UNIVERSITÄT LEIPZIG

Jun.-Prof. Dr. Mira Schedensack
Mathematatisches Institut
Universität Leipzig

Seminar for the lecture "Advanced Computational Partial Differential Equations"

summer term 2026

Topic: Crouzeix-Raviart FEM for right-hand sides in $H^{-1}(\Omega)$

1. Make yourself familiar with the implementation of the CR-FEM for the Poisson problem in [Bar16, Section 7.2.6].

2. Modify the right-hand side in the code to realize the discrete problem: Find $u_{\mathrm{CR}} \in \mathrm{CR}_0^1(\mathcal{T})$ such that

$$\int_\Omega \nabla_{\mathrm{pw}} u_{\mathrm{CR}} \cdot \nabla_{\mathrm{pw}} v_{\mathrm{CR}}\, dx = G(J_1 v_{\mathrm{CR}}) \qquad \text{for all } v_{\mathrm{CR}} \in \mathrm{CR}_0^1,$$

where $J_1 : \mathrm{CR}_0^1(\mathcal{T}) \to S_0^1(\mathcal{T})$ denotes the enriching operator defined by

$$(J_1 v_{\mathrm{CR}})(z) = \frac{1}{\#\mathcal{T}(z)} \sum_{T \in \mathcal{T}(z)} (v_{\mathrm{CR}}|_T)(z) \qquad \text{for all } z \in \mathcal{N}(\Omega)$$

and $G \in (H_0^1(\Omega))'$ is a right-hand side of the form

$$G(v) = \int_S gv\, ds,$$

where $S \subseteq \bigcup\{F | F \in \mathcal{F}(\mathcal{T}_0)\}$ and $\mathcal{T}_0$ denotes the initial coarse triangulation and $g \in L^2(S)$.

3. Run numerical experiments to test the method.

4. Optional: Compare the method with the standard CR-FEM for some G as above with $S = F \in \mathcal{F}(\mathcal{T}_0)$, where G is replaced by G_ε defined by

$$G_\varepsilon(v) := \frac{1}{\varepsilon} \int_\Omega g_\varepsilon v\, ds \qquad \text{with} \qquad g_\varepsilon(x) := \begin{cases} g(\pi_S(x)) & \text{if } \operatorname{dist}(x - S) < \varepsilon, \\ 0 & \text{else.} \end{cases}$$

Here, π_S denotes the projection onto S.

literature

[Bar16] Sören Bartels. *Numerical approximation of partial differential equations*, volume 64 of *Texts Appl. Math.* Cham: Springer, 2016.