projective n-space, we saw that $c_1(\mathcal{O}_{\mathbb{P}^n}(d))$ is the rational equivalence class of a hypersurface of degree d in $\mathbb{P}^n$, which is rationally equivalent to $d \cdot \zeta$, where $\zeta = [H]$ is the rational equivalence class of a hyperplane $H \subset \mathbb{P}^n$.

5.2. Characterizing Chern classes

The first Chern class of a globally generated vector bundle E on X is essentially a reduction to the case of line bundles. Let r be the rank of E. Then $\Lambda^r E$ is a line bundle on X and we define the first Chern class of E to be $c_1(\Lambda^r E) \in A_{\dim(X)-1}(X)$. Intuitively, this measures the locus of points in X where global sections $\tau_1, \dots, \tau_r$ of E become linearly dependent. Pick general global sections $\tau_1, \dots, \tau_r$ of E (assuming that E has enough global sections for now) and consider the map $\tau\colon \mathcal{O}_X^r \to E$ defined by sending the ith copy of $\mathcal{O}_X$ to $\mathcal{O}_X \tau_i$. Then $\tau_1 \wedge \tau_2 \wedge \ldots \wedge \tau_r$ is the determinant of this map, which vanishes if and only if the sections τ_i are linearly dependent. (Clearly, we have not worried about the determinant of τ being rationally equivalent to $c_1(\Lambda^r E)$.)

This point of view can be attempted for any $k \in \{1, \dots, r\}$: we can consider the vanishing scheme of a section $\tau_1 \wedge \tau_2 \wedge \ldots \wedge \tau_k \in \Lambda^k E$ and hope to get a well defined equivalence class in $A(X)$. This is the intuition (in the globally generated case) behind the definition of the ith Chern class of E (for $i = r + 1 - k$).

The main ingredient from commutative algebra for this approach is Macaulay's Unmixedness Theorem stating the following.

5.2.1 Theorem (Eagon, Northcott; Macaulay). *Let M be a $p \times q$ matrix with entries in a noetherian ring R. Let P be a minimal prime of the ideal of R generated by the $k \times k$ minors of M. Then P has codimension at most $(p - k + 1)(q - k + 1)$.*

As a consequence from this dimension result and Kleiman's Theorem, we get that the above construction does indeed make sense for globally generated vector bundles.

5.2.2 Lemma. *Let E be a vector bundle of rank r on an irreducible and smooth variety X. Let i be an integer between 1 and r and let $\tau_0, \tau_1, \dots, \tau_{r-i}$ be global sections of E. Let $D = V(\tau_0 \wedge \tau_1 \wedge \ldots \wedge \tau_{r-i})$ be the degeneracy locus where these sections are linearly dependent.*

(a) No irreducible component of D has codimension bigger than i.
(b) If the τ_i are general elements of a vector space W of the vector space of global sections of E that generates E, then D is generically reduced and has codimension i in X.

Proof. To show part (a), we interpret the vanishing of $\tau_0 \wedge \tau_1 \wedge \ldots \wedge \tau_{r-i}$ in terms of minors as follows. Locally on an open cover $\{U\}$ of X, the global sections τ_j are given by vectors $f_{j\ell}$ of rational functions on U of length r. The outer product $\Lambda_{j=0}^{r-i} \tau_j$ is 0 if and only if all $(r-i+1) \times (r-i+1)$ minors of the $(r-i+1) \times r$ matrix M with entries $(f_{j\ell})$ vanish. By Macaulay's (generalized) Unmixedness Theorem 5.2.1 with $k = p = r - i + 1$ and $q = r$, we get that every minimal prime of the ideal of maximal minors of M has codimension at most i.

Now using Kleiman's Theorem 4.3.1, we get part (b). Let $m = \dim(W)$ and consider the morphism $\varphi\colon X \to \mathrm{Gr}(m - r, W)$ sending a point $p \in X$ to the kernel of the evaluation map $W \to E_p$. (This linear map is surjective for every $p \in X$ because of the assumption that W globally generates E.) Let U be the subspace of dimension $r - i + 1$ spanned by general elements

$\tau_0, \tau_1, \ldots, \tau_{r-i}$ of W. Then the set $V(\tau_0 \wedge \tau_1 \wedge \ldots \wedge \tau_{r-i})$ is the preimage $\varphi^{-1}(\Sigma)$ of the Schubert cycle

$$\Sigma_i(U) = \{[\Lambda] \in \mathrm{Gr}(m-r, W) \colon \Lambda \cap U \neq \{0\}\}.$$

Part (b) of Theorem 4.3.1 says precisely that this preimage is generically reduced of the expected codimension i for general U. ■

The desired properties of Chern classes are summarized in the following theorem. We will prove parts of it, at least for globally generated vector bundles, and discuss applications of and intuitions behind these properties.

5.2.3 Theorem. *Let X be a smooth, irreducible, quasi-projective variety. There is a unique way of assigning to each vector bundle E on X a class $c(E) = 1 + c_1(E) + c_2(E) + \ldots \in A(X)$ called the **Chern class** of E such that the following properties hold.*

(a) *(Line bundles) If $\mathcal{L}$ is a line bundle on X then the Chern class of $\mathcal{L}$ is $1 + c_1(\mathcal{L})$, where $c_1(\mathcal{L}) \in A^1(X)$ is the associated Weil divisor class.*
(b) *(Bundles with enough sections) If $\tau_0, \ldots, \tau_{r-i}$ are global sections of E such that the degeneracy locus $D = V(\tau_0 \wedge \tau_1 \wedge \ldots \wedge \tau_{r-i})$ has codimension i, then $c_i(E)$ is the rational equivalence class of D in $A^i(X)$.*
(c) *(Whitney's formula) For any short exact sequence of vector bundles*

$$0 \to E \to F \to G \to 0$$

we have $c(F) = c(E) \cdot c(G) \in A(X)$.
(d) *(Functoriality) For any morphism $\varphi \colon Y \to X$ of smooth varieties we have*

$$\varphi^*(c(E)) = c(\varphi^*(E)).$$

Let's look at a few consequences of these properties.

5.2.4 Corollary. *If $E = \bigoplus \mathcal{L}_i$ is a direct sum of line bundles $\mathcal{L}_i$, then*

$$c(E) = \prod c(\mathcal{L}_i) = \prod (1 + c_1(\mathcal{L}_i)).$$

Concretely, $c_i(E)$ is the ith elementary symmetric polynomial evaluated in $c_1(\mathcal{L}_i)$.

Proof. This follows by applying Whitney's formula Theorem 5.2.3(c) inductively. ■

5.2.5 Corollary. *Any globally generated vector bundle E on X whose rank is greater than* $\dim(X)$ *has a nowhere vanishing global section.*

Proof. This follows from the dimension statement Theorem 5.2.3(b). ■

The functoriality in Theorem 5.2.3(d) allows us to reformulate Lemma 5.2.2 in terms of Schubert calculus. The following statement is proved above, see proof of part (b) of Lemma 5.2.2.

5.2.6 Proposition. *Let E be a vector bundle of rank r on an irreducible and smooth quasi-projective variety X. Let W be an m-dimensional vector space of global sections of E generating E. Let $\varphi \colon X \to$*

$\mathrm{Gr}(m-r, W)$ *be the morphism that sends a point* $p \in X$ *to the kernel of the evaluation* $W \to E_p$, *then*

$$c_i(E) = \varphi^*(\sigma_i).$$

This approach works well for globally generated bundles and can in fact be used to prove at least parts of Theorem 5.2.3 in this special case.

More generally, the **splitting principle** (based on the splitting construction) is a main technical tool. The basic idea is based on projectivized bundles.

Definition. Let X be a variety (more generally this works even for schemes). Let E be a vector bundle of rank $r+1$ on X and let $\mathcal{E}$ be the associated sheaf of sections of E. The **projectivization** of E is

$$\pi_{\mathcal{E}} \colon \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}^*)) \to X.$$

We write $\mathbb{P}\mathcal{E}$ for the total space $\mathrm{Proj}(\mathrm{Sym}(\mathcal{E}^*))$. A **projective bundle** over X is any morphism $\pi \colon Y \to X$ that is $\pi_{\mathcal{E}}$ for some locally free sheaf $\mathcal{E}$ over X.

This is a local construction packed into the language of sheaves again. The vector bundle associated to a locally free sheaf $\mathcal{E}$ is $\mathrm{Spec}(\mathrm{Sym}(\mathcal{E}^*))$, which just means that we cover X by open subsets U over which $\mathcal{E}|_U \cong \mathcal{O}_X(U)^{r+1}$ so that the ring of sections of the sheaf $\mathrm{Sym}(\mathcal{E}^*)|_U$ over U is isomorphic to the polynomial ring $\mathcal{O}_X(U)[x_0, \ldots, x_r]$. And this is nothing other than the coordinate ring of $U \times \mathbb{A}^{r+1}$. Using now Proj instead of Spec amounts to locally attach a homogeneous coordinate ring, namely this construction locally describes $U \times \mathbb{P}^r$. In this sense, we attach to every point $p \in X$ not the fiber $\pi^{-1}(p)$ but rather the projective space $\mathbb{P}(\pi^{-1}(p))$ (where π is here the bundle map $\pi \colon E \to X$ of the vector bundle E over X).

We also need to construct the tautological bundle $\mathcal{S}_{\mathcal{E}}$ on $\mathbb{P}\mathcal{E}$. We do this via its associated graded ring. The homogeneous coordinate ring of $\mathbb{P}^n$ is $K[x_0, \ldots, x_n]$ with its usual grading assigning every variable x_i degree $\deg(x_i) = 1$. The line bundle $\mathcal{O}(1)$ has global sections, namely the linear forms on $\mathbb{P}^n$. More generally, the vector space of global sections of $\mathcal{O}(d)$ are the forms of degree d. In this way, we get an isomorphism $K[x_0, x_1, \ldots, x_n] \cong \bigoplus_{i \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{O}(i))$. It turns out that we can reconstruct the line bundle $\mathcal{O}$ of $\mathbb{P}^n$ (in this case the structure sheaf) from this associated graded ring: $\mathbb{P}^n = \mathrm{Proj}(K[x_0, \ldots, x_n])$. This also works for any other line bundle $\mathcal{O}(d)$ on $\mathbb{P}^n$: the associated graded ring $\bigoplus_{i \in \mathbb{Z}} H^0(\mathbb{P}^n, \mathcal{O}(i+d))$ is again $K[x_0, x_1, \ldots, x_n]$ with the only difference in grading: now we have $\deg(x_i) = -d + 1$ and $\deg(1) = -d$. The associated graded ring of the tautological bundle $\mathcal{O}(-1)$ of $\mathbb{P}^n$ is $K[x_0, \ldots, x_n]$ with $\deg(x_i) = 2$ and $\deg(1) = 1$. We can do the same, namely shift the grading of $\mathrm{Sym}(\mathcal{E}^*)$ by 1, to obtain the associated graded ring of a line bundle $\mathcal{S}_{\mathcal{E}} = \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1)$ on $\mathbb{P}\mathcal{E}$.

The main result that make projective bundles over X useful to define Chern classes of vector bundles on X is the following.

5.2.7 Theorem. *Let E be a vector bundle of rank r on an irreducible and smooth variety X. Let $\pi_{\mathcal{E}} \colon \mathbb{P}\mathcal{E} \to X$ be the projectivization of E and let ζ be the first Chern class of $\mathcal{S}_{\mathcal{E}}^*$, the dual of the tautological line bundle on $\mathbb{P}\mathcal{E}$.*

(a) The flat pullback map $\pi^* \colon A(X) \to A(\mathbb{P}\mathcal{E})$ is injective.

(b) The element $\zeta = c_1(\mathcal{S}_{\mathcal{E}}^*) \in A(\mathbb{P}\mathcal{E})$ satisfies a unique monic polynomial $f(\zeta)$ of degree r with coefficients in $\pi^*(A(X))$.

Here, the flat pullback is simply $\pi^*([A]) = [\pi^{-1}(A)]$ because the map $\pi_{\mathcal{E}}$ is flat. If the morphism $f\colon Y \longrightarrow X$ is not flat, then the pullback $f^*\colon A(X) \longrightarrow A(Y)$ is more complicated.

This result leads to the following way to define Chern classes.

Definition. Let E be a vector bundle of rank r on a smooth variety X. The Chern classes $c_i(E)$ are the unique elements of $A(X)$ such that

$$f(\zeta) = \zeta^r + \pi^*(c_1(E))\zeta^{r-1} + \ldots + \pi^*(c_r(E)).$$

This definition in fact entails that $A(\mathbb{P}\mathcal{E})$ is isomorphic to $A(X)[\zeta]/(f(\zeta))$.

5.3. The splitting principle

The splitting principle says that any identity among Chern classes of vector bundles that holds for direct sums of line bundles is true in general. It is based on the following use of projectivizations of vector bundles.

5.3.1 Lemma (Splitting construction). *Let X be a smooth and irreducible variety. Let E be a vector bundle of rank r on X. There exists a smooth variety Y and a morphism $\varphi\colon Y \longrightarrow X$ such that the following holds.*

(a) The pullback map $\varphi^*\colon A(X) \longrightarrow A(Y)$ is injective.

(b) The pulled back bundle $\varphi^*(E)$ on Y admits a filtration

$$0 = E_0 \subset E_1 \subset \ldots \subset E_{r-1} \subset E_r = \pi^*(E)$$

by vector subbundles E_i such that the successive quotients E_i/E_{i-1} are line bundles.

Proof. The argument is essentially an iterated projectivization of bundles. First, set $Y_1 = \mathbb{P}\mathcal{E}$ which carries the tautological bundle $\mathcal{S}_1 := \mathcal{S}_{\mathcal{E}}$ which is a subbundle of $\pi^*_{\mathcal{E}}(E)$. Let $\mathcal{Q}_1$ be the quotient of $\pi^*_{\mathcal{E}}(E)$ by $\mathcal{S}_{\mathcal{E}}$ so that we have the exact sequence $0 \longrightarrow \mathcal{S}_{\mathcal{E}} \longrightarrow \pi^*_{\mathcal{E}}(E) \longrightarrow \mathcal{Q}_1 \longrightarrow 0$ on Y_1. Setting $Y_2 = \mathbb{P}\mathcal{Q}_1$ and pulling this exact sequence back to Y_2 and repeating the construction, we get the following two exact sequences on Y_2:

$$0 \longrightarrow \pi^*_{\mathcal{Q}_1}(\mathcal{S}_1) \longrightarrow \pi^*_{\mathcal{Q}_1}\pi^*_{\mathcal{E}}(E) \longrightarrow \pi^*_{\mathcal{Q}_1}(\mathcal{Q}_1) \longrightarrow 0 \text{ and}$$
$$0 \longrightarrow \mathcal{S}_2 \longrightarrow \pi^*_{\mathcal{Q}_1}(\mathcal{Q}_1) \longrightarrow \mathcal{Q}_2 \longrightarrow 0,$$

where $\mathcal{S}_2$ is the tautological bundle on $Y_2 = \mathbb{P}\mathcal{Q}_1$. The line bundle $\mathcal{S}_2$ together with the line bundle $\pi^*_{\mathcal{Q}_1}(\mathcal{S}_1)$ gives a subbundle of rank 2 of $\pi^*_{\mathcal{Q}_1}\pi^*_{\mathcal{E}}(E)$ with quotient $\mathcal{Q}_2$. Continuing like this, we get a space $Y = Y_r$ and a morhpism $\varphi\colon Y \longrightarrow X$ where $\varphi = \pi_{\mathcal{E}} \circ \pi_{\mathcal{Q}_1} \circ \ldots \circ \pi_{\mathcal{Q}_r}$ such that $\varphi^*(E)$ has a filtration into subbundles $\mathcal{E}_i$ of rank i obtained from the pullbacks of $\mathcal{S}_1, \ldots, \mathcal{S}_i$ to Y_r. By construction, the quotients $\mathcal{E}_i/\mathcal{E}_{i-1}$ are line bundles.

The injectivity of the pullback map $\varphi^*\colon A(X) \longrightarrow A(Y)$ follows from the above Theorem 5.2.7 and the Push-Pull Formula. ■

For a vector bundle E on X, we can use this splitting construction to obtain $\varphi\colon Y \longrightarrow X$. Whitney's formula Theorem 5.2.3(c) implies that the Chern class $c(\varphi^*(E))$ of the pull back of E

in $A(Y)$ is $\prod c(\mathcal{E}_i/\mathcal{E}_{i-1})$ and these quotients are line bundles. So any identity among the Chern classes of line bundles also holds for $c(\varphi^*(E))$. The injectivity of the map $\varphi^*\colon A(X) \to A(Y)$ implies that this class $c(\varphi^*(E)) \in A(Y)$ uniquely determines the Chern class $c(E) \in A(X)$.

This approach directly gives several useful observations.

5.3.2 Proposition. *Let E be a vector bundle of rank r on a smooth and irreducible variety X. Then the following hold.*

(a) $c_i(E) = 0$ *for any* $i > r$*;*
(b) $c_i(E^*) = (-1)^i c_i(E)$ *for all* $i \geq 1$*;*
(c) $c_1(\Lambda^r E) = c_1(E)$*.*

Proof. By the splitting principle, it suffices to check these statements for vector bundles that are direct sums of line bundles. So write $E = \bigoplus_{i=1}^r \mathcal{L}_i$. The Chern class of a line bundle is $c(\mathcal{L}_i) = 1 + c_1(\mathcal{L}_i)$ and so $c(\mathcal{L}_i^*) = 1 - c_1(\mathcal{L}_i)$ (because c_1 is a group isomorphism from $\mathrm{Pic}(X)$ to $A^1(X)$. So part (a) follows from Whitney's formula Theorem 5.2.3(c) showing that

$$c(E) = \prod_{i=1}^r c(\mathcal{L}_i) = \prod_{i=1}^r (1 + c_1(\mathcal{L}_i)).$$

There is no term of degree higher than r in the expansion of this product. For part (b), we use the same computation:

$$c(E^*) = c\left(\bigoplus_{i=1}^r \mathcal{L}_i^*\right) = \prod_{i=1}^r c(\mathcal{L}_i^*) = \prod_{i=1}^r (1 - c_1(\mathcal{L}_i))$$

which shows that $c_i(E^*) = (-1)^i c(E)$ by comparing the two results writing the Chern class of $c(E)$ and $c(E^*)$ in terms of elementary symmetric polynomials in the $c_1(\mathcal{L}_i)$.

To show part (c), first check that $\Lambda^r \left(\bigoplus_{i=1}^r \mathcal{L}_i\right) = \bigotimes_{i=1}^r \mathcal{L}_i$ which are line bundles. Their first Chern class is $\sum_{i=1}^r c_1(\mathcal{L}_i)$, again because $c_1\colon \mathrm{Pic}(X) \to A^1(X)$ is a group homomorphism. The first Chern class of E is also $\sum_{i=1}^r c_1(\mathcal{L}_i)$, since the elementary symmetric polynomial of degree 1 is the sum of all variables. ∎

Another useful application of the splitting principle is to tensor products. Let us first take the tensor product of any vector bundle with a line bundle.

5.3.3 Proposition. *Let E be a vector bundle of rank r on a smooth and irreducible variety X. Let $\mathcal{L}$ be a line bundle on X. The kth Chern class $c_k(E \otimes \mathcal{L})$ of $E \otimes \mathcal{L}$ is given by the formula*

$$c_k(E \otimes \mathcal{L}) = \sum_{j=0}^k \binom{r-j}{k-j} c_1(\mathcal{L})^{k-j} c_j(E) = \sum_{i=0}^k \binom{r-k+i}{i} c_1(\mathcal{L})^i c_{k-i}(E).$$

Proof. We assume that $E = \bigoplus_{i=1}^r \mathcal{M}_i$ is a direct sum of line bundles. For simplicity, let's write $\alpha_i = c_1(\mathcal{M}_i)$ and $\beta = c_1(\mathcal{L})$. With this notation, $c(E) = \prod_{i=1}^r (1 + \alpha_i)$ so that the evaluation $e_k(\alpha_1, \ldots, \alpha_r) = c_k(E)$ of the elementary symmetric polynomial e_k of degree k in r variables is exactly the kth Chern class of E. Whitney's formula applied for $E \otimes \mathcal{L} = \bigoplus(\mathcal{M}_i \otimes \mathcal{L})$ gives

$$c(E \otimes \mathcal{L}) = \prod_{i=1}^{r}(1 + \alpha_i + \beta).$$

The claim now follows from collecting terms of degree k on the right hand side which we can also write as

$$\prod_{i=1}^{r}(1 + \alpha_i + \beta) = \sum_{j=0}^{r} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le r} (1 + \beta)^{r-j} \alpha_{i_1} \ldots \alpha_{i_j} = \sum_{j=0}^{r}(1 + \beta)^{r-j} c_j(E).$$

This directly implies the first formula for $c_k(E \otimes \mathcal{L})$. The second is a simple shift in indexing setting $i = k - j$. ∎

The tensor product of two bundles is a more involved computation. The first Chern class is simple to write down.

5.3.4 Proposition. *Let E and F be vector bundles of rank e and f, respectively, on an irreducible and smooth variety X. The first Chern class of $E \otimes F$ is*

$$c_1(E \otimes F) = f c_1(E) + e c_1(F).$$

Proof. If both $E = \bigoplus \mathcal{L}_i$ and $F = \bigoplus \mathcal{M}_j$ are direct sums of line bundles, then $E \otimes F$ is $\bigoplus_{i,j} \mathcal{L}_i \otimes \mathcal{M}_j$. Whitney's formula strikes again and gives

$$c(E \otimes F) = \prod_{i=1}^{e} \prod_{j=1}^{f} (1 + c_1(\mathcal{L}_i) + c_1(\mathcal{M}_j)).$$

This formula can, in principle, be stared at to compute $c_k(E \otimes F)$. We just read off the terms of degree 1 to get

$$c_1(E \otimes F) = \sum_{i=1}^{e} \sum_{j=1}^{f} (c_1(\mathcal{L}_i) + c_1(\mathcal{M}_j)) = f c_1(E) + e c_1(F).$$

∎

Exercise 5.3.5. Let E be a vector bundle of rank 3 on a smooth and irreducible variety X. Express the Chern class of $\Lambda^2 E$ in terms of the Chern class of E (using the splitting principle and Whitney's formula).

5.4. Examples: Chern classes of some bundles

5.4.1 Projective space

5.4.1 Example. Since the first Chern class of a line bundle is the associated Weil divisor, we have $c_1(\mathcal{O}_{\mathbb{P}^n}(d)) = d\zeta \in A(\mathbb{P}^n)$ for any positive integer d, where $\zeta = [H]$ is the rational equivalence class of a hyperplane.

Using Whitney's formula, this implies the following.

5.4.2 Proposition. *The Chern class of the universal quotient bundle Q on $\mathbb{P}^n$ is*

$$c(Q) = 1 + \zeta + \zeta^2 + \ldots + \zeta^n \in \mathbb{Z}[\zeta]/(\zeta^{n+1}) \cong A(\mathbb{P}^n).$$

Proof. The universal quotient bundle Q fits in the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}^n} \to Q \to 0,$$

where we write $V = K^{n+1}$ so that $\mathbb{P}^n = \mathbb{P}(V)$. Whitney's Formula Theorem 5.2.3(c) gives $c(V \otimes \mathcal{O}_{\mathbb{P}^n}) = c(\mathcal{O}_{\mathbb{P}^n}(-1))c(Q)$. Since $\mathcal{O}_{\mathbb{P}^n}(-1) = \mathcal{O}_{\mathbb{P}^n}^\vee$, we have that $c(\mathcal{O}_{\mathbb{P}^n}(-1)) = 1 - \zeta$. The Chern class of the trivial bundle $V \otimes \mathcal{O}_{\mathbb{P}^n}$ is 1. Using the geometric sum formula, we get (abusing notation by formal division)

$$c(Q) = \frac{1}{c(\mathcal{O}_{\mathbb{P}^n}(-1))} = \frac{1}{1-\zeta} = 1 + \zeta + \zeta^2 + \ldots + \zeta^n.$$

■

Exercise 5.4.3. Give an alternative proof of Proposition 5.4.2 by considering any element $v \in V$ as a global section of Q. Determine the degeneracy locus of sections $\sigma_1, \ldots, \sigma_k$ determined by a collection $v_1, \ldots, v_k \in V$ of linearly independent vectors (which is $c_{n-k+1}(Q)$).

Similarly, we can also compute the Chern class of the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ of $\mathbb{P}^n$.

5.4.4 Proposition. *The Chern class of the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ of projective n-space is*

$$c(\mathcal{T}_{\mathbb{P}^n}) = (1+\zeta)^{n+1} \in \mathbb{Z}[\zeta]/(\zeta^{n+1}) \cong A(\mathbb{P}^n).$$

Proof. The tangent bundle fits in the Euler sequence (compare Example 4.1.11)

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1) \otimes V \to \mathcal{T}_{\mathbb{P}^n} \to 0$$

which implies (by Whitney's Formula again) that $c(\mathcal{T}_{\mathbb{P}^n}) \cdot c(\mathcal{O}_{\mathbb{P}^n}) = c(\mathcal{O}_{\mathbb{P}^n}(1) \otimes V)$. This implies the claim because $c(\mathcal{O}_{\mathbb{P}^n}) = 1$ and $c(\mathcal{O}_{\mathbb{P}^n}(1)^{n+1}) = (1+\zeta)^{n+1}$. ■

Exercise 5.4.5. What is the Chern class of the tangent bundle of $\mathbb{P}^m \times \mathbb{P}^n$?

With a bit more technique, we can also compute the tangent bundle of a hypersurface (or more generally, a complete intersection) $X \subset \mathbb{P}^n$.

5.4.6 Proposition. *Let $X = \mathcal{V}_+(f) \subset \mathbb{P}^n$ be a smooth hypersurface of degree d in $\mathbb{P}^n$. The Chern class of the tangent bundle of X in $A(X)$ is*

$$c(\mathcal{T}_X) = \left(1 + (n+1)\zeta + \binom{n+1}{2}\zeta^2 + \ldots \binom{n+1}{2}\zeta^{n-1}\right)(1 - d\zeta + d^2\zeta^2 - + \ldots + (-1)^{n-1}d^{n-1}\zeta^{n-1}),$$

where ζ is the class of a hyperplane section of X.

Proof. Again we apply Whitney's formula but this time to the exact sequence

$$0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}^n}|_X \to \mathcal{N}_{X/\mathbb{P}^n} \to 0.$$

The normal bundle $\mathcal{N}_{X/\mathbb{P}^n}$ of $X \subset \mathbb{P}^n$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(X)|_X = \mathcal{O}_X(d)$, which is part of a proof of the **adjunction formula**. Taking this for granted at the moment, this implies the claim:

$$(1+\zeta)^{n+1} = c(\mathcal{T}_{\mathbb{P}^n}|_X) = c(\mathcal{T}_X)c(\mathcal{O}_X(d)) = c(\mathcal{T}_X)(1+d\zeta)$$

Since X has dimension $n-1$, the expansions become 0 in degree n and higher. ■

5.4.2 Grassmannian

Again, we will compute the Chern classes of the universal sub- and quotient bundles as well as (at least the first) Chern class of the tangent bundle, this time for Grassmannians.

5.4.7 Proposition. *The Chern class of the universal quotient bundle $\mathcal{Q}$ on $\mathrm{Gr}(k,n)$ is*

$$c(\mathcal{Q}) = 1 + \sigma_1 + \sigma_2 + \ldots + \sigma_{n-k} \in A(\mathrm{Gr}(k,n)).$$

The Chern class of the universal subbundle $\mathcal{S}$ on $\mathrm{Gr}(k,n)$ is

$$c(\mathcal{S}) = 1 - \sigma_1 + \sigma_{1,1} - + \ldots + (-1)^k \sigma_{1^k}.$$

In particular, the Chern class of $\mathcal{S}^*$ is $c(\mathcal{S}^*) = 1 + \sum_{i=1}^k \sigma_{1^i}$.

Proof. The universal quotient bundle $\mathcal{Q}$ is globally generated: a vector $v \in V$ gives a global section σ_v of $\mathcal{Q}$ by setting $\sigma_v(\Lambda) = (v + \Lambda) \in V/\Lambda$. Now take linearly independent vectors $v_1, \ldots, v_m \in V$. The degeneracy locus $\mathcal{V}(\sigma_{v_1} \wedge \sigma_{v_2} \wedge \ldots \wedge \sigma_{v_m})$ of the corresponding sections consists of those k-planes $\Lambda \subset V$ such that the intersection $\Lambda \cap \mathrm{span}\{v_1, \ldots, v_m\}$ is nontrivial. This is the Schubert cycle $\Sigma_{n-k+1-m}(\mathrm{span}\{v_1, \ldots, v_m\})$. This degeneracy locus is the jth Chern class of $\mathcal{Q}$ for $j = \mathrm{rank}(\mathcal{Q}) + 1 - m$ and $\mathrm{rank}(\mathcal{Q}) = n - k$. So $c_j(\mathcal{Q}) = \sigma_j$ showing that $c(\mathcal{Q}) = 1 + \sum_{i=1}^{n-k} \sigma_i$ as claimed. We can now compute $c(\mathcal{S})$ by Whitney's formula because we have the exact sequence

$$0 \to \mathcal{S} \to \mathcal{O}^n_{\mathrm{Gr}(k,n)} \to \mathcal{Q} \to 0.$$

The Chern class of $\mathcal{O}^n_{\mathrm{Gr}(k,n)}$ is 1 so that Pieri's formula in the form Corollary 4.5.2 implies the claim. Since $c_i(E^*) = (-1)^i c_i(E)$, we get the Chern class of $\mathcal{S}^*$ from this as well. ■

Exercise 5.4.8. Compute the Chern class of the dual $\mathcal{S}^*$ of the universal subbundle using degeneracy loci as follows. A linear form $\ell \in V^*$ defines a global section σ_ℓ of $\mathcal{S}^*$ by setting $\sigma_\ell(\Lambda) = \ell|_\Lambda \in \Lambda^*$. The rational equivalence class of the degeneracy locus $\mathcal{V}(\sigma_{\ell_1} \wedge \sigma_{\ell_2} \wedge \ldots \wedge \sigma_{\ell_m})$ is $\sigma_{1^{k-m+1}}$.

These results determine the first Chern class of the tangent bundle of $\mathrm{Gr}(k,n)$ by Proposition 5.3.4 because $\mathcal{T}_{\mathrm{Gr}(k,n)} = \mathcal{H}om(\mathcal{S}, \mathcal{Q}) \cong \mathcal{S}^* \otimes \mathcal{Q}$.

5.4.9 Proposition. *The first Chern class of the tangent bundle $\mathcal{T}_{\mathrm{Gr}(k,n)}$ is*

$$c_1(\mathcal{T}_{\mathrm{Gr}(k,n)}) = n\sigma_1.$$

Proof.

$$c_1(\mathcal{T}_{\mathrm{Gr}(k,n)}) = \mathrm{rank}(\mathcal{Q})c_1(\mathcal{S}^*) + \mathrm{rank}(\mathcal{S}^*)c_1(\mathcal{Q}) = (n-k)\sigma_1 + k\sigma_1 = n\sigma_1$$

∎

Exercise 5.4.10. Determine the Chern class of the tangent bundle of $\mathrm{Gr}(2,4)$.

With more commutative algebra, it is possible to show that the Chern classes of the universal subbundle generate the Chow ring as a (graded) $\mathbb{Z}$-algebra in the following sense.

5.4.11 Theorem. *The Chow ring of the Grassmannian* $\mathrm{Gr}(k,n)$ *is*

$$A(\mathrm{Gr}(k,n)) = \mathbb{Z}[c_1,\dots,c_k]/I,$$

where $c_i \in A^i(\mathrm{Gr}(k,n))$ is the ith Chern class of the universal subbundle $\mathcal{S}$ (i.e. $c_i = (-1)^i\sigma_{1^i}$). The ideal I is generated by the terms of total degree $j \in \{n-k+1,\dots,n\}$ in the power series expansion (geometric series)

$$\frac{1}{1+c_1+\ldots+c_k} = 1 - (c_1+\ldots+c_k) + (c_1+\ldots+c_k)^2 - +\ldots \in \mathbb{Z}[[c_1,\dots,c_k]].$$

This graded $\mathbb{Z}$-algebra is a complete intersection.

5.4.3 Chern class and Euler characteristic

The topological Euler characteristic of a manifold (or simplicial complex) generalizes the Euler Formula $E - K + F = 2$ for 3-dimensional polytopes. We write it as $\chi_{top}(M)$ for a manifold M and use it for smooth projective varieties $X \subset \mathbb{P}(\mathbb{C}^{n+1})$. An alternative definition in terms of Chern classes is the following result.

5.4.12 Theorem (essentially Poincaré-Hopf Theorem). *For a smooth m-dimensional projective variety $X \subset \mathbb{P}^n$ (over $\mathbb{C}$), the topological Euler characteristic is determined by the tangent bundle of X by the formula*

$$\chi_{top}(X) = \deg(c_m(\mathcal{T}_X)).$$

With the above computations of tangent bundles, this gives a tool to compute the Euler characteristic in some cases.

5.4.13 Example. The Chern class $c(\mathcal{T}_{\mathbb{P}^n})$ of the tangent bundle of $\mathbb{P}^n$ is $(1+\zeta)^{n+1} \in A(\mathbb{P}^n)$ so that $\chi_{top}(\mathbb{P}^n) = n+1$.

5.4.14 Example. The computation of the Euler characteristic of a hypersurface of degree d in $\mathbb{P}^n$ is a little more involved but it only depends on the degree! Above, we saw for a hypersurface $X \subset \mathbb{P}^n$ of degree d that

$$c(\mathcal{T}_X) = \frac{(1+\zeta)^{n+1}}{(1+d\zeta)}$$

and we need to find the part of degree $n-1$ of the exapnsion by the geometric sum formula. This gives

$$c_{n-1}(\mathcal{T}_X) = \sum_{i=0}^{n-1}(-1)^i \binom{n+1}{n-1-i} d^i \zeta^{n-1}.$$

The degree of ζ^{n-1} is $d = \deg(X)$ so that the topological Euler characteristic is

$$\chi_{top}(X) = \deg(c_{n-1}(\mathcal{T}_X)) = \sum_{i=0}^{n-1}(-1)^i \binom{n+1}{n-1-i} d^{i+1}.$$

5.5. Fano schemes and enumerative problems

We will use the formalism of Chern classes to show that a smooth cubic surface in $\mathbb{P}^3$ contains 27 lines. For this, we compute the class of the **Fano scheme** of lines on a cubic surface as the Chern class of a bundle derived from the universal subbundle of $\mathrm{Gr}(2,4)$. The final equation for this example will look like this

$$\deg(c_4(\mathrm{Sym}^3 \mathcal{S}^*)) = 27.$$

More generally, the Fano scheme $F_k(X)$ for a projective variety $X \subset \mathbb{P}^n$ is the set of k-planes in $\mathbb{P}^n$ that are contained in X as a subset of $\mathbb{G}(k,n) = \mathrm{Gr}(k+1, n+1)$. Let us first focus on set theoretic questions and later worry about scheme structures and transversality of intersections.

Definition. Fix positive integers n, d, and k with $k \leq n$ and define the **universal Fano scheme** of k-planes on hypersurfaces of degree d in $\mathbb{P}^n$ as

$$\Phi(n,d,k) = \left\{(X, [\Lambda]) \in \mathbb{P}^N \times \mathbb{G}(k,n) \colon \Lambda \subset X\right\},$$

where $\mathbb{P}^N = \mathbb{P}(K[x_0, \ldots, x_n]_d)$ for $N = \binom{n+d}{d} - 1$ is the projective space of hypersurfaces $X = \mathcal{V}_+(f) \subset \mathbb{P}^n$ of degree d.

5.5.1 Proposition. *The universal Fano scheme $\Phi(n,d,k)$ is a closed subset of $\mathbb{P}^N \times \mathbb{G}(k,n)$. It is a smooth and irreducible variety of dimension*

$$\dim(\Phi(n,d,k)) = \left(\binom{n+d}{d} - 1\right) + (k+1)(n-k) - \binom{k+d}{d}.$$

Proof. The point is to consider the projection to the second factor and its fibers. If we fix the k-plane $\Lambda \subset \mathbb{P}^n$, the space of hypersurfaces of degree d that contain Λ is a projective space, namely the projective space of all homogeneous polynomials $f \in K[x_0, \ldots, x_n]$ of degree d that vanish identically when restricted to $\Lambda \subset \mathbb{P}^n$.

To see that $\Phi(n,d,k)$ is closed in $\mathbb{P}^N \times \mathbb{G}(k,n)$, we cover $\mathbb{G}(k,n)$ with the standard open affine charts U_Γ. The argument is the same for all covers so let us assume for simplicity that a subspace $\Lambda \in U_\Gamma$ is the rowspace of a unique matrix of the form

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 & a_{0,k+1} & a_{0,k+1} & \dots & a_{0,n+1} \\ 0 & 1 & & 0 & a_{1,k+1} & a_{1,k+1} & \dots & a_{1,n+1} \\ \vdots & & \ddots & & \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 & a_{k,k+1} & a_{k,k+1} & \dots & a_{k,n+1} \end{pmatrix}.$$

We can write any point $x \in \Lambda$ as $(s_0, s_1, \dots, s_k)A$ for a unique point $s = (s_0, s_1, \dots, s_k) \in \mathbb{P}^k$. So a hypersurface $X \subset \mathbb{P}^n$ defined by $f \in K[x_0, x_1, \dots, x_n]_d$ contains Λ if and only if every coefficient of the polynomial $f_\Lambda \in K[s_0, s_1, \dots, s_k]$ obtained from f by substituting x_i by the ith entry of $(s_0, s_1, \dots, s_k)A$ in f is the zero polynomial. (Concretely, we substitute x_i by s_i for $i < k+1$ and by $\sum s_j a_{j,i}$ for $i \geq k+1$.) The coefficients of f_Λ are bihomogeneous polynomials in the coefficients of f and the Plücker coordinates $a_{i,j}$ of Λ and define the universal Fano scheme in $\mathbb{P}^N \times \mathbb{G}(k,n)$.

For the other claims, let us rewrite the restriction map in line bundle notation. The restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \to H^0(\Lambda, \mathcal{O}_\Lambda(d))$ is a surjection (of K-vector spaces) and the hypersurfaces containing Λ are in one-to-one correspondence with the elements in the projective space over the kernel of the restriction map. This kernel has dimension $\binom{n+d}{d} - \binom{k+d}{d}$ by the dimension formula in linear algebra. This shows that the variety $\Phi(n,d,k)$ is irreducible of dimension $\dim(\mathbb{G}(k,n)) + \binom{n+d}{d} - \binom{k+d}{d} - 1$, which is the same as the claimed one because $\dim(\mathbb{G}(k,n)) = (k+1)(n-k)$. By Cramer's rule, $\Phi(n,k,d)$ is the projectivization of a vector bundle over $\mathbb{G}(k,n)$ (exercise) and so it is smooth. ∎

The dimension of the universal Fano scheme gives bounds on the dimension of the variety (rather the Fano scheme) of k-planes contained in a general hypersurface of degree d in $\mathbb{P}^n$.

5.5.2 Corollary. *Fix positive integers n ,d, and k with $k \leq n$ and set*

$$\varphi(n,d,k) = (k+1)(n-k) - \binom{k+d}{d}.$$

(a) If $\varphi(n,d,k) < 0$, then a general hypersurface of degree d in $\mathbb{P}^n$ does not contain any k-plane.
(b) If $\varphi(n,d,k) \geq 0$ and a general hypersurface of degree d in $\mathbb{P}^n$ contains a k-plane, then every hypersurface of degree d contains k-planes. Moreover, every irreducible component of the family of k-planes on a general hypersurface of degree d has dimension exactly $\varphi(n,d,k)$.

Proof. The main point is that $\varphi(n,d,k) = \dim(\Phi(n,d,k)) - N$, where N is the dimension of the projective space of hypersurfaces of degree d in $\mathbb{P}^n$. So part (a) is immediate because the projection to the first factor $\mathbb{P}^N$ of $\Phi(n,d,k)$ cannot be dominant. The assumption in part (b) that a general hypersurface contains a k-plane implies that the projection to the second factor is dominant. Since the universal Fano scheme is projective, the image of the projection is closed showing that every hypersurface contains a k-plane. The claim about the dimension of the fiber is a theorem that holds for general fibers of a dominant morphism. ∎

Definition. For a hypersurface $X \subset \mathbb{P}^n$ of degree d, the Fano scheme $F_k(X)$ of k-planes in X is the fiber over X of the projection of $\Phi(n,d,k)$ to the first factor $\mathbb{P}^N$.

Exercise 5.5.3. Show that the Fano scheme of lines on a generat quadratic hypersurface in $\mathbb{P}^3$ is the disjoint union of two curves (each of degree 2 in the Plücker embedding). (It suffices to do this computation for $\mathcal{V}_+(x_0x_3 - x_1x_2)$.) Then show that the Fano scheme $F_1(\mathcal{V}_+(x_0^2 - x_1^2 - x_2^2)) \subset \mathbb{G}(1,3)$ of lines on a quadratic cone in $\mathbb{P}^3$ is everywhere nonreduced.

We now rewrite the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(\Lambda, \mathcal{O}_\Lambda(d))$ in terms of the universal subbundle of $\mathbb{G}(k,n)$ which allows us to express the class $[F_k(x)] \in A(\mathbb{G}(k,n))$ of the Fano variety of X in terms of Chern classes.

5.5.4 Proposition. *Fix positive integers n, d, and k with $k \leq n$. Let $\mathcal{S}$ be the universal subbundle of $\mathbb{G}(k,n) = \mathrm{Gr}(k+1, n+1)$. A form f of degree d on $\mathbb{P}^n$ gives rise to a global section σ_f of the bundle $\mathrm{Sym}^d\,\mathcal{S}^*$ whose zero locus is the Fano scheme $F_k(\mathcal{V}_+(f))$.*

If the Fano scheme $F_k(X)$ has the expected codimension $r = \binom{k+d}{d}$ in $\mathbb{G}(k,n)$ for a hypersurface $X \subset \mathbb{P}^n$ of degree d, then

$$[F_k(X)] = c_r\left(\mathrm{Sym}^d\,\mathcal{S}^*\right) \in A^r(\mathbb{G}(k,n)).$$

Proof. This is essentially the same trick that we have used before to compute the Chern classes of the universal sub- and quotient-bundles of $\mathbb{G}(k,n)$. A polynomial f of degree d on $\mathbb{P}^n$ is an element of $\mathrm{Sym}^d V^*$, where $V = K^{n+1}$. Given $[\Lambda] \in \mathbb{G}(k,n)$, we can restrict such a polynomial f to $\Lambda \subset \mathbb{P}^n$, which defines the restriction map $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \longrightarrow H^0(\Lambda, \mathcal{O}_\Lambda(d))$. The target vector space $H^0(\Lambda, \mathcal{O}_\Lambda(d))$ is $\mathrm{Sym}^d \Lambda^*$. (In fact, the restriction map $\mathrm{Sym}^d V^* \longrightarrow \mathrm{Sym}^d \Lambda^*$ is the map induced on symmetric powers by the linear map on dual spaces coming from the inclusion $\Lambda \subset V$.) With this point-wise definition, we globally get the map $\mathrm{Sym}^d V^* \longrightarrow H^0(\mathbb{G}(k,n), \mathrm{Sym}^d\,\mathcal{S}^*)$. Denote by σ_f the image of $f \in \mathrm{Sym}^d V^*$ under this map. We can check that this global section σ_f of $\mathrm{Sym}^d\,\mathcal{S}^*$ defines the Fano scheme $F_k(\mathcal{V}_+(f))$ locally on an open cover of $\mathbb{G}(k,n)$. But on an affine open U_Γ, the section σ_f defines $F_k(\mathcal{V}_+(f))$ by construction (see the computation above showing that $F_k(X)$ is closed in the proof of Proposition 5.5.1).

If we know that the codimension is correct, then the appropriate Chern class is the degeneracy locus of the section (see Theorem 5.2.3(b)), which is the Fano scheme. ■

With this result, we can compute the number of lines on a general cubic surface in $\mathbb{P}^3$.

5.5.5 Corollary. *The number of lines on a general cubic surface in $\mathbb{P}^3$ is 27.*

Proof. Let us first do the Chern class computation. By the above result, we need to compute $c_r(\mathrm{Sym}^3\,\mathcal{S}^*) \in A^r(\mathbb{G}(1,3))$, where r is the expected codimension $\binom{4}{3} = 4$ of the Fano scheme of lines on a cubic. The Chern class of $\mathcal{S}^*$ on $\mathbb{G}(1,3)$ is $c(\mathcal{S}^*) = 1 + \sigma_1 + \sigma_{1,1}$ by Proposition 5.4.7. To compute $c_4(\mathrm{Sym}^3\,\mathcal{S}^*)$, we use the splitting principle to compute $c_4(\mathrm{Sym}^3(\mathcal{L} \oplus \mathcal{M}))$. Write $c(\mathcal{L}) = 1 + \alpha$ and $c(\mathcal{M}) = 1 + \beta$. Since we want $c(\mathcal{L} \oplus \mathcal{M}) = 1 + \sigma_1 + \sigma_{1,1}$, Whitney's formula implies $\alpha + \beta = \sigma_1$ and $\alpha \cdot \beta = \sigma_{1,1}$.

The fourth Chern class of $\mathrm{Sym}^3(\mathcal{L} \oplus \mathcal{M})$ is determined by the splitting

$$\mathrm{Sym}^3(\mathcal{L} \oplus \mathcal{M}) = \mathcal{L}^3 \oplus (\mathcal{L}^2 \otimes \mathcal{M}) \oplus (\mathcal{L} \otimes \mathcal{M}^2) \oplus \mathcal{M}^3,$$

which implies by Whitney's formula

$$c(\mathrm{Sym}^3(\mathcal{L} \oplus \mathcal{M})) = (1 + 3\alpha) \cdot (1 + 2\alpha + \beta) \cdot (1 + \alpha + 2\beta) \cdot (1 + 3\beta).$$

So the top Chern class of this vector bundle is given by

$$c_4(\mathrm{Sym}^3(\mathcal{L} \oplus \mathcal{M})) = 3\alpha \cdot (2\alpha + \beta) \cdot (\alpha + 2\beta) \cdot 3\beta.$$

Simplifying this equation and using $\alpha + \beta = \sigma_1$ and $\alpha \cdot \beta = \sigma_{1,1}$, we get

$$c_4(\mathrm{Sym}^3(\mathcal{L} \oplus \mathcal{M})) = 9 \cdot \sigma_{1,1} \cdot (2\sigma_1^2 + \sigma_{1,1}) = 27\sigma_{2,2}.$$

By the splitting principle, the same identity applies to $\mathcal{S}^*$ so that we get

$$\deg(c_4(\mathrm{Sym}^3 \mathcal{S}^*)) = \deg(27\sigma_{2,2}) = 27.$$

This computation shows that every cubic surface must contain some lines, simply because this class is non-zero. Indeed, otherwise, a general cubic would not contain any lines and the degeneracy locus of a general section σ_g would be empty. ∎

So far, to be precise, this method of proof only shows that a general cubic surface in $\mathbb{P}^3$ contains finitely many lines and that their number is 27 if counted with multiplicity. The question of whether there are actually 27 distinct lines on such a surface is related to the reducedness of the degeneracy locus which is to say the Fano scheme $F_1(\mathcal{V}_+(f))$. With a little more theory, it follows that the Fano scheme $F_1(S)$ for a smooth cubic surface $S \subset \mathbb{P}^3$ is always 0-dimensional and reduced so that every such surface contains precisely 27 distinct lines.

The main result in this direction is the following.

5.5.6 Theorem. *Let $\Lambda \subset X \subset \mathbb{P}^n$ be a k-plane in a smooth variety $X \subset \mathbb{P}^n$ and write $[\Lambda]$ for the corresponding point in $F_k(X) \subset \mathbb{G}(k,n)$. The Zariski tangent space of $F_k(X)$ at $[\Lambda]$ is isomorphic to $H^0(L, \mathcal{N}_{L/X})$.*

We will not give a proof of this statement. The proof in 3264 uses deformation theory. For cubic surfaces, we can do a direct computation to show that $F_1(X)$ is reduced for a smooth cubic surface $X \subset \mathbb{P}^3$.

5.5.7 Proposition. *Let X be a smooth cubic surface in $\mathbb{P}^3$ and let $\Lambda \subset X$ be a line in $\mathbb{P}^3$ so that $(X, [\Lambda]) \in \Phi(3,3,1)$. Then the differential $d\pi \colon T_{(X,[\Lambda])}\Phi(3,3,1) \to T_X\mathbb{P}^{19}$ is surjective. In particular, the fiber $\pi^{-1}(X)$ intersects $\Phi(3,3,1)$ transversely at $(X, [\Lambda])$ which implies that $F_1(X)$ is reduced at $(X, [\Lambda])$.*

Proof. After a change of coordinates, we assume that Λ is the rowspan of

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

which is the origin in the affine chart $U_\Gamma \subset \mathbb{G}(1,3)$ for $\Gamma = \mathrm{span}\{e_2, e_3\}$, the coordinate subspace spanned by the last two coordinate vectors in K^4. Every point of U_Γ is the rowspan of a unique matrix of the form

$$\begin{pmatrix} 1 & 0 & a_1 & a_2 \\ 0 & 1 & b_1 & b_2 \end{pmatrix}.$$

Let $F_c = \sum c_\alpha x^\alpha \in K[x_0, x_1, x_2, x_3]_3$ be the cubic equation given by the coefficient vector $c \in \mathbb{P}^{19}$. On $U_\Gamma \cong \mathbb{A}^4$ the condition that the subspace with coordinates (a_1, a_2, b_1, b_2) is contained in $\mathcal{V}_+(F_c)$ simply means that the restriction of F_c to it is 0. Concretely, these are 4 algebraic constraints obtained by expanding

$$F_c(s_0, s_1, s_0a_1 + s_1b_1, s_0a_2 + s_1b_2) = g_0(a, b, c)s_0^3 + g_1(a, b, c)s_0^2s_1 + g_2(a, b, c)s_0s_1^2 + g_3(a, b, c)s_1^3$$

is the zero polynomial. The polynomials g_i depend on c and the Plücker coordinates (a_1, a_2, b_1, b_2) and they are the defining equations of $\Phi(3, 3, 1) \cap (\mathbb{P}^{19} \times U_\Gamma)$. So the tangent space to $\Phi(3, 3, 1)$ at $(X, [\Lambda])$ is the kernel of the Jacobian

$$J = \left(\frac{\partial g_i}{\partial z_j}\right)_{i=1,2,3,4;j=0,\ldots,23}$$

where $z_j = c_{\alpha_i}$ for $i = 0, \ldots, 19$ and some monomial order $\alpha_0, \ldots, \alpha_{19}$ on the cubic monomials in variables x_0, x_1, x_2, x_4 and $z_{20} = a_1$, $z_{21} = a_2$, $z_{22} = b_1$, and $z_{23} = b_2$. Let $(X, [\Lambda])$ be a pair of a smooth cubic surface $X = \mathcal{V}_+(f) \subset \mathbb{P}^3$ and the line $\Lambda \subset X$, $\Lambda = \operatorname{span}\{e_0, e_1\}$ corresponding to the origin in U_Γ. In the Jacobian $J(X, [\Lambda])$ evaluated at this pair, the last 4×4 block has full rank (which implies the claim). Let's compute the entries of $J(X, [\Lambda])$ in this 4×4 block. Write G for the vector $(g_0, g_1, g_2, g_3)^\top$.

$$\begin{aligned}\partial_{a_1} G(c, [\Lambda]) &= \partial_{a_1} F_c(s_0, s_1, s_0a_1 + s_1b_1, s_0a_2 + s_1b_2)(c, [\Lambda]) \\ &= s_0\frac{\partial F}{\partial x_2}(s_0, s_1, s_0a_1 + s_1b_1, s_0a_2 + s_1b_2)(c, [\Lambda])\end{aligned}$$

An analogous computation for the other three variables shows that the 4×4 block $J_{(a_1,a_2,b_1,b_2)}(X, [\Lambda])$ corresponding to the Grassmannian is

$$\left(s_0\tfrac{\partial f}{\partial x_2}(s_0, s_1, 0, 0) \quad s_0\tfrac{\partial f}{\partial x_3}(s_0, s_1, 0, 0) \quad s_1\tfrac{\partial f}{\partial x_2}(s_0, s_1, 0, 0) \quad s_1\tfrac{\partial f}{\partial x_3}(s_0, s_1, 0, 0)\right)$$

If this block did not have full rank there would be a nontrivial linear relation among the columns, say

$$(\lambda_1 s_0 + \lambda_3 s_1)\frac{\partial f}{\partial x_2}(s_0, s_1, 0, 0) + (\lambda_2 s_0 + \lambda_4 s_1)\frac{\partial f}{\partial x_3}(s_0, s_1, 0, 0) = 0.$$

This is an identity of cubic forms in $K[s_0, s_1]$. By factoring each summand into linear factors, we see that the two quadratic forms $\partial_{x_i} f(s_0, s_1, 0, 0)$ must have a common linear factor. This corresponds to a point $p = (p_0, p_1, 0, 0) \in \Lambda$ where both derivatives of f vanish: $\partial_{x_2} f(p) = 0 = \partial_{x_3} f(p)$. Since $\Lambda \subset X$, we also have $f(x_0, x_1, 0, 0) = 0$, which implies $\partial_{x_1} f(p) = 0$ and $\partial_{x_2} f(p) = 0$. So overall, we have found that the gradient of f is 0 at p, which contradicts our assumption that $X = \mathcal{V}_+(f)$ is smooth.

Since this block has full rank, the intersection of $T_{(X,[\Lambda])}\Phi(3, 3, 1)$ with $\{0\} \times K^4$ is trivial showing that the differential $d\pi$ is surjective as claimed. ∎

Let us take a look at when we expect $F_1(X)$ to be 0-dimensional for a hypersurface $X \subset \mathbb{P}^n$. Heuristically, a defining equation $g \in K[x_0, x_1, \ldots, x_n]$ of a hypersurface $X \subset \mathbb{P}^n$ gives a global section σ_g of $\operatorname{Sym}^d \mathcal{S}^*$ on $\mathbb{G}(1, n)$ as in Proposition 5.5.4 that defines the Fano scheme $F_1(X)$ if it has the expected codimension $(d + 1)$. If we want this to be dimension 0 in $\mathbb{G}(1, n)$, which has dimension $2(n-1)$, then we need $d = 2n-3$. For $n = 3$, this is the case of a cubic surface in $\mathbb{P}^3$. The next case is a quintic threefold in $\mathbb{P}^4$ (then a fourfold of degree 7 in $\mathbb{P}^5$ and so on). The same approach as for cubic surfaces works again to count the number of lines on these hypersurfaces. However, the question of transversality is more complicated. The Fano scheme $F_1(X)$ is not necessarily reduced anymore if the hypersurface is smooth. To show how the count for cubic surfaces can be generalized, we do the quintic threefold case explicitly. For this, we compute $c_{d+1}(\operatorname{Sym}^d \mathcal{S}^*)$ as before, using the splitting principle. For this, we need to know how to intersect Schubert classes on $\mathbb{G}(1,n)$. The intersection product in this case follows from Pieri's formula.

On $\operatorname{Gr}(2, n+1)$, Schubert varieties are indexed by sequences (a_1, a_2) of nonnegative integers of length 2 with $n-1 \geq a_1 \geq a_2 \geq 0$. The Schubert variety Σ_{a_1,a_2} is

$$\Sigma_{a_1,a_2}(\mathcal{V}) = \{[\Lambda] \in \operatorname{Gr}(2, n+1) \colon \Lambda \cap V_{n-a_1} \neq \{0\} \text{ and } \Lambda \subset V_{n+1-a_2}\}.$$

5.5.8 Proposition. *Let (a_1, a_2) and (b_1, b_2) be two Schubert indices for* $\operatorname{Gr}(2, n+1)$*. Assume that $a_1 - a_2 \geq b_1 - b_2$. Then the following identity holds.*

$$\begin{aligned}\sigma_{a_1,a_2} \cdot \sigma_{b_1,b_2} = &\sum_{|c|=|a|+|b|, a_1+b_1 \geq c_1 \geq a_1+b_2} \sigma_{c_1,c_2} \\ &\sigma_{a_1+b+1,a_2+b_2} + \sigma_{a_1+b_1-1,a_2+b_2+1} + \ldots + \sigma_{a_1+b_2,a_2+b_1}\end{aligned}$$

If the index c_1 is larger than $n+1-2 = n-1$, the class σ_{c_1,c_2} is 0.

Proof. First, set $b_1 = b_2 = b$. Since the indices are equal, the jumps in dimension in the induced flag on Λ are consecutive for all $[\Lambda] \in \Sigma_{b,b}$ so that this really means $\Lambda \subset V_{n+1-b}$. Let us intersect the two Schubert varieties $\Sigma_{a_1,a_2}(\mathcal{V})$ and $\Sigma_{b,b}(\mathcal{W})$ with respect to general flags $\mathcal{V}$ and $\mathcal{W}$ in K^{n+1}: this intersection contains all $[\Lambda]$ such that $\Lambda \cap V_{n-a_1} \neq \{0\}$, $\Lambda \subset V_{n+1-a_2}$ and $\Lambda \subset W_{n+1-b}$. These conditions are equivalent to $\Lambda \cap (V_{n-a_1} \cap W_{n+1-b}) \neq \{0\}$ and $\Lambda \subset (V_{n+1-a_2} \cap W_{n+1-b})$. Since the flags are transversal, the intersections $V_i \cap W_j$ have the expected dimension meaning $\operatorname{codim}(V_{n-a_1} \cap W_{n+1-b}) = a_1 + b + 1$ and $\operatorname{codim}(V_{n+1-a_2} \cap W_{n+1-b}) = a_2 + b$. Set $U_{n-a_1-b} = V_{n-a_1} \cap W_{n+1-b}$ and $U_{n+1-a_2-b} = V_{n+1-a_2} \cap W_{n+1-b}$. These conditions show that the intersection $\Sigma_{a_1,a_2}(\mathcal{V}) \cap \Sigma_{b,b}(\mathcal{W})$ is the Schubert variety $\Sigma_{a_1+b,a_2+b}(U_{n-a_1-b}, U_{n+1-a_2-b})$. By transversality in the Grassmannian (say characteristic 0 and Kleiman's Theorem), we get the desired identity

$$\sigma_{a_1,a_2} \cdot \sigma_{b,b} = \sigma_{a_1+b,a_2+b}$$

in $A(\mathbb{G}(1,n))$. From this case, we derive the general case by Pieri's rule using the following trick. With what we have just shown, we can write

$$\begin{aligned}\sigma_{a_1,a_2} \cdot \sigma_{b_1,b_2} &= (\sigma_{a_1-a_2,0} \cdot \sigma_{a_2,a_2}) \cdot (\sigma_{b_1-b_2,0} \cdot \sigma_{b_2,b_2}) \\ &= \sigma_{a_1-a_2,0} \cdot \sigma_{b_1-b_2,0} \cdot \sigma_{a_2+b_2,a_2+b_2}\end{aligned}$$

using associativity and commutativity and the above multiplication rule. The product of the first two terms is given by Pieri's rule Proposition 4.5.1 as

$$\sigma_c \cdot \sigma_d = \sigma_{c+d,0} + \sigma_{c+d-1,1} + \ldots + \sigma_{c,d}$$

for any positive integers $c \geq d$. This implies the claim using the above multiplication rule one more time. ■

5.5.9 Proposition. *A general quintic threefold $X = \mathcal{V}_+(g) \subset \mathbb{P}^4$ (meaning that $g \in K[x_0, x_1, \dots, x_4]_5$ is a general polynomial) contains exactly 2875 distinct lines.*

Proof. We approach this the same way as for cubic surfaces in Corollary 5.5.5: we first compute $c_6(\operatorname{Sym}^5 \mathcal{S}^*)$, where $\mathcal{S}$ is the universal subbundle of $\mathbb{G}(1,4)$, by the splitting principle. So let $\mathcal{E} = \mathcal{L} \oplus \mathcal{M}$ be a direct sum of two line bundles with Chern classes $c(\mathcal{L}) = 1+\alpha$ and $c(\mathcal{M}) = 1+\beta$. Then $\operatorname{Sym}^5(\mathcal{E})$ decomposes as before

$$\operatorname{Sym}^5(\mathcal{L} \oplus \mathcal{M}) = \mathcal{L}^5 \oplus (\mathcal{L}^4 \otimes \mathcal{M}) \oplus (\mathcal{L}^3 \otimes \mathcal{M}^2) \oplus (\mathcal{L}^2 \otimes \mathcal{M}^3) \oplus (\mathcal{L} \otimes \mathcal{M}^4) \oplus \mathcal{M}^5.$$

This implies that the Chern class of $\operatorname{Sym}^5(\mathcal{E})$ is

$$c(\operatorname{Sym}^5(\mathcal{L} \oplus \mathcal{M})) = (1+5\alpha)(1+4\alpha+\beta)(1+3\alpha+2\beta)(1+2\alpha+3\beta)(1+\alpha+4\beta)(1+5\beta).$$

We are interested in the term of degree 6 expressed in terms of the elementary symmetric polynomials $\alpha + \beta$ and $\alpha\beta$ (because, as before, we want to apply the resulting formula to $\mathcal{S}^*$ instead of $\mathcal{E}$ by the splitting principle, which corresponds to the substitutions $\alpha + \beta = \sigma_1$ and $\alpha\beta = \sigma_{1,1}$). To multiply this out, we use the two formulas

$$\begin{aligned}(4\alpha+\beta)(\alpha+4\beta) &= 4(\alpha+\beta)^2 + 9\alpha\beta \\ (3\alpha+2\beta)(2\alpha+3\beta) &= 6(\alpha+\beta)^2 + \alpha\beta\end{aligned}$$

These computations show the following formula for the desired Chern class:

$$\begin{aligned}c_6(\operatorname{Sym}^5 \mathcal{S}^*) &= 25\sigma_{1,1} \cdot (4\sigma_1^2 + 9\sigma_{1,1}) \cdot (6\sigma_1^2 + \sigma_{1,1}) \\ &= 225 \cdot \sigma_{1,1}^3 + 1450 \cdot \sigma_{1,1}^2 \cdot \sigma_1^2 + 600 \cdot \sigma_{1,1} \cdot \sigma_1^4.\end{aligned}$$

To finish, we have to see how many points each of the three monomials in Schubert classes contribute. We know that $\sigma_{1,1}^3$ is the class of a point (see Corollary 4.3.8). To evaluate $\sigma_1^2\sigma_{1,1}^2$, we apply the above multiplication rule in $A(\mathbb{G}(1,n))$ (or Pieri's formula) first to $\sigma_1^2 = \sigma_{1,1} + \sigma_2$ so that $\sigma_1^2\sigma_{1,1}^2 = \sigma_{1,1}^3 + \sigma_{1,1}^2\sigma_2$. Then we compute $\sigma_{1,1}^2\sigma_2$, which turns out to be 0. So the monomial $\sigma_1^2\sigma_{1,1}^2$ has degree 1 (meaning it is the class of one point). The monomial $\sigma_{1,1}\sigma_1^4$, however, has degree 2: with the computation of σ_1^2, this monomial is equal to $\sigma_{1,1}(\sigma_{1,1}^2 + 2\sigma_{1,1}\sigma_2 + \sigma_2^2)$. The mixed term $\sigma_{1,1}\sigma_2$ becomes 0 when multiplied with $\sigma_{1,1}$. The first term becomes the class $\sigma_{1,1}^3$ of a point. The last term is $\sigma_{1,1}\sigma_2^2$, which is equal to $\sigma_{1,1}(\sigma_{3,1} + \sigma_{2,2})$. Here, the first term $\sigma_{1,1}\sigma_{3,1}$ is 0 and the second term $\sigma_{1,1}\sigma_{2,2} = \sigma_{3,3}$ is the class of a point (again, see Corollary 4.3.8). To summarize, this computation shows

$$\deg(c_6(\operatorname{Sym}^5 \mathcal{S}^*)) = 225 + 1450 + 2 \cdot 600 = 2875.$$

Since this class is non-zero, this implies that a general quintic threefold contains only finitely many lines. If counted with multiplicity, the number of lines is 2875. To show that $F_1(X)$ is reduced for a general quintic threefold requires some additional work (or the computation of an explicit example). ■

This computation of the degree of the appropriate Chern class is a little involved but can be done with a computer. The next few numbers are 698 005 lines on a hypersurface of degree 7 in $\mathbb{P}^5$, 305 093 061 lines in degree 9 in $\mathbb{P}^6$, 210 480 374 951 in degree 11 in $\mathbb{P}^7$ and so on.

Chapter 6

Parameter Spaces: Five Conics Problem

We want to exemplify the problem of choosing/finding a good parameter space on which to use the techniques of intersection theory to solve an enumerative problem. Our example problem is as follows. We are given five general, smooth conics $C_1, \dots, C_5 \subset \mathbb{P}^2$ defined by quadratic forms $Q_i \in K[x_0, x_1, x_2]_2$ of rank 3. We want to count the number of conics defined by q such that $\mathcal{V}(q)$ is tangent to each C_i at some point. Let us begin naively: each C_i is defined by a quadratic form so that $Q_i \in \mathbb{P}^5 = \mathbb{P}(K[x_0, x_1, x_2]_2)$. We are looking for an element $q \in \mathbb{P}^5$ as well. Fix one of the conics C_i and consider the incidence

$$\Sigma_i = \{(q, p) \in \mathbb{P}^5 \times C_i : q(p) = 0 \text{ and } T_p\mathcal{V}_+(q) \supset T_pC_i\}.$$

Let π_2 be the projection to the second factor. Then the fiber over any point $p \in C_i$ is a linear space in $\mathbb{P}^5$ of codimension 2. For example, it is defined by the 2×2 minors of the 2×3 matrix

$$M_i = \begin{pmatrix} \partial_{x_0} q(p) & \partial_{x_1} q(p) & \partial_{x_2} q(p) \\ \partial_{x_0} Q_i(p) & \partial_{x_1} Q_i(p) & \partial_{x_2} Q_i(p) \end{pmatrix}$$

which are linear in the coefficients of q. So Σ_i has dimension 4 and we expect $Z_i = \pi_1(\Sigma_i)$, the projection to the parameter space $\mathbb{P}^5$ for our conic q to be a hypersurface. We computed above (using the Riemann-Hurwitz formula for morphisms of algebraic curves) that the degree of Z_i is 6. This is the reason that people have arrived at the number $6^5 = 7776$ for the solution of our enumerative five conics problem. However, this number is incorrect due to excess intersection: every square of a linear form ℓ^2 for $\ell \in K[x_0, x_1, x_2]_1$ lies on every Z_i. Indeed, every line $\mathcal{V}_+(\ell)$ intersects C_i in two points. Let p be one of them. Then $(\ell^2, p) \in \Sigma_i$ and therefore $\ell \in Z_i$. It turns out that $Z_1 \cap Z_2 \cap \ldots \cap Z_5 = \nu_2(\mathbb{P}^2) \cup S$, where $\nu_2(\mathbb{P}^2)$ is the Veronese surface of squares in $\mathbb{P}^5$ and S is a finite set (a reduced 0-dimensional scheme) with 3264 elements.

One way to compute the length of S is to use the excess intersection formula that we will take a look at later. Another approach is to change the parameter space for q to make the excess intersection go away. Our naive parameter space $\mathbb{P}^5$, the space of coefficients of q, does not distinguish if the defined conic is smooth (so q has rank 3) or not. Other approaches keep more careful track of this and succeed at eliminating the above excess intersection.

The new problem that we get ourselves into by doing this is that we have to find the intersection ring on this new parameter space (or at least characterize a subring containing the classes that we want to intersect). In detail, we discuss the approach using projective duality where we construct the space of **complete conics** (or more generally, the space of complete quadrics). Other options are **Kontsevich space**, which is another approach to keep track of tangency essentially by considering maps $f\colon C \to \mathbb{P}^2$ instead of equations $q \in \mathbb{P}^5$. A general purpose approach is via blow ups: we can always blow up the parameter space along the excess intersection and hope to improve the situation. In our example, it turns out that the blow up of $\mathbb{P}^5$ along the Veronese surface $\nu_2(\mathbb{P}^2)$ of squares is isomorphic to the space of complete conics.

6.1. *The space of complete conics*

Let us assume that the characteristic of our algebraically closed field K is not 2. We start with an informal discussion and then supply details later. The idea is based on projective duality: for a curve $C \subset \mathbb{P}^2$, we construct the **dual curve** C^* as the Zariski closure as the set of tangent lines $[T_pC] \subset (\mathbb{P}^2)^*$ to smooth points $p \in C$. Generically, this is again a curve. In particular, we saw that C^* is again a conic if C is a conic of rank 3, namely the one defined by the inverse of the Gram matrix of a defining equation of C (see Example 1.3.1). However, if the conic $\mathcal{V}_+(Q)$ has rank 2, then it is the union of two lines so that $\mathcal{V}_+(Q)^*$ is 0-dimensional and consists of two points. The above definition of a dual curve does not make sense of $\mathcal{V}_+(\ell^2)$, a conic of rank 1, because this 1-dimensional scheme does not have any smooth point (it is nowhere reduced). It turns out that it is not clear how to uniquely assign a dual conic to a double line $\mathcal{V}_+(\ell^2)$. It turns out that we don't need to though.

To construct the space of complete conics, we start with

$$U = \{(C, C^*) \in \mathbb{P}^5 \times (\mathbb{P}^5)^* \colon C \subset \mathbb{P}^2 \text{ is a smooth conic, } C^* \text{ its dual}\}$$

and define the space of complete conics X as the closure

$$X = \overline{U} \subset \mathbb{P}^5 \times (\mathbb{P}^5)^*.$$

We will show below that X is a smooth and irreducible variety of dimension 5.

On this space, a smooth conic C_i defines again a divisor Z_{C_i}, this time as the closure of the set of conics $(C, C^*) \in U$ such that C is tangent to C_i at some point $p \in C_i$. It turns out that the divisors Z_{C_i} intersect transversely on X in 3264 distinct points, solving our problem.

6.1.1 Heuristics

To get a feeling for the space $X \subset \mathbb{P}^5 \times (\mathbb{P}^5)^*$ of complete conics, let us look closely at U and at a few points in the boundary by naive degeneration. On U, the pair (C, C^*) is uniquely determined by C or C^* because the dual curve of a smooth conic is given by the inverse of its Gram matrix. In particular, this implies that $(C^*)^* = C$ and that C is smooth if and only if C^* is smooth. The second point shows that the set U is invariant under exchanging $\mathbb{P}^5$ and $(\mathbb{P}^5)^*$, that is $(C, C^*) \mapsto (C^*, C)$.

The boundary of X (in the context of parameter spaces) is defined as $X \setminus U$ and therefore not a topological boundary in any sense: it consists of the points that are added by taking the closure of the "nice" set U to get a complete parameter space. Any point on the boundary is a limit of points in U (in the sense of a degeneration). We will do this part naively.

Let us begin with a family of smooth conics C_t approaching a conic C_0 of rank 2 in the limit. Explicitly, set

$$C = \{(t; x_0 : x_1 : x_2) \in B \times \mathbb{P}^2 \colon x_0^2 - x_1^2 - tx_2^2 = 0\},$$

where $B = \mathbb{A}^1$. (In deformation theory, we would usually take the base B to be a discrete valuation ring like the local ring of the origin in $\mathbb{A}^1$.) For $t = 0$, this conic is a union of lines $\mathcal{V}_+((x_0 + x_1)(x_0 - x_1))$ through the point $p = (0 : 0 : 1)$. Any collection of lines $\{L_t\}$ tangent to C_t for $t \neq 0$ approaches a line through p. Conversely, it is also true that any line through p is such a limit. The dual of C_0 in the sense of the space of complete conics is then, because it must be a conic, the double line in $(\mathbb{P}^2)^*$ of lines through p: $(\mathcal{V}_+(x_0^2 - x_1^2), \mathcal{V}_+((p^\perp)^2)) \in X$.

This picture becomes more interesting if the limit has rank 1. To study this case, we write our family of conics C_t as rational curves that is the image of morphisms $\varphi_t \colon \mathbb{P}^1 \to \mathbb{P}^2$ given by three homogeneous polynomials (f_t, g_t, h_t) of degree 2 on $\mathbb{P}^1$. If the conic C_t is smooth the three polynomials f_t, g_t, and h_t are linearly independent. In the limit, they parametrize a line so they are linearly dependent and span a 2-dimensional subspace W of $H^0(\mathcal{O}_{\mathbb{P}^1}(2))$. Assume for now that this subspace W has no basepoints (which means that for every $p \in \mathbb{P}^1$ there is a polynomial $f \in W$ with $f(p) \neq 0$). Then we have a morphism $\varphi_W \colon \mathbb{P}^1 \to \mathbb{P}^1 = \mathbb{P}W^*$ sending x to $(f \mapsto f(x)) \in W^*$. Riemann-Hurwitz (or, in this special case, the quadratic discriminant) shows that this map is ramified at two points, say u and v. Let $p = \varphi_0(u)$ and $q = \varphi_0(v)$ be their images in $\mathbb{P}^2$. The "dual conic" is then $p^\perp \cup q^\perp \subset (\mathbb{P}^2)^*$, the set of lines through p and the lines through q. The heuristic is based on the observation that the linear system $W_t \subset H^0(\mathcal{O}_{\mathbb{P}^1}(2))$ obtained from C_t by a general projection with center $r \in \mathbb{P}^2$ approaches $W = W_0$ so that any tangent line to C_t through r approaches a line through p (and r) or a line through q (and again r).

In case that the two ramification points u and v of φ_W are the same, the dual C' is a double line.

This geometric heuristic suggests that there are four types of complete conics $(C, C') \in X$.

(a) $(C, C') \in U$: those are the complete conics where both C and C' are smooth and $C' = C^*$,
(b) $C = L_1 \cup L_2$ has rank 2 and $C' = 2p^*$ is the double line dual to the point $p = L_1 \cap L_2$,
(c) $C = 2L$ has rank 1 and $C' = p^* \cup q^*$ is the union of lines dual to two points p and q on L,
(d) $C = 2L$ has rank 1 and $C' = 2p^*$ is the double line dual to a point p on L.

The cases (b) and (c) are interchanged by the symmetry of X switching C and C'.

6.1.2 Rigorous approach

For the proofs in this section, we identify a quadratic form Q on a finite-dimensional K-vector space V with its representing Gram matrix in a coordinate free way. Since $\operatorname{char}(K) \neq 2$, the following three K-vector spaces are isomorphic.

$$\begin{aligned}
BF &= \{\varphi \colon V \to V^* \colon (\varphi(v))(w) = (\varphi(w))(v) \text{ for all } v, w \in V\} \\
QF &= \{q \colon V \to K \colon q(\lambda x) = \lambda^2 x \text{ for all } \lambda \in K \text{ and } x \in V \text{ and } b(x, y) \text{ is bilinear}\} \\
\operatorname{Sym}^2 V^* &= \operatorname{span}\{\ell \otimes \ell \colon \ell \in V^*\} \subset V^* \otimes V^*
\end{aligned}$$

where $b(x, y) = q(x + y) - q(x) - q(y)$ is defined by polarization. The isomorphism $BF \to QF$ takes φ to the quadratic form $x \mapsto (\varphi(x))(x)$. Conversely, a quadratic form q gives the symmetric linear map $v \mapsto (w \mapsto b(v, w))$ (sometimes written as $v \mapsto b(v, -)$). Since we assume that the characteristic of K is not 2, the space QF of quadratic forms is isomorphic to the space $\mathrm{Sym}^2 V^*$ of quadratic polynomials on V. Concretely, $\ell \otimes \ell$ gives the quadratic form $x \mapsto \ell(x)^2$ on V.

The quadratic hypersurface defined by a quadratic form Q identified with a symmetric linear map $\varphi\colon V \to V^*$ is

$$\mathcal{V}_+(Q) = \{v \in \mathbb{P}V : Q(v) = 0\} = \{v \in \mathbb{P}V : (\varphi(v))(v) = 0\}.$$

We give a description of the dual variety in terms of the **cofactor map** $\varphi^c\colon W \to V$ defined by a linear map $\varphi\colon V \to W$ between vector spaces of the same dimension n. In fixed bases of V and W, the cofactor map is represented by the cofactor matrix. The cofactor matrix of A is the transpose of the adjugate matrix $\det(A)A^{-1}$. Abstractly, it therefore satisfies $\varphi \circ \varphi^c = \det(\varphi)\mathrm{id}_W$ and $\varphi^c \circ \varphi = \det(\varphi)\mathrm{id}_V$. It is the map

$$W \cong \Lambda^{n-1}W^* \to \Lambda^{n-1}V^* \cong V$$

given by $\Lambda^n\varphi^*$, where the identification of W with $\Lambda^{n-1}W^*$ depends on the choice of a vector in $\Lambda^n W^*$. This is the usual identification: for an n-dimensional K-vector space U and a choice of $d \in \Lambda^n U$, we get an isomorphism of $\Lambda^n U$ with K by sending $\lambda \in K$ to λd. An element $L \in \Lambda^{n-1}U$ then gives a linear form on U by sending $u \in U$ to $u \wedge L \in \Lambda^n U \cong K$. So after choosing $d \in \Lambda^n U$, we get an isomorphism $\Lambda^{n-1}U \to U^*$. In coordinates, with respect to a basis $(e_1, e_2, \dots, e_n)$ of U, choosing $d = e_1 \wedge e_2 \wedge \ldots \wedge e_n$ corresponds to the normalization $\det(I_n) = 1$ that the identity matrix has determinant 1.

Using this notation and approach by abstract multilinear algebra, we can describe the duality of quadratic hypersurfaces as follows.

6.1.1 Proposition. *Let V be an $(n+1)$-dimensional K-vector space and $Q \subset \mathbb{P}(V)$ a quadratic hypersurface corresponding to the symmetric linear map $\varphi\colon V \to V^*$. The tangent hyperplane to Q at a point $[v] \in Q$ is*

$$\mathbb{P}T_{[v]}Q = \{w \in \mathbb{P}(V) : (\varphi(v))(w) = 0\}.$$

So the dual of Q is

$$Q^* = \{\varphi(v) \in \mathbb{P}(V^*) : [v] \in Q \text{ and } \varphi(v) \neq 0\}.$$

In particular, if Q is nonsingular (so that φ is an isomorphism), then Q^* is the image $\varphi(Q)$ of Q under the map $\varphi\colon \mathbb{P}(V) \to \mathbb{P}(V^*)$ induced on projective spaces by φ and Q^* is the quadratic hypersurface corresponding to the cofactor map φ^c. If the rank of φ is n (so that φ has a 1-dimensional kernel), then the quadratic hypersurface Q^c corresponding to the cofactor map φ^c is the unique double hyperplane containing Q^*.

Proof. For any $w \in V \setminus \{0\}$, the line span $v, w \subset \mathbb{P}(V)$ spanned by v and w is tangent to Q at v if and only if the restriction of Q to it has (at least) a double zero at v meaning that

$$Q(v + \varepsilon w) = (\varphi(v + \varepsilon w))(v + \varepsilon w) = 0 \bmod (\varepsilon^2).$$

Using $(\varphi(v))(v) = 0$ as well as symmetric of φ and $\operatorname{char}(K) \neq 2$, this simplifies to

$$(\varphi(v))(w) = 0.$$

This implies the description of the tangent space in the claim as well as the fact that $Q^* = \varphi(Q)$.

From now on, we assume that $\dim(\ker(\varphi)) \leq 1$ which is equivalent to the cofactor map φ^c not being the zero map. The quadric Q^c defined by the cofactor map is the set of all $w \in \mathbb{P}(V^*)$ such that $(\varphi^c(w))(w) = 0$. Using $\varphi^c \circ \varphi = \det(\varphi)\mathrm{id}_V$, this implies $\varphi(Q) \subset Q^c$ because

$$(\varphi^c(\varphi(v)))(\varphi(v)) = \det(\varphi)(\varphi(v), v) = 0.$$

Now suppose that the rank of φ is equal to the dimension of V so that φ is an isomorphism. Then $\varphi(Q) \subset \mathbb{P}(V^*)$ is again a quadratic hypersurface and the containment $\varphi(Q) \subset Q^c$ implies $\varphi(Q) = Q^c$ as claimed. If $\dim(\ker(\varphi)) = 1$, then $\varphi^c \circ \varphi = 0$ but φ^c is not the zero map. So φ^c is a linear map of rank 1. In terms of the associated quadratic hypersurface, this means that Q^c is a double hyperplane. The variety Q^* is a quadratic hypersurface contained in a hyperplane in $\mathbb{P}(V^*)$, namely the hyperplane dual to the unique singular point of Q, its cone point. Since $\varphi(Q) \subset Q^c$, we get the claim in this case as well. ■

6.1.2 Corollary. *Two smooth quadratic hypersurfaces Q_1 and Q_2 have the same tangent hyperplane $\mathcal{V}_+(\ell) \subset \mathbb{P}(V)$ at a point of intersection $[v] \in Q_1 \cap Q_2$ if and only if the dual hypersurfaces Q_1^* and Q_2^* have the common tangent hyperplane $\mathcal{V}_+(p) \subset \mathbb{P}(V^*)$ at the point $[\ell] \in Q_1^* \cap Q_2^*$.*

Proof. Let φ_i be the symmetric linear map corresponding to Q_i. The above proposition then shows that $[\ell] = [\varphi_i(v)] \in \mathbb{P}(V^*)$. Using that φ_i has full rank, we get $[\varphi_i^c(\ell)] = [\varphi_i^c(\varphi_i(v))] = [\det(\varphi_i)v] = [v] \in \mathbb{P}(V)$. The description of tangent spaces now implies the claim. ■

From now on, we are back in the projective plan (meaning that $\dim(V) = 3$).

6.1.3 Proposition. *Let* $\dim(V) = 3$. *The variety of complete conics*

$$X \subset \mathbb{P}(\operatorname{Sym}^2 V^*) \times \mathbb{P}(\operatorname{Sym}^2 V) \cong \mathbb{P}^5 \times (\mathbb{P}^5)^*$$

is smooth and irreducible. Its vanishing ideal I in the coordinates (φ, ψ) of pairs of symmetric matrices $\varphi\colon V \longrightarrow V^*$ and $\psi\colon V^* \longrightarrow V$ is generated by the eight bilinear equations specifying that the product $\psi \circ \varphi$ has diagonal entries equal to one another and off-diagonal entries equal to 0.

Proof. Using any computer algebra system, it is straightforward to check that the ideal I in the claim is prime and defines a 5-dimensional variety $Y = \mathcal{V}_+(I)$ in $\mathbb{P}^5 \times (\mathbb{P}^5)^*$.

We next show $X \subset Y$: X is by definition the closure of the set U of pairs $(\varphi, \psi) \in \mathbb{P}^5 \times (\mathbb{P}^5)^*$ where φ has rank 3 and, using Proposition 6.1.1, $\psi = \varphi^{-1}$, at least up to scaling. Each such pair satisfies the equations generating I so that we get $X \subset Y$.

Since the set U has dimension 5 as well, the chain of inclusion $U \subset X \subset Y$ combined with the irreducibility of Y also gives the other inclusion $Y \subset X$.

Let us now discuss smoothness of Y. The tangent space to Y at a point (φ, ψ) is the set of pairs (α, β) such that the matrix

$$(\psi + \varepsilon\beta) \circ (\varphi + \varepsilon\alpha) \bmod (\varepsilon^2)$$

has equal diagonal entries and off diagonal entries equal to 0. If $(\varphi, \psi) \in U$ so that both φ and ψ have rank 3, then the linear part of this equation says that $\psi \circ \alpha + \beta \circ \varphi$ had to be a diagonal matrix with equal diagonal entries, say λI_3. Then we can solve for α and get

$$\alpha = \lambda \varphi^{-1} + \psi^{-1} \circ \beta \circ \varphi.$$

Since β is chosen from a 5-dimensional tangent space and α is only determined up to scaling, the set of pairs (α, β) satisfying these equations is 5-dimensional so that Y is smooth along U.

We discuss the case that φ has rank 2. The equations in I then imply that ψ is, up to scaling, equal to the cofactor map φ^c. Up to an orthogonal change of coordinates and rescaling, we can assume that

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ so that } \varphi^c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In this case, the matrix $\varphi^c \circ \alpha + \beta \circ \varphi$ must again be diagonal with equal diagonal entries, say λI_3. Computing with the above matrices (as well as $\alpha = (\alpha_{i,j})$ and $\beta = (\beta_{i,j})$) results in

$$\begin{pmatrix} \beta_{1,1} & \beta_{1,2} & 0 \\ \beta_{2,1} & \beta_{2,2} & 0 \\ \alpha_{3,1} + \beta_{3,1} & \alpha_{3,2} + \beta_{3,2} & \alpha_{3,3} \end{pmatrix} = \lambda I_3.$$

The diagonal imposes 2 conditions up to recaling of α and β and we have three linear conditions in the off-diagonal, namely $\beta_{1,2} = 0$ (which, by symmetry of β implies $\beta_{2,1} = 0$), $\alpha_{3,1} + \beta_{3,1} = 0$, and $\alpha_{3,2} + \beta_{3,2} = 0$. Since these five linear conditions are linearly indepenndent, they cut out a 5-dimensional space that contains the tangent space to X at (φ, ψ), meaning that X is smooth at such points.

If both φ and φ have rank 1, the linear term $\psi \circ \alpha + \beta \circ \varphi$ has rank at most 2. Since the off-diagonal entries of this matrix must be 0, the diagonal entries must also be 0 for the above condition to hold (modulo ε^2), which means that the equation

$$\psi \circ \alpha + \beta \circ \varphi = 0$$

holds for every (α, β) in the tangent space to Y at (φ, ψ). Up to an orthogonal change of coordinates on V followed by rescaling, we can assume

$$\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } \psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Writing $\alpha = (\alpha_{i,j})_{1 \le i,j \le 3}$ and $\beta = (\beta_{i,j})_{1 \le i,j \le 3}$, the above condition becomes

$$0 = \psi \circ \alpha + \beta \circ \varphi = \begin{pmatrix} \beta_{1,1} & 0 & 0 \\ \beta_{2,1} + \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} \\ \beta_{3,1} & 0 & 0 \end{pmatrix}.$$

which imposes 5 linearly independent linear conditions $\beta_{1,1} = 0$, $\beta_{2,1} + \alpha_{2,1} = 0$, $\alpha_{2,2} = 0$, $\alpha_{2,3} = 0$, and $\beta_{3,1} = 0$ on the entries of the pair (α, β) of matrices. So the tangent space of Y has codimension 5 at such a pair (φ, ψ). Since X has dimension 5, this implies smoothness at such a point where both φ and ψ have rank 1. ∎

Exercise 6.1.4. Complete the corresponding proof in Eisenbud, Harris: 3264 and all that (Proposition 8.3).

This proposition gives a complete characterization of the points in the space of complete quadrics for $\dim(V) = 3$.

6.1.5 Corollary. *If $(\varphi, \psi) \in X$, then one of the following holds.*

(a) *If φ has rank 3, the ψ is the inverse of φ up to scaling.*
(b) *If φ has rank 2, then ψ is the cofactor map up to scaling, which can be characterized as the unique linear map from V^* to V whose kernel is the image of φ and whose image is the kernel of φ.*
(c) *If φ has rank 1, then ψ can have rank 1 or rank 2; in the latter case, ψ has rank 2 and $\varphi = \psi^c$ up to scaling.*
(d) *If both φ and ψ have rank 1, then the kernel of φ contains the image of ψ and vice versa.*

Proof. We distinguish now by rank: If φ has rank 3, then the defining equations of Y imply that $\psi = \varphi^{-1}$ up to scaling (because $\psi = 0$ is not allowed in projective space). So (φ, ψ) is a pair of a smooth conic in $\mathbb{P}^2$ and its dual and hence in X. A symmetric argument applies if ψ has rank 3. If φ has rank 2, then the equations in I imply $\psi \circ \varphi = 0$. This implies that ψ has rank 1 and is therefore φ^c up to scaling. Symmetrically, we can argue if ψ has rank 2. If both φ and ψ have rank 1, then again $\psi \circ \varphi = 0$ and $\varphi \circ \psi = 0$, which is case (d). (Here, we use that the space of complete conics X is invariant under switching φ and ψ.) ∎

We now go through the following steps to compute the number of conics tangent to five general conics. Let $Z_i \subset X$ be the set of complete conics $(C, C') \in X$ such that C is tangent to a general conic D_i for $i = 1, \ldots, 5$.

(1) First, we show that for any $(C, C') \in \bigcap_{i=1}^{5} Z_i$, the conic $C \subset \mathbb{P}^2$ is smooth (so that $C' = C^*$) (see Proposition 6.1.6).
(2) Second, we show in Proposition 6.1.10 that the intersection of the Z_i is transverse at every intersection point.
(3) Third, we compute the class ξ of Z_i in the Chow ring of X and compute the intersection $\xi^5 \in A(X)$ in Theorem 6.1.13.

The degree of this class ξ^5 then is our answer 3264.

6.1.6 Proposition. *Let $D_i \subset \mathbb{P}^2$ be five general conics and let $Z_i \subset X$ be the set of complete conics (C, C') such that C is tangent to D_i at some point. For any complete conic $(C, C') \in \bigcap_{i=1}^{5} Z_i$, the conic C is smooth and $C' = C^*$.*

Proof. We distinguish the cases of complete conics described in Corollary 6.1.5. If $C = L \cup M$ is a conic of rank 2 tangent to D_i, then either one of the lines L or M is tangent to D_i or the point $p \in L \cap M$ lies on D_i. Since the conics D_i are general, the point $p \in L \cap M$ can lie on at most 2 out of the five (no three of the D_i intersect in a point). We distinguish now three cases.

(a) *p does not lie on any of the D_i:* Since the D_i are general, the dual conics D_i^* are also general. Concretely what we need here is that no three of the D_i^* meet in a point, meaning that no line in $\mathbb{P}^2$ is tangent to three of the D_i. So the conic $C = L \cup M$ can be tangent to at most 4 of the D_i, not all of 5.

(b) *p lies on exactly one of the conics D_i, say D_1:* Since the other conics $D_2, \ldots, D_5$ are general, D_1 does not contain any of the intersection points of two lines tangent to two of the conics $D_2, \ldots, D_5$. So the lines L and M through p can each only be tangent to one of the other conics so that C is, in total, tangent to at most 3 and not all 5.

(c) *p lies on two of the conics D_i, say D_1 and D_2:* In this case, we use that none of the finitely many lines that are tangent to two conics out of D_3, D_4, and D_5 passes through any of the intersection point of D_1 and D_2. This implies that the lines L and M through $p \in D_1 \cap D_2$ can each only be tangent to at most one of the conics D_3, D_4, D_5 so that C is, in total, tangent to at most 4 of the conics, not all 5.

Since X is symmetric under switching coordinates $(C, C') \mapsto (C', C)$, there is no complete conic $(C, C') \in \bigcap Z_i$ where C' has rank 2.

We now have to check the last case where both conics (C, C') have rank 1, say $C = 2L$ and $C' = 2q^\perp$ with $q \in L$. Such a point lies on a divisor Z_i if and only if $q \in D_i$ or $L \in D_i^*$. We see this via a limit argument: let $(C_t, C_t^*) \subset U \cap Z_i$ be a family of smooth conics C_t such that $\lim_{t\to 0} C_t = 2L$ and $\lim_{t\to 0} C_t^* = 2q^\perp$. Let p_t be a point where C_t is tangent to D_i at p_t so that $T_{p_t} C_t = T_{p_t} D_i$. Set $p = \lim_{t\to 0} p_t$. The family of tangent lines $T_{p_t} D_i = T_{p_t} C_t \in C_t^*$ converges to a line M in $q^\perp$ so that we have $q \in M$. We also have $p \in M$ because p_t converges to p. Since $p \in \lim_{t\to 0} C_t = 2L$, we also have $p \in L$ so that by assumption we have $p, q \in L$. If $p \neq q$, then it follows that $L = M$. If $p = q$ then $q \in D_i$. This proves one implication of the claimed equivalence. The other implication follows by taking the complete conic as the appropriate tangent line to D_i (or D_i^*).

Both cases, $q \in D_i$ and $L \in D_i^*$, can occur at most two times by the generality of the conics D_i and so such a complete conic cannot lie in the intersection of all 5 divisors Z_i. ■

To work towards transversality, we first aim to describe the tangent space of the divisor $Z_D \subset X$ defined by a smooth conic $D \subset \mathbb{P}^2$.

6.1.7 Proposition. *Let $\mathbb{P}^d = \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$ be the projective space over homogeneous polynomials in 2 variables of degree d. Let $\mathcal{D}$ be the discriminant hypersurface, which is the set of all polynomials with a multiple root on $\mathbb{P}^1$. A polynomial $F \in H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ with one double root at p and $d-2$ simple roots is a smooth point of $\mathcal{D}$ and the tangent space $T_{[F]}\mathcal{D}$ is the hyperplane of all polynomials vanishing at p.*

Proof. Consider the incidence correspondence

$$\Phi = \{(F, p) \in \mathbb{P}^d \times \mathbb{P}^1 : \operatorname{ord}_p(F) \geq 2\}.$$

In local coordinates $(a, t) \in \mathbb{P}^d \times \mathbb{P}^1$, it is defined by the equations

$$\begin{aligned} R(a,t) &= a_d t^d + a_{d-1} t^{d-1} + \ldots + a_1 t + a_0 \text{ and} \\ S(a,t) &= d a_d t^{d-1} + (d-1) t^{d-1} + \ldots a_1. \end{aligned}$$

Without loss of generality, we can consider the point $t = 0$ so that $a_0 = a_1 = 0$ are the local equations coming from R and S. It suffices to look at the following submatrix of the Jacobian of R and S at this point to prove the claim. The matrix

$$\begin{pmatrix} \frac{\partial R}{\partial a_1} & \frac{\partial R}{\partial a_0} & \frac{\partial R}{\partial t} \\ \frac{\partial S}{\partial a_1} & \frac{\partial S}{\partial a_0} & \frac{\partial S}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2a_2 \end{pmatrix}$$

has full rank 2 so that Φ is smooth at that point. Assuming $\mathrm{char}(K) \neq 2$, we see that the differential $d\pi \colon T_{(a,0)}\Phi \longrightarrow T_a\mathbb{P}^d$ of the projection $\pi \colon \mathbb{P}^d \times \mathbb{P}^1 \longrightarrow \mathbb{P}^d$ to the first factor is injective. The image is the hyperplane $a_0 = 0$. The map π is one-to-one locally at a point as in the assumptions so that the image $\mathcal{D} = \pi(\Phi)$ is smooth with tangent hyperplane as claimed. ■

6.1.8 Lemma. *Let $D \subset \mathbb{P}^2$ be a smooth conic. Let $Z_D^o \subset \mathbb{P}^5$ be the quasi-projective variety of smooth plane conics C tangent to D. If $C \cap D$ consists of three points, one with multiplicity 2, then Z_D^o is smooth at $[C]$. In this case, the tangent space $T_{[C]}Z_D^o$ is the hyperplane $H_p \subset \mathbb{P}^5$ of conics containing the point p of tangency between C and D.*

Proof. First, write D as a rational curve $\mathbb{P}^1 \longrightarrow D \subset \mathbb{P}^2$ and consider the restriction map

$$H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \longrightarrow H^0(D, \mathcal{O}_D(2)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)).$$

This map is surjective. In terms of projective spaces, this gives a rational map

$$\pi_D \colon \mathbb{P}^5 = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \dashrightarrow \mathbb{P}H^0(\mathcal{O}_D(2)) = \mathbb{P}^4,$$

the linear projection of $\mathbb{P}^5$ from the point $D \in \mathbb{P}^5$. Since a conic C is tangent to D if and only if the corresponding quartic polynomial on $\mathbb{P}^1$ has a double root, the closure of Z_D^o in $\mathbb{P}^5$ is the cone with vertex $D \in \mathbb{P}^5$ over the discriminantal hypersurface $\mathcal{D} \subset \mathbb{P}^4$. The previous Proposition 6.1.7 implies that $\mathcal{D}$ is smooth at a point C that is tangent to D at one point and intersects it transversely at 2 other points. The tangent space is the hyperplane of all quartic polynomials vanishing at the double root which correspond to conics in $\mathbb{P}^2$ passing through the point of tangency of C and D, as claimed. ■

6.1.9 Lemma. *Let $D_1, \ldots, D_5$ be five general conics in $\mathbb{P}^2$ and let $C \subset \mathbb{P}^2$ be a smooth conic that is tangent to all five. Then each conic D_i is simply tangent to C at a point p_i (meaning the intersection multiplicity is 2) and is otherwise transverse to C_i; the five points p_i are distinct.*

Proof. We prove this by dimension count. Let $U \subset \mathbb{P}^5$ be the set of smooth conics in $\mathbb{P}^2$ (so that $U = \mathbb{P}^5 \setminus \mathcal{V}_+(\det)$). Consider the incidence correspondence

$$\begin{aligned} \Phi &= \left\{(D_1, D_2, D_3, D_4, D_5; C) \in U^5 \times U \colon \text{ each } D_i \text{ is tangent to } C\right\} \\ \subset \Phi' &= \left\{(D_1, D_2, D_3, D_4, D_5; C) \in (\mathbb{P}^5)^5 \times U \colon \text{ each } D_i \text{ is tangent to } C\right\}. \end{aligned}$$

The set Φ is an open subset of Φ'. Since U is irreducible of dimension 5 and the projection map $\Phi' \longrightarrow U$ to the last factor has irreducible fibers $(Z_C)^5$ of dimension 20, it follows that Φ' is irreducible of dimension 25. The conditions in the claim are open conditions (the intersection is as transverse as possible given that C is tangent to each D_i) and the set of such conics is in fact non-empty: indeed, fix C first and pick five general points p_i on C and a general conic D_i tangent to C at p_i with multiplicity 2, otherwise transverse. The set of conics in Φ for which the assumptions are not satisfied are therefore contained in a proper closed subset of Φ, which has dimension at most 24. The projection to U^5 restricted this set in Φ cannot be dominant, which proves the claim. ■

6.1.10 Proposition. *Let $D_1, \dots, D_5 \subset \mathbb{P}^2$ be five general conics. The divisors Z_i in X intersect transversely on X and every intersection point $(C, C') \in \bigcap_{i=1}^5 Z_i \subset X$ is a pair of a smooth conic C and its dual $C' = C^*$.*

Proof. This follows directly from putting the above results together. First, Proposition 6.1.6 tells us that $(C, C') \in \bigcap Z_i$ is a pair of a smooth conic $C \subset \mathbb{P}^2$ and its dual $C' = C^*$. Second, Lemma 6.1.9 shows that we can apply Lemma 6.1.8 so that we have a description of the tangent space to each divisor Z_i. This description shows that

$$\bigcap_{i=1}^{5} T_{(C,C')} Z_i = \bigcap_{i=1}^{5} H_{p_i} = \{[C]\}$$

where H_{p_i} is the hyperplane of conics through p_i and p_i is the point of tangency of C with D_i. The conic C is the unique conic through these five points, giving the last equation in the above chain. So the five tangent spaces intersect in one point, a 0-dimensional set, which means exactly that the intersection is transverse. ■

We have now established that the divisors Z_i on the space X of complete conics intersect in exactly the points that we want to count and transversely at each of the points. It remains to determine their number, which we now can do using the intersection product in $A(X)$ by the defining property.

To do this, it is enough to consider the pullbacks of the hyperplane classes in $\mathbb{P}^5$ and $(\mathbb{P}^5)^*$ to $X \subset \mathbb{P}^5 \times (\mathbb{P}^5)^*$. We set

$$\alpha = [\{(C, C') \in X \colon p \in C\}] \text{ and } \beta = [\{(C, C') \in X \colon [L] \in C'\}]$$

for any point $p \in \mathbb{P}^2$ and any line $L \subset \mathbb{P}^2$. So the class α is the class of the pullback of the hyperplane of all conics through p to X and β is the class of the pullback of the hyperplane of all conics in the dual projective plane containing the point corresponding to the line $L \subset \mathbb{P}^2$.

The class α^4 can be represented by the set of points $(C, C') \in X$ such that C contains four general points in $\mathbb{P}^2$. This means that α^4 is the pullback of the line in $\mathbb{P}^5$ determined by four general points. We write γ for α^4. Dually, we write $\beta^4 = \varphi$ for the class represented by the set of points (C, C'), where C' contains four general points in $(\mathbb{P}^2)^*$. Let us compute some intersections.

6.1.11 Lemma. *The Picard group $A^1(X)$ of the space of complete conics has rank 2 and is generated over $\mathbb{Q}$ by the classes α and β. We have* $\deg(\alpha^5) = \deg(\beta^5) = 1$ *and* $\deg(\alpha^4\beta) = \deg(\alpha\beta^4) = 2$.

Proof. The open set $U \subset X$ of complete conics (C, C') where C is smooth and $C' = C^*$ is isomorphic to the complement of a hypersurface in $\mathbb{P}^5$. This implies that its Picard group is torsion by excision (as in Proposition 3.2.4, see Corollary 3.2.9). This implies that for any line bundle on X, a suitable power of L is trivial on U and hence supported on the complement $X \setminus U$. This complement of U has two irreducible components, namely the complete conics of type (b) and (c) (meaning one of the two conics in (C, C') has rank 2). So any divisor class on X is a rational linear combination of these two divisors: the dimension of $\operatorname{Pic}(X) \otimes \mathbb{Q}$ is at most 2, which means that the rank of $\operatorname{Pic}(X)$ is at most 2.

That the dimension of $\operatorname{Pic}(X) \otimes \mathbb{Q}$ is in fact 2 follows from the degrees of intersection in the claim because the corresponding 2×2 matrix has rank 2.

We compute the degrees of intersection geometrically, case by case. The first, $\deg(\alpha^5) = 1$ just means that there is a unique conic through 5 general points in $\mathbb{P}^2$. The equation $\deg(\beta^5) = 1$ is dual to it. The remaining two $\deg(\alpha^4\beta) = 2$ and $\deg(\alpha\beta^4) = 2$ are also dual and it suffices to prove one of them. The class α^4 is represented by a pencil M of conics in $\mathbb{P}^2$ determined by four general points in the plane. Representing the class β by

$$\{(C, C') \in X \colon [L] \in C'\}$$

as above shows that $\alpha^4\beta$ is the number of conics in the pencil M whose restriction to L is singular. This number is 2 because a conic in $\mathbb{P}^2$ is tangent to a line $L \subset \mathbb{P}^2$ if and only if the restriction of the conic to L has a double root, which boils down to the vanishing of the determinant of a 2×2 matrix. ■

This information about the Chow ring of X is in fact enough to compute the class ξ of the divisor Z_D by the method of undetermined coefficients.

6.1.12 Proposition. *The class ξ of the divisor $Z_D \subset X$ determined by a smooth conic $D \subset \mathbb{P}^2$ is*

$$\xi = 2\alpha + 2\beta \in A(X).$$

Proof. By the above Lemma 6.1.11, we can write $\xi = a\alpha + b\beta \in A^1(X) \otimes \mathbb{Q}$ for some $a, b \in \mathbb{Q}$. We again use the fact that the open subset $U \subset X$ of pairs (C, C') where $C \subset \mathbb{P}^2$ is smooth and $C' = C^*$ is isomorphic to its projection to the first factor. The image of $Z_D \cap U$ under this isomorphism is a sextic hypersurface. This implies that $\deg(\xi\alpha^4) = 6$; indeed, the pencil of conics through 4 general points in $\mathbb{P}^2$ contains three conics of rank 2 and those are generically not tangent to D because the four points determine only 6 lines contained in the three conics of rank 2 in the pencil. Dually, it also follows that $\deg(\xi\beta^4) = 6$. By the above rules for intersection, this gives the equations

$$\begin{aligned} 6 &= \deg(\xi\alpha^4) = a \deg(\alpha^5) + b \deg(\alpha^4\beta) = a + 2b \\ 6 &= \deg(\xi\beta^4) = a \deg(\alpha\beta^4) + b \deg(\beta^5) = 2a + b. \end{aligned}$$

The unique solution to these equations is the claimed $\xi = 2\alpha + 2\beta$. ■

6.1.13 Theorem. *There are 3264 conics in the plane that are tangent to five general conics.*

Proof. We have to compute the degree of the class $\xi^5 = 32(\alpha + \beta)^5 \in A(X)$, by Proposition 6.1.12. By the binomial theorem and the linearity of deg, we only need to compute the degrees of the monomials $\alpha^{5-i}\beta^i$ for $i = 0, \ldots, 5$. The cases $i = 0, 1, 4, 5$ are already done in Lemma 6.1.11. So we only need to do the case $i = 2$ (the case $i = 3$ is, again, dual so that a symmetric argument gives the same degree as for $i = 2$). We use the same geometric approach as before: $\deg(\alpha^3\beta^2)$ is the number of conics in the plane through 3 general points that are tangent to two general lines. The conics passing through 3 general points in $\mathbb{P}^2$ form a projective plane in $\mathbb{P}^5$. In this plane, the set of conics being tangent to a general line is again a conic. Indeed, a conic in $\mathbb{P}^2$ is tangent to a line if and only if its restriction to the line has a double root. Since we want the conics tangent to two general lines, we intersect two conics in the plane of conics through 3 points in $\mathbb{P}^5$, which gives 4 points by Bézout's Theorem. (We can apply Bézout's Theorem here because of the following exercise.) So overall, we know have

$$\begin{aligned}(\alpha+\beta)^5 &= \binom{5}{0}\cdot 1 + \binom{5}{1}\cdot 2 + \binom{5}{2}\cdot 4 + \binom{5}{3}\cdot 4 + \binom{5}{4}\cdot 2 + \binom{5}{5}\cdot 1\\ &= 1 + 10 + 40 + 40 + 10 + 1\\ &= 102.\end{aligned}$$

Overall, the answer to our five conics problem is $\deg(\xi^5) = 32(\alpha+\beta)^5 = 32 \cdot 102 = 3264$. ■

Exercise 6.1.14. Find three points p_1, p_2, and p_3 as well as two lines L_1, L_2 in $\mathbb{P}^2$ such that there exists a conic through the points p_i tangent to L_1 but not L_2.

CHAPTER 7

Excess Intersection

The goal of this chapter is to give a definition of Segre classes from Chern classes. Fulton's book uses the reverse approach: he defines Segre classes first and then usem them to define Chern classes. We go through this so that we can state the excess intersection formula in terms of Segre classes. We will not give a proof but apply it in examples to see it in action.

7.1. Pushforward

In this section, we need to assume properness of our maps and we finally introduce this notion.

Definition. A morphism $f\colon X \to Y$ of algebraic varieties is **closed** if the image of any closed subset of X is closed in Y. It is **universally closed** if it is closed and for any morphism $Z \to Y$ the corresponding morphism $X \times_Y Z \to Z$ obtained by base extension is closed. A morphism is **proper** if it is **universally closed**.

In general, for schemes, we also require f to be of **finite type** which is built into our notion of varieties because their coordinate rings are finitely generated algebras over fields. It means that for every open affine subset $U \subset Y$ the preimage $f^{-1}(U)$ is quasi-compact and $\mathcal{O}_X|_U(V)$ is finitely generated over $\mathcal{O}_Y|_U(U)$ for every affine open subset $V \subset f^{-1}(U)$. Moreover, we also get for free that f is **separated** which means that $X \to X \times_Y X$ is a closed immersion.

7.1.1 Example. The structure morphism $\mathbb{A}^1 \to \operatorname{Spec}(K)$ coming from the ring homomorphism $K \to K[x]$ is not proper. It is clearly closed because it is constant. However, it is not universally closed: $\mathbb{A}^1 \times_K \mathbb{A}^1 \to \mathbb{A}^1$ is the coordinate projection $\mathbb{A}^2 \to \mathbb{A}^1$ which is not closed.

In most cases that we are encountering here, our maps are proper, essentially due to the Main Theorem of Elimination Theory.

7.1.2 Theorem. *A projective morphism is proper.*

7.1.3 Remark. In topology, a continuous map $f\colon X \to Y$ between topological spaces is called proper if the preimage of any compact set in Y is compact in X. On reasonable topological spaces (it suffices that Y is Hausdorff and locally compact), this is equivalent to f being closed with compact fibers. If X is also Hausdorff (and Y Hausdorff and locally compact) it is also equivalent to being universally closed. This latter definition is the one used in algebraic geometry.

Let $f\colon X \to Y$ be a proper map of schemes and let $A \subset X$ be an irreducible subvariety. Since f is proper, the image $f(A) \subset Y$ is closed (and irreducible). The pull back of regular functions $f^*\colon \mathcal{O}_Y|_{f(A)} \to \mathcal{O}_X|_A$ induces a field homomorphism $K(f(A)) \to K(A)$ from the function field of $f(A)$ to the function field of A. This means that $K(A)$ is a field extension of $K(f(A))$. If $f|_A\colon A \to f(A)$ is **generically finite** (which happens if and only if $\dim(A) = \dim(f(A))$), then these fields have the same transcendence degree over K and this extension is algebraic. In fact, it is finite and $\deg(f) = [K(A) : K(f(A))]$ is the number of elements in a generic fiber $f^{-1}(x)$ for a general $x \in f(A)$. We use this number to define the pushforward map on Chow rings coming from a porper map of schemes.

Definition. Let $f\colon X \to Y$ be a proper map of schemes. The **pushforward** $f_*\colon A(X) \to A(Y)$ is defined on the classes of irreducible subvarieties $A \subset X$ by the following conditions.

- ⋄ If $\dim(f(A)) < \dim(A)$, then $f_*([A]) = 0$.
- ⋄ If $\dim(f(A)) = \dim(A)$, then $f_*([A]) = [K(A) : K(f(A))] \cdot [f(A)]$.

We get $f_*\colon A(X) \to A(Y)$ by linearly extending the above definition to $\mathbb{Z}$-linear combinations of classes of irreducible subvarieties of X.

The point of the definition $f_*([A]) = \deg(f)[f(A)]$ is to preserve rational equivalence of cycles. One can show (with this definition) the following result.

7.1.4 Theorem. *If $f\colon X \to Y$ is a proper map of schemes, then the map $f_*\colon Z(X) \to Z(Y)$ defined above preserves rational equivalence so that it induces a map $f_*\colon A_k(X) \to A_k(Y)$ on Chow groups for each k.*

7.1.5 Example. To see that the modification $f_*([A]) = \deg(f)[f(A)]$ by the degree of f is necessary in order for this map to preserve rational equivalence of cycles, we consider the following example the projection of a plane curve. Let $C \subset \mathbb{A}^2$ be the curve defined by the equation $x - (y-2)y(y+2)$, which is the graph of a cubic polynomial rotated by 90 degrees. As a map, we take the projection to the x-axis $f\colon C \to \mathbb{A}^1$. The fiber over 0 consists of the three points $(0,2)$, $(0,0)$, and $(0,-2)$. However, over a branch point ($x = \pm 2/\sqrt{3}$), the fiber has only two points. On $\mathbb{A}^1$, the two points are rationally equivalent but a set of two points cannot be rationally equivalent to a set of three points without assigning multiplicities.

This result on pushforwards shows the existence of a degree map on 0-dimensional cycles for a proper scheme (in particular for smooth projective varieties, which is a case where we used such a map, see Grassmannians).

7.1.6 Corollary. *Let X be a proper scheme over an algebraically closed field. There is a unique map* $\deg\colon A(X) \to \mathbb{Z}$ *taking the class $[p]$ of a closed point $p \in X$ to 1 and vanishing on the class of any cycle of pure dimension greater than 0.*

Proof. That X is a proper scheme over an algebraically closed field K means, by definition, that the map $X \to \operatorname{Spec}(K)$ is proper. The dimension of $\operatorname{Spec}(K)$ is 0 and its Chow ring is $\mathbb{Z}$ so that the claim follows directly from the previous Theorem 7.1.4. ∎

In particular, we have used the fact that if $A \subset X$ is an irreducible, k-dimensional subvariety of a smooth projective variety X and $B \subset X$ is irreducible of codimension k such that $A \cap B$ is finite and non-empty, then the map $\deg\colon A_k(X) \to \mathbb{Z}$, $[Z] \to \deg([Z] \cdot [B])$ sends the class of A to a non-zero integer. This implies that no multiple of $[A]$ can be 0 in $A(X)$.

7.2. Pullback

A pullback on Chow rings exists for smooth quasi-projective varieties and it is geometric in the following sense. If $f\colon X \to Y$ is a morphism of smooth quasi-projective varieties and $A \subset Y$ is an irreducible subvariety of codimension c in Y, then $f^*([A]) = [f^{-1}(A)]$ under the additional assumption that the preimage $f^{-1}(A) \subset X$ is generically reduced of the expected codimension c in X. This is also a result that requires work (in particular when it comes to preserving rational equivalence classes). We briefly summarize the main results.

Definition. Let $f\colon X \to Y$ be a morphism of smooth varieties. We call an irreducible subvariety $A \subset Y$ **generically transverse** to f if the preimage $f^{-1}(A)$ is generically reduced and the codimension of $f^{-1}(A)$ in X is equal to the codimension of A in Y.

7.2.1 Theorem. *Let $f\colon X \to Y$ be a morphism of smooth and quasi-projective varieties.*

(a) *There is a unique map of groups $f^*\colon A^c(Y) \to A^c(X)$ such that $f^*([A]) = [f^{-1}(A)]$ for every generically transverse irreducible subvariety $A \subset Y$. Moreover, the map f^* is a ring homomorphism and a contravariant functor from the category of smooth projective varieties to the category of graded rings.*

(b) *(Push-pull formula) For all $\beta \in A^k(Y)$ and all $\alpha \in A_\ell(X)$ we have*

$$f_*(f^*\beta \cdot \alpha) = \beta \cdot f_*\alpha \in A_{\ell-k}(Y).$$

In fact, the identity $f^*([A]) = [f^{-1}(A)]$ more generally holds for subvarieties $A \subset Y$ that are Cohen-Macaulay (as long as $\operatorname{codim}_X(f^{-1}(A)) = \operatorname{codim}_Y(A)$). Another nice case of morphisms are flat morphisms between schemes. In that case, $f^*([A]) = [f^{-1}(A)]$ holds very generally.

7.3. Segre Classes

It turns out that Segre classes are, in a very concrete sense, inverse to Chern classes. For globally generated vector bundles, we can compute the Chern class by the degeneracy locus of the appropriate number of general global sections. This answers the question on what locus sections become linearly dependent. Segre classes ask the opposite question in linear algebra in this case: when do general sections fail to span the fiber?

Let $\mathcal{E}$ be a line bundle on a smooth projective variety X that is globally generated. The locus where one general global section of $\mathcal{E}$ fails to generate $\mathcal{E}$ locally is the same as the locus where this section is linearly dependent (aka 0), namely the associated divisor $c_1(\mathcal{E}) \in A(X)$. However, we can also ask when i general global sections fail to generate $\mathcal{E}$ locally (at least for any $i \le \dim(H^0(X, \mathcal{E}))$). Since $\mathcal{E}$ is a line bundle, the locus where they fail to generate $\mathcal{E}$ locally is exactly the locus where they all vanish. Generally, this locus has the expected codimension i and is of class $c_1(\mathcal{E})^i \in A(X)$.

Suppose that $\mathcal{E}$ has rank $r > 1$ and is globally generated. Naively, we expect that r general global sections $\tau_1, \tau_2, \dots, \tau_r$ generate $\mathcal{E}$ away from a subset of codimension 1. Let X' be the set of points where they fail to generate $\mathcal{E}$. At a general point $p \in X'$, we might expect exactly one linear relation among the r vectors $\tau_i(p) \in K^r$. If that is true for one point, then there is an non-empty open subset $U \subset X'$ on which this is true. On U, the sections τ_i then generate a subbundle $\mathcal{E}'$ of $\mathcal{E}$ of rank $r-1$ so that the quotient bundle $\mathcal{E}/\mathcal{E}'$ is a line bundle. Adding another section τ_{r+1}, we expect that the induced sections $\overline{\tau_i}$ of $\mathcal{E}/\mathcal{E}'$ $(i = 1, 2, \ldots, r+1)$ generate this line bundle away from a codimension 1 subset of U, which means that they generate $\mathcal{E}$ away from a codimension 2 subset of X. By induction, the heuristic is that $r + i - 1$ general sections of $\mathcal{E}$ generate $\mathcal{E}$ away from a locus of codimension i in X. In particular, the expectation is that $r + \dim(X)$ general sections generate $\mathcal{E}$ globally.

If this heuristic is correct, then the Segre class is (up to sign) the rational equivalence class of the locus where the sections fail to generate $\mathcal{E}$, as we will see below.

To contrast Chern and Segre classes from the point of view of linear algebra, we can summarize the above discussion as follows (at least for globally generated vector bundles).

- The ith Chern class $c_i(\mathcal{E})$ of $\mathcal{E}$ is the set of points where a suitably general map
$$\bigoplus_{i=1}^{r-i+1} \mathcal{O}_X \to \mathcal{E}$$
of vector bundles fails to be injective.
- The ith Segre class $s_i(\mathcal{E})$ is $(-1)^i$ times the set of points where a suitably general map
$$\bigoplus_{i=1}^{r+i-1} \mathcal{O}_X \to \mathcal{E}$$
of vector bundles fails to be surjective.

Let us give a definition of Segre classes.

Definition. Let X be a smooth projective variety. Let $\mathcal{E}$ be a vector bundle of rank r on X and write $\pi\colon \mathbb{P}\mathcal{E} \to X$ be its projectivization. Set $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$ (see Section 5.3). The ith **Segre class** of $\mathcal{E}$ is the class
$$s_i(\mathcal{E}) = \pi_*(\zeta^{r+i-1}) \in A^i(X).$$
The **(total) Segre class** of $\mathcal{E}$ is the sum
$$s(\mathcal{E}) = 1 + s_1(\mathcal{E}) + s_2(\mathcal{E}) + \ldots \in A(X).$$

Segre classes are inverse to Chern classes in the following formal sense.

7.3.1 Proposition. *Let X be a smooth projective variety and $\mathcal{E}$ a vector bundle on X. Then the following equation holds in the Chow ring of X*
$$c(\mathcal{E}) \cdot s(\mathcal{E}) = 1 \in A(X).$$

Sketch of proof. Let $\pi\colon \mathbb{P}\mathcal{E} \to X$ be the projectivization of $\mathcal{E}$. Let $\mathcal{S}$ and $\mathcal{Q}$ be the universal subbundle and the universal quotient bundle of $\mathbb{P}\mathcal{E}$ so that we have the exact sequence
$$0 \to \mathcal{S} = \mathcal{O}_{\mathbb{P}\mathcal{E}}(-1) \to \pi^*(\mathcal{E}) \to \mathcal{Q} \to 0.$$

Then Whitney's formula for Chern classes implies that

$$c(\mathcal{Q}) = \frac{c(\pi^*(\mathcal{E}))}{c(\mathcal{S})} = c(\pi^*(\mathcal{E}))(1 + \zeta + \zeta^2 + \ldots) \in A(\mathbb{P}\mathcal{E}),$$

where we write $c(\mathcal{S}) = 1 - \zeta$ and $\zeta = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1))$. We now pushforward this equation to X. The dimension of $\mathbb{P}\mathcal{E}$ is $\dim(X) + \mathrm{rank}(\mathcal{E}) - 1$. So for $i < r - 1$, the Chern class $c_i(\mathcal{Q}) \in A(\mathbb{P}\mathcal{E})$ is represented by cycles of dimension larger than $\dim(X)$. So by definition, their pushforward is 0. The top Chern class $c_{r-1}(\mathcal{Q})$ maps to a multiple $m[X]$ of the fundamental class of X. In fact, this multiple m is equal to 1 (which can be computed using a push-pull formula).

On the right hand side, using the push-pull formula, we get

$$\pi_* \left(c(\pi^*(\mathcal{E}))(1 + \zeta + \zeta^2 + \ldots)\right) = c(\mathcal{E})\pi_*(1 + \zeta + \zeta^2 + \ldots) = c(\mathcal{E})s(\mathcal{E}).$$

Overall, the computations on both sides show the claim. ■

The identity $c(\mathcal{E}) \cdot s(\mathcal{E}) = 1 \in A(X)$ implies the following relations for Segre classes from the analogues of the case of Chern classes.

7.3.2 Corollary. *Let X be a smooth projective variety and $\mathcal{E}$ a vector bundle on X.*

(1) (Duality) $s_i(\mathcal{E}^*) = (-1)^i s_i(\mathcal{E})$

(2) (Whitney's Formula) For any exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$$

of vector bundles on X we have

$$s(\mathcal{F}) = s(\mathcal{E}) \cdot s(\mathcal{G}).$$

Using this relation between Chern and Segre classes, we can make sense of the above heuristic discussion for globally generated vector bundles. To see this in full detail requires Porteous's Formula. We only sketch part of the proof for the following statement.

7.3.3 Proposition. *Let $\mathcal{E}$ be a vector bundle of rank r on a smooth projective variety X that is globally generated. Let $\tau_1, \tau_2, \ldots, \tau_{r+i-1}$ be general global sections of $\mathcal{E}$ and set X_i to be the subscheme where they fail to generate $\mathcal{E}$. Then X_i has pure codimension i and the class $[X_i]$ is equal to $(-1)^i s_i(\mathcal{E})$.*

Proof. Let $V = H^0(X, \mathcal{E})$ and $n = \dim(V)$. The assumption that $\mathcal{E}$ is globally generated implies that we have a well defined morphism $\varphi\colon X \to \mathrm{Gr}(n - r, V)$ sending each point $p \in X$ to the kernel of the map $V \to \mathcal{E}_p$ which is the same as the subspace of global sections vanishing at p. Via this map, the vector bundle $\mathcal{E}$ on X is isomorphic to the pullback $\varphi^*(\mathcal{Q})$ of the universal quotient bundle on $\mathrm{Gr}(n - r, V)$. We saw in Proposition 5.2.6 that this gives one way to define the Chern classes of the globally generated bundle $\mathcal{E}$ as

$$c_i(\mathcal{E}) = \varphi^*(c_i(\mathcal{Q})) = \varphi^*(\sigma_i).$$

Let us proceed similarly: take general global sections $\tau_1, \tau_2, \ldots, \tau_{r+i-1} \in V$ of $\mathcal{E}$ and let X_i be the variety of points p where they fail to span $\mathcal{E}_p$. Then $X_i = \varphi^*(\Sigma_{1^i}(W))$ whose class is

$\varphi^*(\sigma_{1^i})$, where $W = \operatorname{span}(\tau_1, \tau_2, \ldots, \tau_{r+i-1}) \subset V$. Indeed, $\Sigma_{1^i}(W)$ is the Schubert variety of all $(n-r)$-planes in V that intersect W in dimension at least i.

Now we combine this computation with the identity in Corollary 4.5.2 saying that

$$1 = (1 + \sigma_1 + \sigma_2 + \ldots + \sigma_r)(1 - \sigma_1 + \sigma_{1,1} - +(-1)^{n-r}\sigma_{1^{n-r}})$$

to see that $s(\mathcal{E}) = 1/c(\mathcal{E})$ evaluates, as desired, to

$$\sum_{i=0}^{n-r}(-1)^i[X_i] = \varphi^*\left(\sum_{i=0}^{n-r}\sigma_{1^i}\right) = \varphi^*\frac{1}{\sum_{i=0}^{r}\sigma_i} = \frac{1}{c(\mathcal{E})} = s(\mathcal{E}).$$

■

7.3.4 Example. Let us compute the Segre classes of the tangent bundle $\mathcal{T}_{\mathbb{P}^n}$ on $\mathbb{P}^n$. We have seen in Proposition 5.4.4 that $c(\mathcal{T}_{\mathbb{P}^n}) = (1+\zeta)^{n+1}$, where $\zeta = [H]$ is the class of a hyperplane in $\mathbb{P}^n$. We compute the Segre classes by formally inverting this

$$s(\mathcal{T}_{\mathbb{P}^n}) = \frac{1}{(1+\zeta)^{n+1}} = 1 - (n+1)\zeta + \binom{n+2}{2}\zeta^2 - + \ldots = \sum_{i=0}^{n}(-1)^i\binom{n+i}{i}\zeta^i.$$

(For example, one can compute the Taylor expansion of the function $\frac{1}{(1-t)^{n+1}}$ to compute the above series expansion inductively.)

7.3.1 Secant Varieties

Secant varieties are a special case of varieties "swept out" by linear spaces. Generally, such varieties are defined in the following way. Let $B \subset \mathbb{G}(k,n)$ be an irreducible algebraic variety of k-planes in an n-dimensional projective space. We can then define

$$X_B = \bigcup_{[\Lambda]\in B} \Lambda \subset \mathbb{P}^n,$$

a variety swept out by the family of k-planes in B. To study such varieties, it is often useful to consider an associated incidence correspondence (which is the universal subbundle of $\mathbb{G}(k,n)$ pulled back to B). Concretely, we define

$$\Phi_B = \{([\Lambda], x) \in \mathbb{G}(k,n) \times \mathbb{P}^n \colon [\Lambda] \in B,\ x \in \Lambda\}$$

so that X_B is the projection $\pi_2(\Phi_B)$ of Φ_B to the second factor. (Eisenbud and Harris define Φ_B as the fiber product of B with the universal k-plane Φ, which is the projectivization of the universal subbundle of $\mathbb{G}(k,n)$, see Section 10.2.)

As usual, the advantage of the incidence correspondence is that it is easy to compute its dimension and it is a smooth projective variety (at least if B is smooth). Its dimension is $\dim(\Phi_B) = \dim(B) + k$. So we naively expect the dimension of X_B to be $\dim(B) + k$ as well (and this is true if and only if the projection $\pi_2 \colon \Phi_B \longrightarrow X_B$ is generically finite). If this is the case, we can compute the degree of X_B in terms of Segre classes and the degree of $\pi_2 \colon \Phi_B \longrightarrow X_B$ as follows.

7.3.5 Proposition. *Let $B \subset \mathbb{G}(k,n)$ be a smooth and irreducible projective variety of dimension m. Assume that the projection $\pi_2 \colon \Phi_B \longrightarrow X_B$ is generically finite of degree d. Then the degree of $X_B \subset \mathbb{P}^n$ is given by the mth Segre class of the restriction $\mathcal{E} = \iota^*(\mathcal{S})$ of the universal subbundle $\mathcal{S}$ on $\mathbb{G}(k,n)$ to $\iota \colon B \longrightarrow \mathbb{G}(k,n)$ by*

$$\deg(X_B) = \deg(s_m(\mathcal{E}))/d.$$

Proof. The main point of the proof here is that $\pi_1 : \Phi_B \longrightarrow B$ is actually $\pi : \mathbb{P}\mathcal{E} \longrightarrow B$ which makes Segre classes relevant for this discussion by their very definition of $s_i(\mathcal{E}) = \pi^*(\zeta^{k+i-1})$.

The degree of X_B is by definition the degree of the intersection $X \cap H_1 \cap \ldots \cap H_{\dim(X)}$ for general hyperplanes $H_i \subset \mathbb{P}^n$. We can compute this number via the pullback to Φ_B as the degree of $[\pi_2^*(H)]^{m+k} \in A(\Phi_B)$ up to a factor d for the degree of the map π_2. To compute this class $[\pi_2^*(H)]$, let $L \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ be a linear form which induces a global section σ_L of $\mathcal{S}^*$ by evaluation: $\sigma_L([\Lambda]) \colon \Lambda \longrightarrow K$, $x \longmapsto L(x)$. We also write σ_L for the induced section on the pullback $\mathcal{E}^*$. The pullback $\pi_2^*(H)$ of the hyperplane $H = \mathcal{V}_+(L) \subset \mathbb{P}^n$ to Φ_B is the zero locus of the section σ_L of $\mathcal{E}^*$. This means simply that σ_L is a global section of $\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)$ so that we have $[\pi_2^*(H)] = c_1(\mathcal{O}_{\mathbb{P}\mathcal{E}}(1)) = \zeta$. By definition of Segre classes, this implies the desired formula

$$d \cdot \deg(X_B) = \deg(\zeta^{m+k}) = \deg(s_m(\mathcal{E})).$$

∎

An alternative proof using the relation of Segre classes with the Chern classes of the universal quotient bundle of the Grassmannian $s(\mathcal{S}) = 1/c(\mathcal{S}) = c(\mathcal{Q})$ goes as follows. The degree of X is the degree of the intersection $X_B \cap \Gamma$ for a general linear subspace $\Gamma \subset \mathbb{P}^n$ of dimension $n-m-k$. We can compute this number as the intersection of $B \subset \mathbb{G}(k,n)$ with the Schubert variety $\Sigma_m(\Gamma)$ of all k-planes in $\mathbb{P}^n$ that intersect Γ (again up to a factor of $\deg(\pi_2)$). Using Proposition 5.4.7, this approach gives the formula

$$d \cdot \deg(X_B) = \deg(\iota^*\sigma_m) = \deg(c_m(\iota^*(\mathcal{Q}))) = \deg(s_m(\mathcal{E})).$$

This formula can be used in the case of secant varieties (as stated of smooth curves). Generally, the kth secant variety of a projective variety $X \subset \mathbb{P}^n$ is the closure of the union of all k-planes in $\mathbb{P}^n$ spanned by $k+1$ points on X. The 0th secant variety of X is X itself, the first the closure of the union of all secant lines, and so on. In the above setup, the kth secant variety of X corresponds to the subvariety $B_X \subset \mathbb{G}(k,n)$ obtained as the closure of all k-planes spanned by points on X. This variety can be identified with the $(k+1)$st symmetric product of X, which is the quotient of the $(k+1)$-fold direct product modulo the action of S_{k+1} by permutation. It turns out that this variety $X^{(k+1)}$ for $k > 1$ is smooth if and only if X is smooth and of dimension at most 1.

7.3.6 Example. The kth symmetric product of $\mathbb{P}^1$ is isomorphic to $\mathbb{P}^k$. The isomorphism can be described as follows. Let $(a_i : b_i)$ be homogeneous coordinates on the ith copy of $\mathbb{P}^1$ in $(\mathbb{P}^1)^k$. To the k-tuple of points $(a_1 : b_1, a_2; b_2, \ldots, a_k : b_k)$, we associate the coefficients of the binary form $f = (sb_1 - ta_1) \cdot (sb_2 - ta_2) \cdot \ldots \cdot (sb_k - ta_k) \in K[s,t]$ of degree k with roots $(a_i : b_i) \in \mathbb{P}^1$. Clearly, the polynomial f is invariant under permutation of its roots and its $k+1$ coefficients (essentially the elementary symmetric polynomials in the roots) are coordinates on $\mathbb{P}^k$. This is clearly a bijection between the kth symmetric power $(\mathbb{P}^1)^{(k)}$ of $\mathbb{P}^1$ and $\mathbb{P}^k$. In fact this is also an isomorphism of projective varieties.

Using the isomorphism $(\mathbb{P}^1)^{(k)} \cong \mathbb{P}^k$ and the formula for the degree of secant varieties Proposition 7.3.5 (in case of expected dimension), one can compute the degree of secant varieties of any rational normal curve in $\mathbb{P}^n$. The result of this computation is the following (see Theorem 10.16 in Eisenbud, Harris).

7.3.7 Proposition. *Let $C \subset \mathbb{P}^d$ be a rational normal curve and m a positive integer with $2m+1 \leq d$. Then the degree of the mth secant variety of C is*

$$\binom{d-m}{m+1}.$$

7.4. Excess Intersection Formula

The goal of this section is to make sense of the excess intersection formula (developed in this form mostly in work by Fulton and MacPherson) and try some computations of examples. In its full glory, the formula can be stated as follows.

7.4.1 Theorem. *Let X be a smooth and irreducible projective variety. Let S and T be irreducible subvarieties and assume that T is locally a complete intersection (which is implied by T being smooth, for example). Then the intersection product $[S] \cdot [T]$ is equal to the sum*

$$\sum_C (\iota_C)_*(\gamma_C)$$

taken over all connected componenta C of $S \cap T$ where $\iota_C \colon C \to X$ is the inclusion morphism and γ_C is the homogeneous part of degree $d = \dim(X) - \operatorname{codim}(S) - \operatorname{codim}(T)$ of the product $s(C, S) \cdot c(\mathcal{N}_{T/X}|_C) \in A(C)$. If S is also a locally complete intersection, then we can write γ_C symmetrically as the degree d part of $s(C, X) \cdot c(\mathcal{N}_{S/X}|_C) \cdot c(\mathcal{N}_{T/X}|_C) \in A(C)$.

This formula clearly needs some digesting. The part we already understand are the Chern classes of the normal bundles $\mathcal{N}_{T/X}$ (and $\mathcal{N}_{S/X}$) pulled back to the connected component C (at least more or less: the trouble being that connected components of $S \cap T$ can, of course, be singular). The main point of using Segre classes in this formula is the term $s(C, X)$, which is the Segre class of the subvariety C of X. This is defined as follows. For a subvariety C of X (more precisely a proper subscheme $C \subset X$), set $\mathcal{S} = \bigoplus_{n \in \mathbb{N}} \mathcal{I}^n_{C/X} / \mathcal{I}^{n+1}_{C/X}$ and set $E = \operatorname{Proj}(\mathcal{S})$. Then we have a map $\pi \colon E \to X$ (which is the restriction of the blow up $\pi \colon \operatorname{Bl}_C(X) \to X$ of X along C to the exceptional divisor E). We then define similarly to the case $E = \mathbb{P}\mathcal{E}$ from before the Segre class of C in X to be

$$s(C, X) = \pi_* \left(\sum_{k \geq 0} c_1(\mathcal{O}_E(1))^k \right) \in A(C).$$

The intersection $c_1(\mathcal{O}_E(1))^k$ are taken in the Chow ring $A(E)$ of E. This definition generalizes the case where X is smooth and C is locally a complete intersection which is more geometric.

Under these assumptions we have

$$\mathcal{I}^n_{C/X}/\mathcal{I}^{n+1}_{C/X} \cong \mathrm{Sym}^n(\mathcal{I}_{C/X}/\mathcal{I}^2_{C/X}) = \mathrm{Sym}^n(\mathcal{N}^*_{C/X})$$

which implies that the Segre class $s(C,X)$ is, in this case, the Segre class $s(\mathcal{N}_{C/X})$ of the normal bundle of C in X, because $\mathrm{Proj}(\bigoplus_n \mathcal{I}^n_{C/X}/\mathcal{I}^{n+1}_{C/X}) = \mathbb{P}\mathcal{N}_{C/X}$. Concretely, this means

$$s(C,X) = s(\mathcal{N}_{C/X}) = \frac{1}{c(\mathcal{N}_{C/X})}.$$

We will now attempt to apply this formula in special cases.

7.4.1 Two curves in the plane

Bézout's Theorem tells us that the intersection of a curve C_1 of degree d_1 with a curve of degree d_2 in $\mathbb{P}^2$ is a scheme of length $d_1 \cdot d_2$ if the two curves have no irreducible components in common. From this, we directly get an expectation for the contribution of excess intersection in case that the two curves have a common component D of degree k: then the 0-dimensional part is the intersection of a curve $C_1 \setminus D$ of degree $d_1 - k$ with a curve $C_2 \setminus D$ of degree $d_2 - k$, which gives a 0-dimensional intersection of degree $(d_1 - k)(d_2 - k) = d_1 d_2 - (d_1 + d_2 - k)k$ – the common component D of degree k must therefore account for the $(d_1 + d_2 - k)k$ points that are missing compared to the case of a generic intersection of two curves of degree d_1 and d_2, respectively. Indeed, the intersection $[C_1] \cdot [C_2]$ has degree $d_1 \cdot d_2$ but is concretely written in the excess intersection formula as a sum of $d_1d_2 - (d_1+d_2-k)k$ points and the class $(\iota_D)_*(\gamma_D)$ corresponding to the common component D of degree k (because every irreducible curve in $\mathbb{P}^2$ is connected and any two curves intersect so that the common component D is always connected). Let us see how the excess intersection formula confirms this expectation, that is $\deg((\iota_D)_*(\gamma_D)) = (d_1+d_2-k)k$.

For this, we have to compute the classes $s(D,\mathbb{P}^2)$, $s(\mathcal{N}_{C_1/X}|_D)$, and $s(\mathcal{N}_{C_2/X}|_D)$ on the common component D of degree k. Here, we assume that C_1, C_2, and therefore D are reduced, which just says that the defining polynomials are squarefree. Every hypersurface is locally a complete intersection, which implies that we can use the symmetric formula in Theorem 7.4.1. The normal bundle of a reduced curve $C \subset \mathbb{P}^2$ is $\mathcal{N}_{C/\mathbb{P}^2}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(C)|_C = \mathcal{O}_C(\deg(C))$ (by adjunction, compare the proof of Proposition 5.4.6). From this formula, we know that $c(\mathcal{N}_{C/\mathbb{P}^2}) = (1+\deg(C)\zeta_C) \in A(C)$, where ζ_C is the restriction of the hyperplane class to C (which therefore has degree $\deg(C)$ as well). Since $s(D,\mathbb{P}^2) = 1/c(\mathcal{N}_{D/\mathbb{P}^2})$ in this case, we have everything we need to get

$$\begin{aligned}
(\iota_D)_*(\gamma_D) &= (\iota_D)_*\left\{\frac{c(\mathcal{N}_{C_i/\mathbb{P}^2}|_D)\cdot c(\mathcal{N}_{C_2/\mathbb{P}^2}|_D)}{c(\mathcal{N}_{D/\mathbb{P}^2})}\right\}_0\\
&= (\iota_D)_*\left\{\frac{(1+d_1\zeta_D)\cdot(1+d_2\zeta_D)}{(1+k\zeta_D)}\right\}_0\\
&= (\iota_D)_*\left\{1+(d_1+d_2-k)\zeta_D\right\}_0\\
&= (\iota_D)_*\left((d_1+d_2-k)\zeta_D\right)\\
&= (d_1+d_2-k)\cdot(\iota_D)_*(\zeta_D) = (d_1+d_2-k)\cdot(k\zeta^2)
\end{aligned}$$

And indeed, as expected from Bézout's Theorem, the degree of this class coming from the excess intersection in the common component D of degree d is $(d_1 + d_2 - k)k$.

7.4.2 Three surfaces in 3-space

We saw a heuristic discussion of excess intersection for surfaces in $\mathbb{P}^3$ in Section 1.3. With the warm up in the case of plane curves, let us now apply the formalism of excess intersection to determine some degrees in this case as well.

The setup is the following: Let S_1, S_2, and S_3 be surface in $\mathbb{P}^3$ of degree s_1, s_2, and s_3, respectively. Suppose that the intersection $S_1 \cap S_2 \cap S_3$ is a 0-dimensional scheme and an irreducible curve of degree d and genus g. In case that we are computing the degree of the dual projective surface for a smooth surface S_1, the surfaces S_2 and S_3 are general polar hypersurfaces to S_1 and we discussed how a line as excess intersection reduces the degree of the dual surface in Section 1.3: If S has degree d, then a line diminishes the degree of the dual surface by $3d - 4$.

Let us begin with a heuristic for a slightly more general case that we have three surfaces S_i with excess intersection equal to a line $L \subset S_1 \cap S_2 \cap S_3$ along which the intersection is smooth, arguing essentially along the lines of the 19th century, well before the development of intersection theory as the field it is today. For simplicity, we assume that S_1 is smooth (which can be shown not to be a restriction). Then $S_1 \cap S_2 = L + D$ for a divisor D on S_1 and $S_1 \cap S_3 = L + E$ for another divisor E on S_2. By the smoothness assumption on L, it follows that $S_1 \cap S_2 \cap S_3$ is the union of L and $\Gamma = D \cap E$. So we can reduce our computation of the degree of the 0-dimensional part of the intersection to the computation of $\deg([D] \cdot [E])$ in $A(S_1)$. (This idea of splitting off the excess intersection by working in the Chow ring of the complete intersection S_1 is developed in Vogel's approach to excess intersection of hypersurfaces, compare section 13.3.6 in Eisenbud, Harris: 3264).

Since $D = S_1 \cap S_2 \setminus L$, we have $[D] = \deg(S_2)[H] - [L]$, where $[H]$ is the class of a hyperplane section of S_1 so that $[D] = s_2[H] - [L]$ and symmetrically $[E] = s_3[H] - [L]$ in $A^1(S_1)$. So we get

$$[D] \cdot [E] = (s_2[H] - [L]) \cdot (s_3[H] - [L]) = s_2 s_3 [H]^2 - (s_2 + s_3)[H] \cdot [L] + [L]^2.$$

Since S_1 has degree s_1, the product $[H]^2$ has degree s_1. The intersection $[H] \cdot [L]$ has degree 1, because $[H]$ is the class of a hyperplane in $\mathbb{P}^3$ and $[L]$ the class of a line. So the main point is to compute the degree of $[L]^2$. It can be shown that $\deg([L]^2) = \deg(c_1(\mathcal{N}_{L/S_1}))$, which is related to the canonical divisor of L and hence can be computed with the adjunction formula. This formula states that the canonical divisor of a divisor $C \subset S$ inside a variety S is $K_C = (K_S + C)|_C$, which is the sum of the canonical divisor K_S of S with the divisor class of C restricted back to C. In our case, it determines the canonical divisor of the hypersurface $S_1 \subset \mathbb{P}^3$ to be $K_{S_1} = \mathcal{O}_{S_1}(s_1 - 4)$. For $L \subset S_1$, it says

$$K_L = [L]|_L + [K_{S_1}]|_L.$$

By assumption, L is a line so that the canonical divisor of L has degree -2. The divisor $[K_{S_1}]|_L$ on L has degree $\deg([K_{S_1}] \cdot [L]) = s_1 - 4$. Since $\deg(K_L) = \deg([L] \cdot [K_L])$ by the adjunction formula $2g - 2 = \deg([C] \cdot ([C + K]))$ for a nonsingular curve C inside a smooth, it follows that $-2 = \deg([L]^2) + s_1 - 4$, which determines $\deg([L]^2)$ to be $2 - s_1$.

To summarize, this computation shows that the degree $\deg(\Gamma)$ of the 0-dimensional intersection of the surfaces S_1 is

$$\deg(\Gamma) = s_1s_2s_3 - (s_2+s_3) + 2 - s_1 = s_1s_2s_3 - (s_1+s_2+s_3-2)$$

The excess intersection along the line L accounts for $s_1+s_2+s_3-2$ intersection points (which recovers the special case that $s_2 = s_3 = s_1 - 1$ that we considered for projective duality in the introduction).

Essentially the same computation can be applied if $S_1 \cap S_2 \cap S_3$ have an excess intersection along a smooth curve C of degree d and genus g (instead of the line L of degree 1 and genus 0). In this case, the degree of $[H]\cdot[C]$ is d and the canonical divisor has degree $2g-2$, which by adjunction gives $\deg([C]^2) = 2g-2-d(s_1-4)$. The summary then becomes

$$\deg(\Gamma) = s_1s_2s_3 - (s_2+s_3)d + 2g - 2 - d(s_1-4) = s_1s_2s_3 - (s_1+s_2+s_3)d + 4d + 2g - 2.$$

The curve in the intersection accounts in this sense for quite a lot of points, namely the difference $d(s_1+s_2+s_3) - 4d - 2g + 2$ from the Bézout bound $s_1s_2s_3$.

Our next goal is to check this number with the help of the excess intersection formula.

In this case, we apply a triple intersection version in $X = \mathbb{P}^3$ out of the box which is

$$[S_1]\cdot[S_2]\cdot[S_3] = \sum_C (\iota_C)_*(\gamma_C),$$

where the sum is taken over all connected components $C \subset S_1 \cap S_2 \cap S_3$ and γ_C is the 0-dimensional part of the product $s(C,\mathbb{P}^3)\cdot\prod_{i=1}^3 c(\mathcal{N}_{S_i/\mathbb{P}^3}|_C)$. We compute the excess intersection along a smooth curve $D \subset \mathbb{P}^3$ of degree d and genus g. Our task is to check

$$\deg((\iota_D)_*(\gamma_D)) = d(s_1+s_2+s_3) - 4d - 2g + 2.$$

Since we assumed D to be smooth, it is locally a complete intersection and we have

$$s(D,\mathbb{P}^3) = \frac{1}{c(\mathcal{N}_{C/\mathbb{P}^3})}.$$

We can compute this class by the exact sequence

$$0 \to \mathcal{N}_{D/S_1} \to \mathcal{N}_{D/\mathbb{P}^3} \to \mathcal{N}_{S_1/\mathbb{P}^3}|_D \to 0$$

of relative normal bundles on D using Whitney's formula, which implies

$$c(\mathcal{N}_{D/\mathbb{P}^3}) = c(\mathcal{N}_{D/S_1})\cdot c(\mathcal{N}_{S_1/\mathbb{P}^3}|_D).$$

Above, we computed $\deg(c_1(\mathcal{N}_{D/S_1})) = 2g-2-d(s_1-4)$ using the adjunction formula. The first Chern class of $\mathcal{N}_{S_1/\mathbb{P}^3}$ has degree s_1 and so its restriction to the curve D of degree d has degree $d\cdot s_1$. In total, we get $\deg(c_1(\mathcal{N}_{D/\mathbb{P}^3})) = \deg(c_1(\mathcal{N}_{D/S_1})) + \deg(c_1(\mathcal{N}_{S_1/\mathbb{P}^3}|_D))$, which is

$$\deg(c_1(\mathcal{N}_{D/\mathbb{P}^3})) = 2g-2-d(s_1-4) + ds_1 = 4d + 2g - 2.$$

This leaves the three terms in the numerator. The degree of the first Chern class of $\mathcal{N}_{S_i/\mathbb{P}^3}|_D$ is

ds_i by the same argument as above. So the degree of the 0-dimensional part of

$$s(C,\mathbb{P}^3)\cdot\prod_{i=1}^{3}c(\mathcal{N}_{S_i/\mathbb{P}^3}|_C)$$

is $d(s_1+s_2+s_3)-4d-2g+2$, as claimed.

7.4.3 Five Conics: 3264

As before, let $D_1, D_2, \ldots, D_5$ be general conics in $\mathbb{P}^2$ and let Z_i be the (closure of the) set of all (smooth) conics C that are tangent to D_i. The divisor $Z_i \subset \mathbb{P}^5$ in the space of equations of conics up to scaling is a hypersurface of degree 6. The (scheme theoretic) intersection of the five divisors Z_i is

$$\bigcap_{i=1}^{5} Z_i = T \cup \Gamma,$$

where Γ is a reduced, 0-dimensional scheme (a set of points – namely 3264, as we know) and the variety underlying the scheme T is the Veronese surface $S = \nu_2(\mathbb{P}^2)$ of conics of rank 1. Here, we have $T \neq S$, because the scheme T carries a non-reduced structure, which we won't get into in much detail. So we sketch how to compute $|\Gamma| = 3264$ by the excess intersection formula, partially relying on computations that we made in Section 6.1.2.

For the connected component $S \subset \bigcap Z_i$, we use the fivefold intersection formula

$$\deg\left\{s(T,\mathbb{P}^5)\cdot\prod_{i=1}^{5}c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S)\right\}_0$$

to compute the contribution of th excess intersection scheme T. So let's go.

(a) *Chern class $c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S)$:* The Veronese embedding $\nu_2\colon \mathbb{P}^2 \longrightarrow \mathbb{P}^5$ is an embedding, so it is an isomorphism $S \cong \mathbb{P}^2$. Write ζ for the class of a line in $\mathbb{P}^2$ in $A(S)$ and η for the hyperplane class in $A(\mathbb{P}^5)$. Since pulling back a hyperplane in $\mathbb{P}^5$ to $\mathbb{P}^2$ via ν_2 gives a conic in $\mathbb{P}^2$, we have $\nu_2^*(\eta) = 2\zeta$. We already know that $Z_i \subset \mathbb{P}^5$ is a hypersurface of degree 6, which implies (by adjunction) that $\mathcal{N}_{Z_i/\mathbb{P}^5} = \mathcal{O}_{Z_i}(6)$ and pulling this line bundle back to S then gives

$$c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S) = 1 + \nu_2^*(6\eta) = 1 + 12\zeta.$$

Our next goal is to compute $s(T,\mathbb{P}^5)$ which we will reduce to the computation $s(\mathcal{N}_{S/\mathbb{P}^5})$. This is possible in this case because the multiplicity structure of T turns out to be nice.

(b) *Multiplicity of Z_i along S:* The goal is to determine how singular Z_i is along S, which can be done by Riemann-Hurwitz again by intersecting Z_i with a general pencil P of conics among those that contains a double line $2L \in S$. We want to compute the ramification points of a quadratic map $\varphi_P\colon \mathbb{P}^1 \longrightarrow D_i \longrightarrow \mathbb{P}^1$, $(s:t) \mapsto (q_0(d_i(s,t)) : q_1(d_i(s,t)))$ where q_0, q_1 is a basis of P and $d_i\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^2$ a parametrization of the rational curve D_i. In total, this is a (generically) finite map of degree 4 and has 6 ramification points by the Riemann-Hurwitz formula (compare Section 1.4). The double line $2L$ in P counts for two intersection points in the Riemann-Hurwitz formula because it is a higher order ramification point. This shows that Z_i has multiplicity 2 along $S \subset Z_i$.

(c) *Chern class of $\mathcal{N}_{S/\mathbb{P}^5}$:* We use the exact sequence

$$0 \to \mathcal{T}_S \to \mathcal{T}_{\mathbb{P}^5}|_S \to \mathcal{N}_{S/\mathbb{P}^5} \to 0$$

of vector bundles on S defining the relative normal bundle and Whitney's formula. We know the tangent bundle of S because $S \cong \mathbb{P}^2$, which implies that $c(\mathcal{T}_S) = (1+\zeta)^3 \in A(S)$. The Chern class of $\mathcal{T}_{\mathbb{P}^5}$ is $(1+\eta)^6$ so that

$$c(\mathcal{T}_{\mathbb{P}^5}|_S) = (1+2\zeta)^6 = 1 + 2 \cdot 6 \cdot \zeta + 4 \cdot 15 \cdot \zeta^2.$$

Applying Whitney's formula gives the Chern class of $\mathcal{N}_{S/\mathbb{P}^5}$ as

$$c(\mathcal{N}_{S/\mathbb{P}^5}) = \frac{1 + 12\zeta + 60\zeta^2}{1 + 3\zeta + 3\zeta^2} = 1 + 9\zeta + 30\zeta^2.$$

We get the Segre class $s(\mathcal{N}_{S/\mathbb{P}^5})$ as the reciprocal of the Chern class in $A(S)$, which gives

$$s(\mathcal{N}_{S/\mathbb{P}^5}) = 1 - 9\zeta + 51\zeta^2.$$

Since S is smooth in $\mathbb{P}^5$, it is locally a complete intersection, which implies that

$$s(S, \mathbb{P}^5) = s(\mathcal{N}_{S/\mathbb{P}^5}) = 1 - 9\zeta + 51\zeta^2.$$

(d) *Segre class $s(T, \mathbb{P}^5)$:* It can be shown that $T = \mathcal{V}_+(I^2_{S/\mathbb{P}^5})$. Believing this and going back to the definition of the Segre class $s(T, \mathbb{P}^5)$ as

$$\pi_* \left(\sum_{k=3}^{5} c_1(\mathcal{O}_E(1)^k) \right)$$

with $E = \operatorname{Proj}(\mathcal{S})$ and $\mathcal{S} = \bigoplus_{n \in \mathbb{N}} \mathcal{I}^n_{T/\mathbb{P}^5} / \mathcal{I}^{n+1}_{T/\mathbb{P}^5}$ (because the dimension of E is 5), we get

$$s_k(T, \mathbb{P}^5) = 2^{k+3} \cdot s_k(S, \mathbb{P}^5).$$

This formula gives us the last ingredient for our computation as

$$s(T, \mathbb{P}^5) = 8 - 16 \cdot 9 \cdot \zeta + 32 \cdot 51 \cdot \zeta^2 = 8 - 144\zeta + 1632\zeta^2.$$

(e) *Putting it all together:* The contribution of T to the intersection $\bigcap_{i=1}^5 Z_i$ is the degree of the 0-dimensional part of

$$s(T, \mathbb{P}^5) \cdot \prod_{i=1}^{5} c(\mathcal{N}_{Z_i/\mathbb{P}^5}|_S) \in A(S)$$

which we can now compute as the coefficient of ζ^2 in

$$\begin{aligned}
&\operatorname{coeff}\left((8-144\zeta+1632\zeta^2)(1+12\zeta)^5,\zeta^2\right) = \\
&\operatorname{coeff}\left((8-144\zeta+1632\zeta^2)(1+12\cdot 5\zeta+12^2\cdot 10\zeta^2)\zeta^2\right) = \\
&8\cdot 1440-144\cdot 60+1632 = 4512.
\end{aligned}$$

That gives the cardinality of the 0-dimensional part Γ as before as $6^5-4512 = 3264$. Comparing this result with the formula

$$3264 = 2^5\cdot(1+10+40+40+10+1)$$

that we found in Section 6.1.2 using the space of complete conics, they appear very different.

Chapter 8

References

This course is based mostly on the following textbooks.

[FIT] William Fulton, *Intersection Theory*. Springer, Second Edition, 1998.

[FIIT] William Fulton, *Introduction to Intersection Theory in Algebraic Geometry*. AMS, Regional Conference Series in Mathematics, 1985.

[3264] David Eisenbud and Joe Harris, *3264 and all that*. Cambridge University Press, 2016.

During the course, we might include further literature on more special topics. As a reference for some of the more advanced algebraic geometry and commutative algebra, the following books might be helpful.

◇ Robin Hartshorne, *Algebraic Geometry*. Springer, 1983.
◇ Joe Harris, *Algebraic Geometry*. Springer, 1992.
◇ David Eisenbud, *Commutative Algebra*. Springer, 1995.
◇ Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay Rings*, Cambridge University Press, 1998.
◇ Hideyuki Matsumura, *Commutative Ring Theory*. Cambridge University Press, 1989.