

Exercise Sheet 1

winter term 2025/26

Discussion on 20.10.2025

Exercise 1. Prove the following theorem from the lecture:

If $f \in \mathcal{C}^m([0, T] \times \mathbb{R}^n)$, then the solution of the ODE $y'(x) = f(x, y(x))$ satisfies $y \in \mathcal{C}^{m+1}([0, T])$. If $m \geq 1$, the solutions of the corresponding initial value problems are unique.

Exercise 2 (Discrete Gronwall-Lemma). Let $(u_k)_{k=0, \dots, K}$ be a sequence of non negative, real numbers and $\alpha, \beta \in \mathbb{R}$ with $\beta \geq 0$. Furthermore let

$$u_\ell \leq \alpha + \tau \sum_{k=0}^{\ell-1} \beta u_k \quad \text{for all } \ell = 0, \dots, K.$$

Show that for all $\ell = 0, \dots, K$

$$u_\ell \leq \alpha e^{\ell \tau \beta}$$

holds. (*Hint: Use $(1 + \tau \beta) \leq e^{\tau \beta}$ inductively.*)

Exercise 3 (Gerschgorin circle theorem). Let $A \in \mathbb{R}^{n \times n}$ and let $\lambda \in \mathbb{C}$ be an eigenvalue of A . For $i \in \{1, \dots, n\}$ define the i -th Gerschgorin disc

$$D_i := \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{\substack{j=1, \dots, n \\ j \neq i}} |a_{ij}| \right\}.$$

Show that

$$\lambda \in \bigcup_{i=1}^n D_i.$$

Exercise 4. Let $D_1 = 1 \frac{N}{m}$, $D_2 = 1.0201 \frac{N}{m}$ be the spring constants of two spring pendulums with the same mass $m = 1 \text{ kg}$. The two pendulums are described by the following ODEs and their initial values

$$\begin{aligned} y_i''(t) &= -\frac{D_i}{m} y_i(t) & i \in \{1, 2\} \text{ and } t \in [0, T] \\ y_i(0) &= 1 & i \in \{1, 2\} \\ y_i'(0) &= 0 & i \in \{1, 2\} \end{aligned}$$

Compute the unique solutions for y_1 and y_2 and their difference at time $T = 100\pi s \approx 314s$.