Exercise Sheet 11

winter term 2025/26

Discussion on 19.01.2026

Exercise 1 (transformation of finite elements)**.** Let $(\widehat{T}, \widehat{\mathcal{P}}, \mathcal{K}_{\mathrm{ref}})$ be a finite element and $\Phi_T : \widehat{T} \to T$ an affine diffeomorphism.

a) Show that $(T, \mathcal{P}, \mathcal{K})$ is a finite element, where

$$\begin{aligned}
&T = \Phi_T(\widehat{T}),\\
&\mathcal{P} = \{\widehat{q} \circ \Phi_T^{-1} \,|\, \widehat{q} \in \widehat{\mathcal{P}}\},\\
&\mathcal{K} = \{\chi | \chi(v) = \widehat{\chi}(v \circ \Phi_T) \text{ for all } v \in \mathcal{C}^\infty(T), \widehat{\chi} \in \widehat{\mathcal{K}}\}.
\end{aligned}$$

b) Show that the corresponding interpolants I_T and $I_{\widehat{T}}$ and any $\widehat{v} \in W^{m,p}(\widehat{T})$ and $v := \widehat{v} \circ \Phi_T^{-1} \in W^{m,p}(T)$ satisfy

$$(I_T v) \circ \Phi_T = I_{\widehat{T}} \widehat{v}.$$

Exercise 2 (barycentric coordinates)**.** Consider a triangle $T = \operatorname{conv}\{P_1, P_2, P_3\}$ and the barycentric coordinates $\lambda_1, \lambda_2, \lambda_3 \in P_1(T)$ defined via $\lambda_j(P_k) = \delta_{jk}$ for $j, k = 1, 2, 3$.

a) Prove that any $\alpha, \beta, \gamma \in \mathbb{N}_0$ satisfy

$$\int_T \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma \, dx = 2|T| \frac{\alpha!\beta!\gamma!}{(2 + \alpha + \beta + \gamma)!}.$$

Hint: You can use integration by parts on the reference triangle.

b) Compute the local mass matrix for $P_1(T)$, i.e.

$$M_1 = \left(\int_T \lambda_i \lambda_j \, dx \right)_{i,j=1,\dots,3} \in \mathbb{R}^{3\times 3}.$$

Exercise 3 (approximation of the right-hand side)**.** Define the L^2 projection to piecewise constants by $\Pi_0 : L^2(\Omega) \to P_0(\mathcal{T})$, i.e., for all $T \in \mathcal{T}$, $\|f - \Pi_0 f\|_{L^2(T)} = \min_{q \in P_0(T)} \|f - q\|_{L^2(T)}$. Let $u \in H^1_0(\Omega)$, $\widetilde{u} \in H^1_0(\Omega)$ and $\widehat{u} \in H^1_0(\Omega)$ be the solutions to

$$\begin{aligned}
\int_\Omega \nabla u \cdot \nabla v \, dx &= \int_\Omega f \, v \, dx \qquad \text{for all } v \in H^1_0(\Omega),\\
\int_\Omega \nabla \widetilde{u} \cdot \nabla v \, dx &= \int_\Omega \Pi_0 f \, v \, dx \qquad \text{for all } v \in H^1_0(\Omega),\\
\int_\Omega \nabla \widehat{u} \cdot \nabla v \, dx &= \sum_{T \in \mathcal{T}} f(x_T) \int_T v \, dx \qquad \text{for all } v \in H^1_0(\Omega),
\end{aligned}$$

where $x_T \in T$ is arbitrary for all $T \in \mathcal{T}$. Define the oscillations

$$\operatorname{osc}(f, \mathcal{T}) := \sqrt{\sum_{T \in \mathcal{T}} \|h_T (f - \Pi_0 f)\|^2_{L^2(T)}}$$

and let $h := \max_{T\in\mathcal{T}} h_T$ be the maximal mesh-size of $\mathcal{T}$. Show that

$$\begin{aligned}\|\nabla(u-\widetilde{u})\| &\lesssim \operatorname{osc}(f,\mathcal{T}),\\ \|\nabla(u-\widehat{u})\| &\lesssim \sqrt{\sum_{T\in\mathcal{T}}\|f-f(x_T)\|^2_{L^2(T)}}.\end{aligned}$$

If $f \in H^1(\Omega)$, then

$$\operatorname{osc}(f,\mathcal{T}) \lesssim \|\nabla f\| h^2.$$

If $f \in H^2(\Omega)$ then

$$\sqrt{\sum_{T\in\mathcal{T}}\|f-f(x_T)\|^2_{L^2(T)}} \lesssim \|\nabla f\| h, \qquad \text{and if } x_T = \operatorname{mid}(T), \text{ then } \|\nabla(u-\widehat{u})\| \lesssim h^2.$$

Is it possible to modify the right-hand side in the P^1 FEM similar as for $\widehat{u}$ with arbitrary $x_T \in T$ without diminishing the convergence rate? Is the same possible for the P_2 FEM?

Exercise 4 (programming exercise)**.** Study the MATLAB implementation of the P_1 FEM from the lecture, that computes the P_1 finite element solution to the Poisson problem for homogeneous Dirichlet boundary data and $f \equiv 1$. Modify it to include general right-hand sides $f \in L^2(\Omega)$, given as function handle.
Hint: You may use the midpoint quadrature rule from Exercise 3 with $x_T = \operatorname{mid}(T)$ for the integration of

$$\int_T g(x)\ dx \approx |T| g(\operatorname{mid}(T)).$$