Exercise Sheet 5

winter term 2025/26

Discussion on 24.11.2025

Exercise 1. Prove Lemma 4.14 from the lecture:
A sequence $(y_k)_{k=1,\ldots}$ is a solution of the difference equation given by $(\alpha_\ell)_{\ell=0,\ldots,m}$ if and only if the vectors

$$Y_k = (y_k, y_{k+1}, \ldots, y_{k+m-1})^\top$$

satisfy $Y_{k+1} = AY_k$ for $k = 0, 1, \ldots$. Here, the companion matrix $A \in \mathbb{R}^{m\times m}$ is defined by

$$A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{m-1} \end{pmatrix}.$$

If A has the linearly independent eigenvectors $v_1, v_2, \ldots, v_m \in \mathbb{R}^m$ with corresponding eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_m$ and if $\gamma_1, \gamma_2, \ldots, \gamma_m \in \mathbb{R}$ are the coefficients of the vector Y_0 with respect to this basis, it follows

$$Y_k = A^k Y_0 = \sum_{j=1}^{m} \lambda_j^k \gamma_j v_j.$$

The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ are exactly the roots of the characteristic polynomial, which reads (up to a sign)

$$q(\lambda) = \lambda^m + \alpha_{m-1}\lambda^{m-1} + \cdots + \lambda\alpha_1 + \alpha_0.$$

Exercise 2. Show that the geometric multiplicity of an Eigenvalue of the companion matrix of a multitstep method, i.e. a matrix like

$$A = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \ldots & -\alpha_{m-1} \end{pmatrix}.$$

has to be 1.

Exercise 3. Verify whether the multistep methods

$$y_{k+2} - 4y_{k+1} + 3y_k = -2\tau f(t_k, y_k)$$

and

$$y_{k+4} - y_k = \frac{h}{3}\left(8f(t_{k+3}, y_{k+3}) - 4f(t_{k+2}, y_{k+2}) + 8f(t_{k+1}, y_{k+1})\right)$$

are convergent and compute the order of convergence.

Exercise 4. Compute the solutions of the explicit and implicit Euler-method of the initial value problem

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \qquad \text{with initial value} \qquad \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

For this you can compute the eigenvalues of the matrix

$$A = \begin{pmatrix} \lambda_1 + \lambda_2 & \lambda_1 - \lambda_2 \\ \lambda_1 - \lambda_2 & \lambda_1 + \lambda_2 \end{pmatrix}$$

and then compute a decomposition of the form $A = C^{-1}DC$ with a diagonal matrix D. Derive a formula of the solution of the explicit and the implicit Euler-method. Are the solutions bounded for $\lambda_1, \lambda_2 < 0$?