Convergence Rates in Monotone Separable Stochastic Networks

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We study bounds on the rate of convergence to the stationary distribution in monotone separable networks which are represented in terms of stochastic recursive sequences. Monotonicity properties of this subclass of Markov chains allow us to formulate conditions in terms of marginal network characteristics. Two particular examples, generalized Jackson networks and multiserver queues, are considered.

CONVERGENCE RATES, MOMENTS, COUPLING, RENOVATING EVENTS, HARRIS ERGODIC MARKOV CHAIN, MONOTONE SEPARABLE NETWORK, GENERALIZED JACKSON NETWORK, MULTISERVER QUEUE

1 Introduction

Let X_n^x, $n = 0, 1, \ldots$, $X_0^x = x$ be a (time-homogeneous) Markov chain (MC) taking values in a state space $\mathcal{X}$ with a countably generated sigma-algebra $\mathcal{B}_{\mathcal{X}}$. Denote by $P(\cdot, \cdot)$ its transition probabilities.

It is known (see, e.g. [20]) that any MC may be represented as a stochastic recursive sequence (SRS)

$$X_{n+1}^x = f(X_n^x,\ Y_{n+1}), \quad n \geq 0. \tag{1.1}$$

Herein, the elements Y_n of the driving sequence are i.i.d. and take values in another measurable state space $(\mathcal{Y}, \mathcal{B}_{\mathcal{Y}})$, and the function $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ is measurable with respect to the product sigma-algebra $\mathcal{B}_{\mathcal{X}} \times \mathcal{B}_{\mathcal{Y}}$ (that is the minimal sigma-algebra which contains all sets $A \times B$, $A \in \mathcal{B}_{\mathcal{X}}$, $B \in \mathcal{B}_{\mathcal{Y}}$). Then

$$P(z, A) = \mathbf{P}(f(z, Y_1) \in A), \quad z \in \mathcal{X},\ A \in \mathcal{B}_{\mathcal{X}}.$$

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Assume further that the MC X_n^x is *Harris ergodic*, i.e. there exists a (stationary) distribution π on $\mathcal{X}$ such that the distribution of X_n^x converges to π in the total variation norm:

$$\|\mathbf{P}(X_n^x \in \cdot) - \pi(\cdot)\| \equiv \sup_{A \in \mathcal{B}_{\mathcal{X}}} |\mathbf{P}(X_n^x \in A) - \pi(A)| \to 0, \quad n \to \infty.$$

Geometric ergodicity of a general MC is treated in [27, Chapter 15]. Various sufficient conditions for subgeometric rates of convergence of an f-ergodic MC are obtained in [32]. The equivalent conditions of [32] are couched in terms of the first hitting time on some set and a sequence of Foster-Lyapunov drift conditions. There have been several attemts to replace the sequence of drift conditions in [32, Theorem 2.1], which are difficult to check, with a single one ([12, 19, 13]). Note that the geometric ergodicity of an MC follows from a single drift condition (see [27, Chapter 15]). The nested drift conditions of [32] were first dropped in [12], where f-ergodicity and the power convergence of a certain Markov process associated with a multiclass queueing network were studied. In [19] a single drift condition for the power convergence of a general MC was constructed. The condition of [19] was generalized in [13], which permitted the proof of subgeometric rates of convergence in a rather straightforward way. All the papers on convergence rates listed above are based on a concept of a 'petite set' (see [27]). An alternative approach was suggested in [21], where subgeometric rates of convergence of an MC in $\mathbb{R}^d$ were studied. In [21] it is assumed that an MC satisfies the local Doeblin condition. This condition together with recurrence type conditions implied irreducibility.

In our paper we consider only convergence in the total variation norm. For our purpose it is more convenient to use sufficient conditions for various convergence rates of $\|\mathbf{P}(X_n^x \in \cdot) - \pi(\cdot)\|$ to 0 in terms of the first hitting time on some set. We first recall these conditions. Then we use representation (1.1) and apply the results to a subclass of networks which are *monotone* and *separable* (see Section 3 for definitions). We introduce a class of networks with an arbitrary initial state which generalizes the class of monotone separable networks described in [4]. For such networks, we give a *unified approach* to obtaining estimates of the convergence rates of $\|\mathbf{P}(X_n^x \in \cdot) - \pi(\cdot)\|$ to 0, where X_n^x describes the behavior of the network. In particular, we get the same bounds on the convergence rate for a number of network characteristics. Our conditions are formulated in terms of the so-called *individual maximal dater* Z. In our main Theorem 3.1, we prove that, given the existence of *renovating events of positive probability*, if $\mathbf{E}e^{g(Z)} < \infty$ for a certain function g, then $e^{\widetilde{g}(n)}\|\mathbf{P}(X_n^x \in \cdot) - \pi(\cdot)\| \to 0$ for another function $\widetilde{g}$; the relation between g and $\widetilde{g}$ is then specified. We consider mainly functions g which grow to infinity slower than linear and faster than logarithmic functions. We apply the general results to two particular examples of monotone and separable networks:

generalized Jackson networks and multiserver queues. In each example, we provide a sample-path representation of a network in terms of SRS, and construct the corresponding renovating events.

Properties of SRS were thoroughly investigated in the stationary-ergodic context in [10, 11]. In particular, the existence of a stationary measure and coupling convergence (for the definition see, for example, [22]) were proved by using renovation techniques. General monotone Markov processes and their various applications were considered in [26, 14].

Monotone separable networks were introduced in [4]. They include generalized Jackson networks, multiserver queues, Petri nets, (max,+) systems, etc. Monotonicity properties of the workload processes and the queue length processes in single- and multiserver queues are described in detail in [25, 29]. Various monotonicity properties of Jackson networks are proved in [23, 17, 28, 16]. A general description of a generalized Jackson network in terms of monotone processes was proposed in [3].

Basic results on convergence rates for the workload process in a stable $GI/GI/1$ queue may be found, e.g., in [7]. For a summary of results on power and exponential convergence rates for various queueing systems, see [8, Chapter 4]). In [30, 31], the intermediate case between power and exponential convergence in single- and multiserver queues was considered. In [24], the rates of convergence were studied by considering a new metric for the input process. The exact rate of convergence for the $M/G/1$ queue with regularly varying distribution of service times was obtained in [1]. First results on rates of convergence in generalized Jackson networks were obtained in [9] under rather restrictive assumptions on the distributions of the inter-arrival and service times. Exponential convergence rates have been obtained for Markovian Jackson networks in [15], and for general monotone regenerative Markov processes in [26]. In [12] the power convergence for multiclass queueing networks was established via a fluid model.

Structure of the paper. In the auxiliary Section 2 we formulate known results on convergence rates for Harris ergodic Markov chains. Section 3 contains a description of a monotone separable network with an arbitrary initial state and the main result on convergence rates, Theorem 3.1. We then consider two applications. The instructive example of a monotone separable network, the generalized Jackson network, is treated in Section 4, see Theorem 4.1. Bounds on the rate of convergence in multiserver queues are studied in Section 5. Finally, the Appendix contains some technical results.

2 Convergence rates in Markov chains

In this section we formulate some results on convergence rates for Harris ergodic Markov chains. The results are applied to monotone separable networks in the next section. The following criterion for ergodicity of Markov chains is well-known (see, for example, [10, Theorem 1]).

Theorem 2.1. *Let $\{X_n\}$ be aperiodic. Assume that there exists a measurable set $B \in \mathcal{B}_{\mathcal{X}}$ such that, for*

$$\tau(x, B) = \min\{n \geq 1 \ : \ X_n^x \in B\},$$

1. $\mathbf{P}(\tau(x, B) < \infty) = 1$ *for all* $x \in \mathcal{X}$*;*

2. $\sup\limits_{x \in B} \mathbf{E}\tau(x, B) < \infty$*;*

3. *there exist an integer $l \geq 1$, a number $p \in (0, 1]$, and a probability measure φ on $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ such that, for all $x \in B$,*

$$\mathbf{P}(X_l^x \in A) \geq p\varphi(A), \quad \textit{for all } A \in \mathcal{B}_{\mathcal{X}}.$$

Then $\{X_n\}$ is Harris ergodic, i.e. there exists a probability measure π on $(\mathcal{X}, \mathcal{B}_{\mathcal{X}})$ such that, for any $x \in \mathcal{X}$,

$$\|\mathbf{P}(X_n^x \in \cdot) - \pi(\cdot)\| \to 0, \quad \textit{as } n \to \infty.$$

Remark 1. Suppose that an MC is represented as an SRS. Then Condition 3 of Theorem 2.1 follows from
Condition $3'$: *for some $l \geq 1$ there exists an event $A \in \sigma(Y_1, \ldots, Y_l)$ of positive probability such that , for any $x \in B$,*

$$X_l^x \mathbf{I}(A) = G(Y_1, \ldots, Y_l)\mathbf{I}(A) \quad \textit{a.s.},$$

where the measurable function G does not depend on x.
Conversely, the standard splitting arguments leed to the following: if Condition 3 holds, then there exists an SRS representation of an MC such that Condition 3' holds (see, e.g. [2]).

From [32, Theorem 2.1], one can deduce the following result.

Theorem 2.2. *Consider a function g on the real line such that, for some $M > 0$, g is bounded for $x \le M$, $g(x) = o(x)$ as $x \to \infty$, there exists the right derivative $g'(x)$ for $x \ge M$ such that $g'(x) > 0$, and the function*

$$\widetilde{g}(x) = \begin{cases} 0, & \text{if } x < 0, \\ g(M) + \ln g'(M), & \text{if } 0 \le x \le M, \\ g(x) + \ln g'(x), & \text{if } x > M \end{cases} \tag{2.1}$$

is subadditive, i.e. $\widetilde{g}(x+y) \le \widetilde{g}(x) + \widetilde{g}(y)$ for all x, y.

Suppose that the conditions of Theorem 2.1 hold. Let $z \in \mathcal{X}$. Suppose also that

$$\mathbf{E}e^{\widetilde{g}(\tau(z,B))} < \infty \tag{2.2}$$

and

$$\sup_{y \in B} \mathbf{E}e^{g(\tau(y,B))} < \infty. \tag{2.3}$$

Then

$$e^{\widetilde{g}(n)} \|\mathbf{P}(X_n^z \in \cdot) - \pi(\cdot)\| \to 0 \quad \text{as} \quad n \to \infty. \tag{2.4}$$

We also use the known result on exponential convergence rates. The following is [27, Theorem 15.0.1]:

Theorem 2.3. *Suppose that the conditions of Theorem 2.1 hold. Let $z \in \mathcal{X}$. If there exists some $\alpha > 0$ such that*

$$\mathbf{E}e^{\alpha\tau(z,B)} < \infty,$$

and

$$\sup_{x \in B} \mathbf{E}e^{\alpha\tau(x,B)} < \infty, \tag{2.5}$$

then

$$e^{\beta n} \|\mathbf{P}(X_n^z \in \cdot) - \pi(\cdot)\| \to 0 \quad \text{as} \quad n \to \infty,$$

for some $\beta > 0$.

3 Rates of convergence in monotone separable stochastic networks

The concept of a monotone separable network was introduced in [4]. Here we consider monotone separable networks *driven by stochastic recursions* with an arbitrary initial state.

We start with a deterministic network *without* exogeneous input. The *state space* of the network is a metric space $(\mathcal{X}, \rho)$ which contains a distinguished element $\mathbf{0}$ (*the empty state*). We assume that the following properties hold:
(i) if the network is in state $\mathbf{0}$ at some time t_0 , then a network stays in this state for all times $t > t_0$;
(ii) the state $\mathbf{0}$ is *attractive*: if the network starts at time t from any initial state $x \in \mathcal{X}$, $x \neq \mathbf{0}$, it eventually reaches the state $\mathbf{0}$ in a finite time, and
(iii) the time to reach $\mathbf{0}$ does not depend on t (so the network dynamics is *time-homogeneous*).

More precisely, we assume that, for any state $x \in \mathcal{X}$, $x \neq \mathbf{0}$, and for any t, if the network starts from the state $x = \Phi(x, 0)$ at time t, then its state at time $t+u$, $u \geq 0$ is $\Phi(x, u)$ where, for any fixed x, the function $\{\Phi(x, u),\ u \geq 0\}$ is a cadlag (right-continuous with left limits) function with at most finite number of jumps such that $W(x) = \min\{u \geq 0\ :\ \Phi(x, u) = \mathbf{0}\}$ is finite.

Let us now describe a *network with a single (exogeneous) stochastic input process.* An input process is a marked point process (Y, T) with points $\{T_n\}$ and marks $\{Y_n\}$ where $T_{n+1} \geq T_n$ a.s. for all n and Y_n take values in a measurable space $\mathcal{Y}$. For all $m \leq n \in \mathbb{N}$, let $(Y, T)_{[m,n]} = \{(Y_l, T_l)\}_{l=m}^n$ be the $[m, n]$ restriction of (Y, T). Let $\tau_n = T_n - T_{n-1}$.

Let $\Psi : \mathcal{X} \times \mathcal{Y} \to \mathcal{X}$ be a measurable function with the following meaning: if, at any time t, network is at state x and then receives an input y, then $\Psi(x, y)$ is its state after that.

Now fix two integers $m \leq n$ and denote by $(x, m, n, (Y, T))$ a network which starts at time T_{m-1} from the state x and receives consequently $n - m + 1$ inputs $Y_m, \ldots Y_n$ at times $T_m, \ldots, T_n$ respectively. If we denote by X_i^x, $i = m, \ldots, n$ the state of this network after receiving an input Y_i, then $X_m^x = \Psi(\Phi(x, T_m - T_{m-1}), Y_m)$ and, by homogeneity, for $i = m + 1, \ldots, n$,

$$X_i^x = \Psi(\Phi(X_{i-1}^x, T_i - T_{i-1}), Y_i).$$

Thus,

$$X_i^x = f(X_{i-1}^x, Y_i, T_i - T_{i-1})$$

where $f(x, y, u) = \Psi(\Phi(x, u), y)$. Now let $W_{[m,n]}^x(Y, T)$ be the time of the *last activity in the network* $(x, m, n, (Y, T))$, i.e.

$$W_{[m,n]}^x(Y, T) = T_n + W(X_n^x) = \min\{t > T_n\ :\ \Phi(X_n^x, t - T_n) = \mathbf{0}\}.$$

In what follows, we write $W_{[m,n]}(T)$ instead of $W_{[m,n]}(Y, T)$.

As in [4], we say that the network is *monotone separable* if the following properties hold for all T.

1. **(causality):** for all $m \leq n$,

$$W^x_{[m,n]}(T) \geq T_n;$$

2. **(external monotonicity):** for all $m \leq n$,

$$W^x_{[m,n]}(T) \leq W^x_{[m,n]}(T'),$$

whenever $T' = \{T'_n\}$ is such that $T_n \leq T'_n$ for all n, a property which we will write $T \leq T'$ for short;

3. **(homogeneity):** for all $c \in \mathbb{R}$ and for all $m \leq n$

$$W^x_{[m,n]}(T + c) = W^x_{[m,n]}(T) + c,$$

where $T + c = \{T_n + c\}$;

4. **(separability):** for all $m \leq l < n$, if $W^x_{[m,l]}(T) \leq T_{l+1}$ then

$$W^x_{[m,n]}(T) = W^0_{[l+1,n]}(T).$$

Note that, in our description of a network, we have assumed the causality and homogeneity properties from the beginning. Henceforth, we assume also that external monotonicity and separability hold.

Definition 3.1. The $[m, n]$ maximal dater is

$$Z^x_{[m,n]}(T) = W^x_{[m,n]}(T) - T_n = W^x_{[m,n]}(T - T_n).$$

For convenience, we denote $Z^x_{[1,0]} = W(x)$.

Further we provide two properties of the maximal dater without proofs (see [4] for details).

External monotonicity. For all $m \leq n$

$$Z^x_{[m,n]}(T) \geq Z^x_{[m,n]}(T'), \tag{3.1}$$

whenever $T' = \{T'_n\}$ is such that $\tau_n \leq \tau'_n$ for all n.

Sub-additive property. For all $m \leq l < n$ and for all T

$$Z^x_{[m,n]}(T) \leq Z^x_{[m,l]}(T) + Z^0_{[l+1,n]}(T).$$

Independence assumptions. In what follows, we assume that $\{Y_n\}$, $\{\tau_n\}$ are *mutually independent sequences of i.i.d. random variables.* Then, for any initial state x, $\{X^x_i\}$, $i = 0, 1, \ldots$, forms a time-homogeneous MC.

Also suppose that

$$\mathbf{E}\tau_1 = a < \infty, \ \mathbf{E}Z_n < \infty,$$

where $Z_n = Z^0_{[n,n]} = W(\Psi(\mathbf{0}, Y_n))$.

Denote by $Q = \{T_n\}$ the degenerate input process with $T_n = 0$ a.s. for all n. Then there exists a non-negative constant $\gamma(0)$ such that

$$\lim_{n\to\infty} \frac{\mathbf{E}Z^0_{[1,n]}(Q)}{n} = \gamma(0). \tag{3.2}$$

We further suppose that

$$\gamma(0) < a. \tag{3.3}$$

Based on ideas from [5], we construct a single server majorant for the maximal dater.

The single server majorant. From (3.2) it follows that there exists $L > 0$ such that

$$\mathbf{E}Z^0_{[1,L]}(Q) < L\mathbf{E}\tau_1. \tag{3.4}$$

Consider the input process $T^+ = \{T^+_k\}$, where $T^+_k = T_{nL}$ for $(n-1)L+1 \le k \le nL$ and for all $n \in \mathbb{N}$. Then

$$Z^x_{[1,nL]}(T) \le Z^x_{[1,nL]}(T^+).$$

We can slightly modify [5, Lemma 5] to get

$$Z^x_{[1,(n+1)L]}(T^+) \le Z^0_{[nL+1,(n+1)L]}(T^+) + (Z^x_{[1,nL]}(T^+) - \widetilde{\tau}_{n+1})^+, \tag{3.5}$$

where $\widetilde{\tau}_{n+1} = \tau_{nL+1} + \ldots + \tau_{(n+1)L}$.
Introduce now an *auxiliary single server queue*. This is a $GI/GI/1$ queue with service times

$$\widetilde{\sigma}_n = Z^0_{[(n-1)L+1,nL]}(T^+) \equiv Z^0_{[(n-1)L+1,nL]}(Q),$$

interarrival times

$$\widetilde{\tau}_n = \tau_{(n-1)L+1} + \ldots + \tau_{nL},$$

and an arbitrary initial value. Let R_n be the sojourn time for the single server queue defined above, i.e.

$$R_{n+1} = \widetilde{\sigma}_n + (R_n - \widetilde{\tau}_n)^+, \tag{3.6}$$

with initial value $R_0 = \overline{R} = const > 0$. The following lemma is a consequence of (3.5) and (3.6).

Lemma 3.1. *For any initial state* $x \in B_{\overline{R}} = \{x \ : \ Z^x_{[1,0]} \le \overline{R}\}$,

$$Z^x_{[1,nL]}(T) \le Z^x_{[1,nL]}(T^+) \le R_n \quad \text{for all } n \ge 0. \tag{3.7}$$

Hence R_n provides a uniform upper bound for all maximal daters $Z^x_{[1,nL]}$ such that $Z^x_{[1,0]} \le \overline{R}$. The following theorem is the main result of the paper.

Theorem 3.1. *Let* $M \ge 0$. *Consider a function* $g(x)$ *such that* $g(x) = 0,\ x \le 0$, *the function* $h(x) = g(x) - \ln^+ x$ *is non-decreasing and concave on* $[M, \infty)$, $h(x) = o(x)$ *as* $x \to \infty$, *and* $e^{g(x)}$ *is convex on* $[M, \infty)$. *Here* $\ln^+ x = (\ln x^+)^+$, *where* $y^+ = \max(0, y)$. *As earlier, let*

$$\widetilde{g}(x) = \begin{cases} 0, & \text{if } x < 0, \\ g(M) + \ln g'(M), & \text{if } 0 \le x \le M, \\ g(x) + \ln g'(x), & \text{if } x > M. \end{cases}$$

Suppose that a network satisfies the causality, external monotonicity, homogeneity, separability, and independence assumptions. Suppose also that the following conditions hold:

1. $\gamma(0) < a$, *where* $a = \mathbf{E}\tau_1 < \infty$, *and* $\gamma(0)$ *is defined in (3.2);*
2. *there exists* $c > 1/(a - \gamma(0))$ *such that*

   $$\mathbf{E}e^{g(cZ_1)} < \infty \quad (\text{or, respectively,} \quad \mathbf{E}e^{\alpha Z_1} < \infty \quad \text{for some } \alpha > 0);$$

3. *for any* $\overline{R} > 0$, *there exist* $l \ge 1$ *and an event* $A \in \sigma((Y_1, \tau_1), \ldots, (Y_l, \tau_l))$ *of positive probability such that , for any* $x \in B_{\overline{R}}$,

   $$X^x_l \mathbf{I}(A) = G((Y_1, \tau_1), \ldots, (Y_l, \tau_l))\mathbf{I}(A) \quad a.s.,$$

   where G *does not depend on* x. *Here* (Y_n, τ_n) *is a driving sequence for* X^x_n.

Then there exists a probability measure π *such that, for any* $x \in \mathcal{X}$,

$$e^{\widetilde{g}(n)} \| \mathbf{P}(X^x_n \in \cdot) - \pi(\cdot) \| \to 0 \quad \text{as} \quad n \to \infty \tag{3.8}$$

$$\Big(\text{or, respectively,} \quad e^{\beta n} \| \mathbf{P}(X^x_n \in \cdot) - \pi(\cdot) \| \to 0 \quad \text{as} \quad n \to \infty, \quad \text{for some} \quad \beta > 0\Big). \tag{3.9}$$

Corollary 3.1. *Let now the initial value* x *be an independent random variable. If, in addition to conditions of Theorem 3.1,* $\mathbf{E}e^{\widetilde{g}(x)} < \infty$ *(or* $\mathbf{E}e^{\alpha x} < \infty$*), then (3.8) (or, respectively, (3.9)) holds.*

Remark 2. 1. The function $\widetilde{g}$ is non-decreasing on $[M, \infty)$ if and only if $e^{g(x)}$ is convex on $[M, \infty)$. Indeed,

$$(e^{g(x)})' = e^{g(x)} g'(x) = e^{\widetilde{g}(x)}.$$

2. The function $\widetilde{g}$ is subadditive. Indeed, from Lemma A.3, it follows that, for $x, y \geq M$, $\widetilde{g}(x+y) \leq \widetilde{g}(x) + \widetilde{g}(y)$.
For $x \geq M$, $0 < y \leq M$, since $\widetilde{g}(z) = 0$ for $z < 0$,

$$\widetilde{g}(x+y) \leq \widetilde{g}(x+M) \leq \widetilde{g}(x) + \widetilde{g}(M) = \widetilde{g}(x) + \widetilde{g}(y).$$

For $x, y \in (0, M)$,

$$\widetilde{g}(x+y) \leq \widetilde{g}(2M) \leq \widetilde{g}(M) + \widetilde{g}(M) = \widetilde{g}(x) + \widetilde{g}(y).$$

Hence $\widetilde{g}(x+y) \leq \widetilde{g}(x) + \widetilde{g}(y)$ for all $x, y \geq 0$. Since $\widetilde{g}(x)$ is non-decreasing for $x \geq 0$ and $\widetilde{g}(x) = 0$ for $x < 0$, it follows that $\widetilde{g}(x+y) \leq \widetilde{g}(x) + \widetilde{g}(y)$ for all x, y.

Proof of Theorem 3.1. First we show that the conditions of Theorem 2.1 hold. Take $L_0 > 0$ such that (3.4) holds for all $L \geq L_0$. From (3.4) and (3.7), there exists $\overline{R}_0 > 0$ such that $B_{\overline{R}}$ is positive recurrent for all $\overline{R} \geq \overline{R}_0$, i.e. Conditions 1 and 2 of Theorem 2.1 hold. By Remark 1 Condition 3 of Theorem 3.1 implies Condition 3 of Theorem 2.1. The MC $\{X_n^x\}$ is aperiodic, since $\mathbf{P}(Z^x_{[1,L]}(T) \leq \overline{R}) > 0$ for all $L \geq L_0$ and $x \in B_{\overline{R}}$.

Fix some $x \in \mathcal{X}$. Take $\overline{R} \geq Z^x_{[1,0]}$. From (3.7),

$$\tau(x, B_{\overline{R}}) \leq L \min\{n \geq 1 \ : \ R_n \leq \overline{R}\} \quad \text{a.s.} \tag{3.10}$$

From the property (3.1), we can assume (w.l.o.g.) that the interarrival times are bounded from above, i.e. that there exists $K > 0$ such that $\tau_n \leq K$ for all n. Hence from (3.6) it follows that, for any $\overline{R} \geq LK$,

$$R_{n+1} = R_n + \widetilde{\xi}_{n+1}, \quad \text{for all } n < \min\{k \geq 1 \ : \ R_k \leq \overline{R}\},$$

where $\widetilde{\xi}_n = Z_{[(n-1)L+1, nL]}(Q) - \widetilde{\tau}_n$. Put $S_n = R_n - \overline{R}$. Then $S_0 = 0$, and $S_{n+1} = S_n + \widetilde{\xi}_{n+1}$ for all $n < \min\{k \geq 1 \ : \ S_k \leq 0\}$. Moreover, from (3.10),

$$\tau(x, B_{\overline{R}}) \leq L \min\{n \geq 1 \ : \ S_n \leq 0\}.$$

From Lemma A.1, applied to the function $g(Lx)$, it follows that, if $\mathbf{E}e^{g(cZ^0_{[1,L]}(Q))} < \infty$ for some $c > L/(La - \mathbf{E}Z^0_{[1,L]}(Q))$, then $\mathbf{E}e^{g(\tau(x, B_{\overline{R}}))} < \infty$.

From Kingman's ergodic theorem,

$$\gamma(0) = \lim_{L\to\infty} \frac{\mathbf{E}Z^0_{[1,L]}(Q)}{L} = \inf_{L>0} \frac{\mathbf{E}Z^0_{[1,L]}(Q)}{L}.$$

Hence for any $c > 1/(a - \gamma(0))$ there exists $L > 0$ such that (3.4) holds and

$$c > \frac{L}{La - \mathbf{E}Z^0_{[1,L]}(Q)} \geq \frac{1}{a - \gamma(0)} > 0. \tag{3.11}$$

Since $Z^0_{[1,L]}(Q) \leq Z_1 + \ldots + Z_L$, where $Z_n = Z^0_{[n,n]}$ are i.i.d. random variables, $\mathbf{E}e^{g(\tau(x,B_{\overline{R}}))} < \infty$ for all $x \in \mathcal{X}$.
One can apply Lemma A.2 to get the corresponding result in the exponential case.

Thus, the conditions of Theorem 2.2 and Theorem 2.3 hold, and the result follows. □

In the following two sections we consider two classes of monotone separable networks: generalized Jackson networks and multiserver queues.

4 Generalized Jackson networks

Consider an open network with r single server stations. Let $T_0 = 0$. Customers arrive at times $T_1, \ldots, T_n$, where $\tau_i = T_i - T_{i-1}$ are i.i.d. non-negative random variables with a finite positive mean $\mathbf{E}\tau_1 = a$. Upon arrival, a customer is directed to station i with probability $p^{0,i}$, where $\sum\limits_{i=1}^{r} p^{0,i} = 1$. At each station, customers are served in FCFS order. Upon service completion at station i, a customer joins the queue at station j with probability $p^{i,j}$ and leaves the network with probability $p^{i,r+1}$, where $\sum\limits_{j=1}^{r+1} p^{i,j} = 1$. The service times at station i are i.i.d. with a finite mean b^i, and are independent for different i. We suppose that any customer eventually leaves the network. Such networks are known as generalized Jackson networks. Let π^i be the mean number of visits to station i by a customer. We assume the network to be stable, i.e. $b = \max\limits_{1\leq i\leq r} b^i\pi^i < a$ (see [17] for details).

In this Section, we apply Theorem 3.1 to find conditions for convergence rates to the stationary regime for various characteristics of a generalized Jackson network.

In the proofs, we use results from the previous Section and also coupling arguments. For the latter, we need a special sample-path construction of the network which preserves certain monotonicity properties.

Thus, we assume that service times and routing decisions are associated with stations but not with customers. At time 0, there are $q_0 = (q_0^1, \ldots, q_0^r)$ customers already at the stations, and χ_0^i is the residual service time of the first customer at station i (if any, otherwise $\chi_0^i = 0$). Further, with any station i, we associate two mutually independent sequences $\{\sigma_n^{(i)}\}_{n\geq 1}$ and $\{v_n^{(i)}\}_{n\geq 1}$. Here the σ's are the service times and the v's are the routing decisions. For each i we assume $\{v_n^{(i)}\}_{n\geq 1}$ to be i.i.d. with $\mathbf{P}(v_n^{(i)} = j) = p^{i,j}$. With regard to the sequence $\{\sigma_n^{(i)}\}_{n\geq 1}$, we assume that (a) if $q_0^i = 0$, then $\{\sigma_n^{(i)}\}_{n\geq 1}$ are i.i.d. with mean b^i; (b) if $q_0^i > 0$, then $\sigma_1^{(i)} = \chi_0^i$ and $\{\sigma_n^{(i)}\}_{n\geq 2}$ are i.i.d. with mean b^i. With the exogeneous input, we associate an i.i.d. sequence of input routing decisions: $\mathbf{P}(v_n^{(0)} = i) = p^{0,i}$.

The initial $|q_0| = q_0^1 + \ldots + q_0^r$ customers are numbered $-|q_0|+1, \ldots, 0$, and the exogeneous customers are numbered $1, 2, \ldots$. The nth customer ($n \geq 1$) is directed (upon its arrival) to the station $v_n^{(0)}$ and joins the queue there.

At any station i, the duration of the jth service is $\sigma_j^{(i)}$. After the service completion the customer is directed to station $v_j^{(i)}$ and joins the queue there (or leaves the network if $v_j^{(i)} = r+1$).

Now we introduce an SRS

$$X_{n+1}^x = f(X_n^x, Y_{n+1}), \quad n \geq 0 \ \ \text{(see previous section)}.$$

First, let us give the description of the initial value x. Consider first the case where there is no exogeneous input.
If $q_0 = (0, \ldots, 0)$, then $x = \mathbf{0}$.
If $|q_0| > 0$, then, for any $i = 1, \ldots, r$ and $j = 1, \ldots, r+1$, let $\Gamma_0^{i,j}(t)$ be the number of transitions of customers from i to j in the time interval (t, ∞). Let $\Gamma_0^{i,j} = \{\Gamma_0^{i,j}(t),\ t \geq 0\}$. Then

$$x = \{\Gamma_0^{i,j},\ 1 \leq i \leq r;\ 1 \leq j \leq r+1\},$$

$$\Phi(x, u) = \{\theta^u \Gamma_0^{i,j},\ 1 \leq i \leq r;\ 1 \leq j \leq r+1\},$$

where $\theta^u \Gamma_0^{i,j} = \{\theta^u \Gamma_0^{i,j}(t),\ t \geq 0\}$ and $\theta^u \Gamma_0^{i,j}(t) = \Gamma_0^{i,j}(t+u)$. We endow the state space with the Skorohod topology (see, e.g. [6]).

Finally $Z_{[1,0]}^x = \max_{i,j} \min\{t\ :\ \Gamma_0^{i,j}(t) = 0\}$. Thus x is a random element.

Let $d^i(0) = \sum\limits_{j=1}^{r+1} \Gamma_0^{i,j}(0)$ for $1 \leq i \leq r$, $d^0(0) = 0$. Let $\sigma(0) = \{\{\sigma_k^i(0)\}_{k=1}^{d^i(0)},\ i = 1, \ldots, r\}$ and let $v(0) = \{\{v_k^i(0)\}_{k=1}^{d^i(0)},\ i = 1, \ldots, r\}$, where $\sigma_k^i(0) = \sigma_k^{(i)}$ and $v_k^i(0) = v_k^{(i)}$ for all $1 \leq k \leq d^i(0)$, $1 \leq i \leq r$.

To define the recursion, we will need some other definitions. Consider an open generalized Jackson network with r stations which is initially empty and which is fed by one customer arriving at time T_1. With any station i, we associate two mutually independent i.i.d. sequences of service times $\sigma^{(i)}_{d^i(0)+j}$ and routing decisions $v^{(i)}_{d^i(0)+j}$ for $j \geq 1$, $1 \leq i \leq r$. The random variable $v^{(0)}_1$ is the first routing decision for the customer (upon its arrival).

Let $d^i(1)$ be the number of departures from the station i in this network. Then $d^0(1) = 1$. We assume that $\max_i d^i(1) < \infty$ a.s., that is, the customer eventually leaves the network. Let $\sigma(1) = \{\{\sigma^i_k(1)\}^{d^i(1)}_{k=1},\ i = 1, \ldots, r\}$ and $v(1) = \{\{v^i_k(1)\}^{d^i(1)}_{k=1},\ i = 0, \ldots, r\}$, where $\sigma^i_k(1) = \sigma^{(i)}_{d^i(0)+k}$, $v^i_k(1) = v^{(i)}_{d^i(0)+k}$ for $1 \leq k \leq d^i(1)$, $1 \leq i \leq r$, and $v^0(1) = v^0_1 = v^0_{d^0(0)+1}$. We call this network Σ_1.

In a similar way one can define networks Σ_n which are initially empty, fed by one customer arrived at time T_n, with service times $\sigma(n)$ and routing decisions $v(n)$, where $\sigma^i_k(n) = \sigma^{(i)}_{d^i(0)+\ldots+d^i(n-1)+k}$ and $v^i_k(n) = v^{(i)}_{d^i(0)+\ldots+d^i(n-1)+k}$ for $k = 1, \ldots, d^i(n)$.

We thus obtain a sequence of independent random variables $\{(\sigma(n), v(n))\}_{n\geq 0}$ such that $(\sigma(n), v(n))$ are identically distributed for $n \geq 1$.

For a network with input process T and initial state x at time T_{m-1}, which is fed by customers $m, \ldots, n$ at times $T_m, \ldots, T_n$, let $\Gamma^{i,j}_{[m,n]}(T, t, x)$ be the number of transitions of customers from i to j in the time interval $(T_n + t, \infty)$. Let $\Gamma_{[m,n]}(T, x) = \{\Gamma^{i,j}_{[m,n]}(T, t, x),\ 1 \leq i \leq r,\ 1 \leq j \leq r + 1;\ t \geq 0\}$. Then

$$\Gamma_{[m,n+1]}(T, x) = \Psi(\Phi(\Gamma_{[m,n]}(T, x), \tau_{n+1}), \xi_{n+1}),$$

where $\xi_n = (\sigma(n), v(n))$. For the definition of Ψ we refer the reader to Appendix B.

If the Γ-processes are known, we are able to define service times, routing decisions, moments of customers arrivals at stations and moments of their departures, queue lengthes, etc. Hence, we can conclude that any generalized Jackson network with input point process T can be described by stochastic recursive sequence

$$X^x_{[1,n+1]}(T) = f(X^x_{[1,n]}(T), Y_{n+1}),\ X^x_{[1,0]}(T) = x, \tag{4.1}$$

where $X^x_{[m,n]}(T) = \{\Gamma^{i,j}_{[m,n]}(T, t, x), 1 \leq i \leq r,\ 1 \leq j \leq r + 1,\ t \geq 0\}$ and $Y_n = (\tau_n,\ \sigma(n),\ v(n))$.

In particular, the maximal dater $Z^x_{[1,n]}(T)$ for the network is

$$Z^x_{[1,n]}(T) = \max_{1\leq i\leq r} \inf\{t \geq 0\ :\ \Gamma^i_{[1,n]}(T, t, x) = 0\},$$

where $\Gamma^i_{[1,n]}(T, t, x) = \sum_{j=1}^{r+1} \Gamma^{i,j}_{[1,n]}(T, t, x)$ is the number of departures from station i after time $T_n + t$.

As was shown in [16, 3], the sequence $X^x_{[1,n]}(T)$ is monotone in T and in x, and $Z^x_{[1,n]}(T)$ satisfies the properties of monotone separable networks.

To check Condition 3 of Theorem 3.1 we introduce the class of feed-forward generalized Jackson networks.

Definition 4.1. A generalized Jackson network is *feed-forward* if there exists a renumbering of the nodes of the network such that the transition matrix $\{p^{k,l}\}$ satisfies

$$p^{k,l} = 0 \ \text{ for all } 0 \le l \le k \le r+1.$$

In other words, there are no loops in a feed-forward network.

Consider a discrete-time Markov chain $\{V(m), m \ge 0\}$, with state space $\{0, \dots, r+1\}$, initial value $V(0) = 0$ and transition matrix $\{p^{k,l}\}$. Let

$$\mu^k = \sharp\{m \ge 1 \ : \ V(m) = k\} \ \text{ and } \ \pi^k = \mathbf{E}\mu^k.$$

Lemma 4.1. *There exists a matrix $\{\widetilde{p}^{k,l}\}$ and a renumbering of states such that*

1. $\widetilde{p}^{k,l} = 0$ *for all* $0 \le l \le k \le r+1$*;*
2. *for all* k, l *if* $\widetilde{p}^{k,l} > 0$ *then* $p^{k,l} > 0$*;*
3. *if* $\widetilde{V}(m)$ *is a Markov chain with initial value* $\widetilde{V}(0) = 0$ *and with transition probabilities* $\{\widetilde{p}^{k,l}\}$*, then* $\widetilde{\pi}^k \le \pi^k$ *for all* $k \in \{1, \dots, r\}$*.*

Lemma 4.1 implies that, for any stable generalized Jackson network, one can obtain a stable feed-forward network by denying some transitions admissible for the original network, and by renumbering nodes. For the proof of Lemma 4.1 see Appendix C.

If a generalized Jackson network is given by (4.1), then we define the corresponding feed-forward network by

$$\widetilde{X}^x_{[1,n+1]}(T) = f(\widetilde{X}^x_{[1,n]}(T), \widetilde{Y}_{n+1}), \ \widetilde{X}^x_{[1,0]}(T) = x, \tag{4.2}$$

where

$$\widetilde{Y}_n = (\tau_n, \ \widetilde{\sigma}(n), \ \widetilde{v}(n)),$$

and $\{\widetilde{v}(n)\}$ is a corresponding sequence of routing decisions. Note that $\widetilde{d^i}(n) \le 1$ for all $i \in \{0, \dots, r+1\}$ and $n \in \mathbb{N}$. If $\widetilde{d^i}(n) = 1$ then $\widetilde{\sigma}^i_1(n) = \sigma^i_1(n)$.

In the following remark we construct a coupling of the original network and the corresponding feed-forward one.

Remark 3. Let

$$s^i = \max_{j\,:\,\widetilde{p}^{i,j}>0} \frac{\widetilde{p}^{i,j}}{p^{i,j}} \geq 1, \quad s = \max_{0\leq i\leq r} s^i \geq 1.$$

Consider a sequence of routing decisions $\widehat{v}(n)$, where

$$\mathbf{P}(\widehat{v}_k^i(n) = j) = \frac{p^{i,j} - \widetilde{p}^{i,j}/s}{1 - 1/s},$$

and sequence of service times $\widehat{\sigma}(n) = \{\{\widehat{\sigma}_k^i(n),\ 1 \leq k \leq \widehat{d}^i(n)\},\ 1 \leq i \leq r\}$, where $\widehat{\sigma}_k^i(n)$ are independent and have the same distribution as the service time at station i in original network. Note that $\widehat{d}^i(n)$ are known, if we know $\widehat{v}(n)$. We can also assume that $\{\widehat{\sigma}(k), \widehat{v}(k)\}$ and $\{\widetilde{\sigma}(k), \widetilde{v}(k)\}$ are independent.

Consider a sequence of i.i.d. random variables $\{\alpha_n\}$ such that

$$\mathbf{P}(\alpha_n = 1) = 1 - \mathbf{P}(\alpha_n = 0) = 1/s,$$

and the α_n are independent of $\{\widetilde{\sigma}(k), \widetilde{v}(k)\}$ and $\{\widehat{\sigma}(k), \widehat{v}(k)\}$.

Let

$$(\sigma(n), v(n)) = \begin{cases} (\widetilde{\sigma}(n), \widetilde{v}(n)), & \text{if } \alpha_n = 1, \\ (\widehat{\sigma}(n), \widehat{v}(n)), & \text{if } \alpha_n = 0. \end{cases}$$

Then $(\sigma(n), v(n))$ have the same distributions as corresponding characteristics of the original network.

Hereafter, we assume that the original and the corresponding feed-forward networks are defined according to the coupling above.

For any $M > 0$, let n_M be such that

$$T_{n_M} \geq M.$$

For any $n \geq 0$, let

$$x_n = X^{\mathbf{0}}_{[1,n]}(Q + T_n) \quad \text{and} \quad \widetilde{x}_n = \widetilde{X}^{\mathbf{0}}_{[1,n]}(Q + T_n).$$

In particular, $x_0 = \widetilde{x}_0 = \mathbf{0}$, where $\mathbf{0}^{k,l}(t) = 0$ for all k, l and t.

Let

$$\widetilde{\zeta}(n) = \min\{m \geq n+1 \ :\ \widetilde{X}^{\widetilde{x}_n}_{[n+1,k]}(T) = \widetilde{X}^{\mathbf{0}}_{[n+1,k]}(T) \text{ for all } k \geq m\}.$$

From [17, Theorem 6] it follows that $\widetilde{\zeta}(n) < \infty$ a.s. for any $n \in \mathbb{N}$.

For $K_1, K_2 > 0$ consider the event

$$C = \{n_M = K_1\} \cap \{\widetilde{\zeta}(K_1) \leq K_1 + K_2\} \cap \bigcap_{m=1}^{K_1+K_2} \{\sigma(m) = \widetilde{\sigma}(m),\ v(m) = \widetilde{v}(m)\}.$$

Note that there exist $K_1, K_2 > 0$ such that $\mathbf{P}(C) > 0$. Indeed, from the coupling construction,

$$\mathbf{P}(C) \geq \mathbf{P}(n_M = K_1)\mathbf{P}(\widetilde{\zeta}(K_1) \leq K_1 + K_2)\mathbf{P}(\alpha_m = 1, \text{ for all } m \in [1, K_1 + K_2]) > 0.$$

The following lemma shows that Condition 3 of Theorem 3.1 holds for generalized Jackson networks.

Lemma 4.2. *There exist $K_1, K_2 > 0$ such that $\mathbf{P}(n_M = K_1) > 0$, $\mathbf{P}(\widetilde{\zeta}(K_1) \leq K_1 + K_2) > 0$, and the event $A = \{Z^x_{[1,0]}(T) \leq M\} \cap C$ satisfies Condition 3 of Theorem 3.1, i.e.*

$$X^x_{[1,K_1+K_2]}(T)\mathbf{I}(A) = G(Y_1, \ldots, Y_{K_1+K_2})\mathbf{I}(A), \tag{4.3}$$

where G does not depend on x, and $\mathbf{P}(A) > 0$.

Proof of Lemma 4.2. It is sufficient to prove only (4.3). Consider an initial state x such that

$$Z^x_{[1,0]}(T) \leq M. \tag{4.4}$$

We show that

$$X^x_{[1,K_1+K_2]}(T)\mathbf{I}(C) = G(Y_1, \ldots, Y_{K_1+K_2})\mathbf{I}(C),$$

for some function G which does not depend on x.

Consider the point process $\widetilde{T} = \{\widetilde{T}_n\}$, where

$$\widetilde{T}_n = \begin{cases} T_{K_1}, & \text{if } 1 \leq n \leq K_1; \\ T_n, & \text{otherwise.} \end{cases}$$

Note that $T \leq \widetilde{T}$. Hence

$$X^x_{[1,n]}(T) \leq X^x_{[1,n]}(\widetilde{T}), \quad \text{for all } n \in \mathbb{N}.$$

From (4.4),

$$X^x_{[1,n]}(\widetilde{T}) = X^{\mathbf{0}}_{[1,n]}(\widetilde{T}) \text{ on } C \text{ for all } n \in \mathbb{N}.$$

From the monotonicity properties of $X^x_{[1,n]}(T)$,

$$X^{\mathbf{0}}_{[K_1+1,n]}(T) \leq X^{\mathbf{0}}_{[1,n]}(T) \leq X^x_{[1,n]}(T) \leq X^x_{[1,n]}(\widetilde{T}) = X^{\mathbf{0}}_{[1,n]}(\widetilde{T}) = X^{x_{K_1}}_{[K_1+1,n]}(T)$$

on C. Let

$$\zeta(n) = \min\{m \geq n+1 \ : \ X^{x_n}_{[n+1,k]}(T) = X^{\mathbf{0}}_{[n+1,k]}(T) \text{ for all } k \geq m\}.$$

Since $\zeta(K_1)\mathbf{I}(C) = \widetilde{\zeta}(K_1)\mathbf{I}(C)$, the result follows. □

Remark 4. Let $b = \max\limits_{1\leq i\leq r} \pi^i b^i$. From [3] it follows that $\gamma(0) = b$.

Remark 5. Note that $Z_1 = \sum\limits_{k=1}^{r}\sum\limits_{i=1}^{d^k(1)} \sigma^k_i(1)$. Moreover, $\mathbf{E}e^{g(Z_1)} < \infty$, provided that $\max\limits_{1\leq k\leq r} \mathbf{E}e^{g(\sigma^k_1(1))} < \infty$. Indeed, let $d = \max\limits_{1\leq k\leq r} d^k(1)$; then there exists $\delta > 0$ such that $\mathbf{P}(d \geq n) \leq (1-\delta)^n$, and d is independent of $\sigma^k_i(1)$ for all $1 \leq k \leq r$ and $i \geq 1$. Let $\eta_i = \sum\limits_{k=1}^{r} \sigma^k_i(1)$. Then η_i are i.i.d. and $\mathbf{E}e^{g(\eta_1)} < \infty$, provided that $\max\limits_{1\leq k\leq r} \mathbf{E}e^{g(\sigma^k_1(1))} < \infty$. It follows from Lemma A.5 that $\mathbf{E}e^{g(Z_1)} < \infty$. Similarly, if $\max\limits_{1\leq k\leq r} \mathbf{E}e^{\widetilde{\alpha}\sigma^k_1(1)} < \infty$ for some $\widetilde{\alpha} > 0$, then $\mathbf{E}e^{\alpha Z_1} < \infty$ for some $\alpha > 0$.

We summarize the results of the section in the following theorem.

Theorem 4.1. *Let the function g satisfy the conditions of Theorem 3.1. If $b < a$ and $\max\limits_{1\leq k\leq r} \mathbf{E}e^{g(c\sigma^k_1(1))} < \infty$ for some $c > 1/(a-b)$ (or, respectively, $\max\limits_{1\leq k\leq r} \mathbf{E}e^{\alpha\sigma^k_1(1)} < \infty$ for some $\alpha > 0$) then there exists a probability measure π such that, for any initial state x with fixed queue lengths and residual service times at stations,*

$$e^{\widetilde{g}(n)}\|\mathbf{P}(X^x_{[1,n]}(T) \in \cdot) - \pi(\cdot)\| \to 0 \quad \text{as} \quad n \to \infty$$

$$\Big(\text{or, respectively,} \quad e^{\beta n}\|\mathbf{P}(X^x_{[1,n]}(T) \in \cdot) - \pi(\cdot)\| \to 0 \quad \text{as} \quad n \to \infty, \quad \text{for some} \quad \beta > 0\Big),$$

where the Markov chain $X^x_{[1,n]}(T)$ is defined in (4.1).

Remark 6. Since $X^x_{[1,n]}(T)$ describes the behavior of the network, the results on convergence rates of a number of natural characteristics of network (e.g., queue length distributions, waiting and sojourn time distributions, etc.) to the corresponding stationary ones follow immediately from Theorem 4.1.

5 $GI/GI/m$ queues

A multiserver queueing system is another example of a monotone separable network. We consider a multiserver queue $GI/GI/m$ with i.i.d. service times $\{\sigma_n\}$ and i.i.d. interarrival times $\{\tau_n\}$. We assume that the stability condition $\mathbf{E}\sigma_1 < \mathbf{E}\tau_1$ holds. Let

$$W_n^x = (W_{n,1}^x, \dots, W_{n,m}^x)$$

denote the total workload at the n-th arrival time, where $\{W_{n,i}^x\}$ are the workloads of the servers, arranged in non-decreasing order, and where x is the initial state of the system, i.e.

$$W_{n+1}^x = \mathcal{R}(W_n^x + \sigma_n \mathbf{e}_1 - \tau_n \mathbf{i})^+, \quad W_0^x = x \tag{5.1}$$

for $n \geq 0$, where $\mathbf{e}_1 = (1, 0, \dots, 0)$ and $\mathbf{i} = (1, \dots, 1)$ are m-dimensional vectors and the operator $\mathcal{R}$ sorts the components of a vector into non-decreasing order. Then

$$Z_{[1,n]}^x = \max(W_{n,1}^x + \sigma_{n+1},\ W_{n,m}^x).$$

Definition 5.1. For two m-dimensional vectors $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$, we write $\mathbf{x} \leq (\geq) \mathbf{y}$ if $x_j \leq (\geq) y_j$ for all $j \in \{1, \dots, m\}$.

Let $k \in \{1, 2, \dots, m\}$ be such that

$$(k-1)\mathbf{E}\tau_1 \leq \mathbf{E}\sigma_1 < k\mathbf{E}\tau_1. \tag{5.2}$$

Let $b = \mathbf{E}\sigma_1$, $a = \mathbf{E}\tau_1$. Fix $M > 0$. Consider the event

$$C = \{\sigma_1 \leq b, \dots, \sigma_{n_0} \leq b,\ \tau_1 \geq a, \dots, \tau_{n_0+k-1} \geq a\}. \tag{5.3}$$

where

$$n_0 = \left(\left[\frac{M}{ma - b} \right] + 1 \right) m. \tag{5.4}$$

Note that

$$p = \mathbf{P}(C) = \mathbf{P}(\sigma_1 \leq b)^{n_0} \mathbf{P}(\tau_1 \geq a)^{n_0+k-1} > 0. \tag{5.5}$$

The following lemma shows that W_n^x satisfies Condition 3 of Theorem 3.1.

Lemma 5.1. *Let $A = \{Z_{[1,n]}^x \leq M\} \cap C$. Then*

$$W_{n_0+k-1}^x = \big(0, \dots, 0, \mathcal{R}(\sigma_{n_0+1} - \tau_{n_0+1} - \dots - \tau_{n_0+k-1}, \dots, \sigma_{n_0+k-1} - \tau_{n_0+k-1})^+\big)$$

on A, where k is defined in (5.2) and n_0 is defined in (5.4).

Proof. Consider the process $\widetilde{W}_n$: $\widetilde{W}_0 = (M-b,\dots,M)$,

$$\begin{aligned} \widetilde{\sigma}_n = b, \quad \widetilde{\tau}_n = a \qquad & \text{for} \quad n \le n_0, \\ \widetilde{\sigma}_n = \sigma_n, \quad \widetilde{\tau}_n = \tau_n \qquad & \text{otherwise.} \end{aligned}$$

Then $\widetilde{Z}_{[1,0]} = M$ and

$$W^x_{[1,n]} \le \widetilde{W}_n \quad \text{on} \quad A. \tag{5.6}$$

Note that the service discipline of the process $\widetilde{W}_n$ is cyclic until the n_0-th arrival. Hence the n_0-th customer (where n_0 is defined in (5.4)) faces the system with $(k-1)$ busy stations such that

$$\widetilde{W}_{n_0} \le (0,\dots,0,\ a,\dots,(k-1)a) \quad \text{on} \quad C, \tag{5.7}$$

because of the choice of k in (5.2). Since (5.6) and (5.7),

$$W^x_{n_0} \le (0,\dots,0,\ a,\dots,(k-1)a) \quad \text{on} \quad A. \tag{5.8}$$

The following inequalities hold

$$\tau_{n_0+1} \ge a,\dots,\tau_{n_0+k-1} \ge a \quad \text{on} \quad A. \tag{5.9}$$

Hence the result of the lemma follows from (5.8) and (5.9). □

Remark 7. 1. From [4] it follows that $\gamma(0) = \mathbf{E}\sigma_1/m = b/m$.

2. It is clear that $Z_1 = \sigma_1$.

To summarize the results of the section, we provide the following theorem.

Theorem 5.1. *Let the function g satisfy the conditions of Theorem 3.1. If $b < ma$ and $\mathbf{E}e^{g(c\sigma_1)} < \infty$ for some $c > 1/(a-b/m)$ (or, respectively, $\mathbf{E}e^{\alpha\sigma_1} < \infty$ for some $\alpha > 0$) then there exists a probability measure π such that, for any initial workload x,*

$$e^{\widetilde{g}(n)} \|\mathbf{P}(W^x_n \in \cdot) - \pi(\cdot)\| \to 0 \quad \textit{as} \quad n \to \infty$$

$$\left(\textit{or, respectively,} \quad e^{\beta n} \|\mathbf{P}(W^x_n \in \cdot) - \pi(\cdot)\| \to 0 \quad \textit{as} \quad n \to \infty, \quad \textit{for some} \quad \beta > 0\right).$$

Appendix A: Some results for random walks

Consider a random walk $S_n = \sum_{i=1}^{n} \xi_i$, $S_0 = 0$, where $\{\xi_n\}$ is a sequence of i.i.d. random variables such that $\xi_n \geq -N$ a.s. for some $N > 0$, $\mathbf{E}\xi_1 = -a < 0$. Let $\tau(X) = \min\{n > 0 \,:\, X + S_n \leq 0\}$, where $X \geq 0$.

Let $M \geq 0$. We consider functions $g(x)$ such that $g(x) = 0$, $x \leq 0$ and the function $h(x) = g(x) - \ln^+ x$ is non-decreasing and concave on $[M, \infty)$, and $h(x) = o(x)$ as $x \to \infty$. Here $\ln^+ x = (\ln x^+)^+$, where $y^+ = \max(0, y)$. Let $\widetilde{g}(x)$ be defined as in (2.1).

The following lemma is Theorem 2.2 from [18].

Lemma A.1. *If* $\mathbf{E}e^{g(c\xi^+)} < \infty$ *for some* $c > 1/a$, *where* $a = -\mathbf{E}\xi_1$, *then* $\mathbf{E}e^{g(\tau(X))} < \infty$.

The following lemma is a well known result from the theory of random walks (see, e.g. [18] for a proof and also the references therein).

Lemma A.2. *Suppose that* $\mathbf{E}e^{\alpha\xi_1} < \infty$ *for some* $\alpha > 0$. *Let* $\theta > 0$ *be such that* $e^{-\theta} = \inf_{\alpha>0} \mathbf{E}e^{\alpha\xi_1}$. *Then* $\mathbf{E}e^{\theta\tau(X)} < \infty$.

Lemma A.3. *For all* $x, y \geq M$,

$$\widetilde{g}(x+y) \leq \widetilde{g}(x) + \widetilde{g}(y).$$

Proof of Lemma A.3 requires the following lemma.

Lemma A.4. *Let function* f *be such that* $f(x)/x$ *is non-increasing for* $x \geq M$. *Then, for all* $x, y \geq M$,

$$f(x+y) \leq f(x) + f(y).$$

Proof of Lemma A.4.

$$f(x) + f(y) = \frac{f(x)}{x}x + \frac{f(y)}{y}y \geq \frac{f(x+y)}{x+y}x + \frac{f(x+y)}{x+y}y = f(x+y).$$

□

Proof of Lemma A.3. Note that $\widetilde{g}(x) = h(x) + \ln(1 + xh'(x))$. Let $\widehat{h}(x) = xh'(x)$. Since $h(x)$ is concave on $[M, \infty)$, $\widehat{h}(x)/x$ is decreasing on $[M, \infty)$. From Lemma A.4 it follows that $\widehat{h}(x+y) \leq \widehat{h}(x) + \widehat{h}(y)$ for all $x, y \geq M$. Hence

$$\begin{aligned} \widetilde{g}(x+y) &= h(x+y) + \ln(1 + \widehat{h}(x+y)) \leq h(x) + h(y) + \ln(1 + \widehat{h}(x) + \widehat{h}(y)) \\ &\leq \Big(h(x) + \ln(1 + \widehat{h}(x))\Big) + \Big(h(y) + \ln(1 + \widehat{h}(y))\Big) = \widetilde{g}(x) + \widetilde{g}(y). \end{aligned}$$

□

Lemma A.5. *Let G be any subadditive function such that $G(x) = 0$ for $x \le 0$, and $G(x) = o(x)$ as $x \to \infty$. Let $\{\eta_n\}$ be a sequence of i.i.d. random variables on $\mathbb{R}_+$, let μ be a random variable which is independent of η_n and such that $\mathbf{P}(\mu = k) = p(1-p)^k$ for some $p \in (0,1)$. Then*

$$\mathbf{E}e^{G(\eta_1)} < \infty \quad \textit{iff} \quad \mathbf{E}e^{G(\sum_{i=1}^{\mu} \eta_i)} < \infty.$$

Proof. The 'if'-statement is obvious. To prove the 'only if'-statement consider

$$\mathbf{E}e^{G(\sum_{i=1}^{\nu} \eta_i)} = \frac{p}{1-p} \sum_{n=1}^{\infty} \mathbf{E}e^{G(\sum_{i=1}^{n} \eta_i)} (1-p)^n.$$

Let $X_n = \sum_{i=1}^{n} \eta_i$ and let $q = \ln \frac{1}{1-p} > 0$. Then

$$\mathbf{E}e^{G(\sum_{i=1}^{n} \eta_i)} (1-p)^n = \mathbf{E}e^{G(X_n) - qn} = \mathbf{E}e^{G(X_n) - \frac{2}{3}qn} e^{-\frac{1}{3}qn}.$$

Hence it is sufficient to show that

$$\sup_n \mathbf{E}e^{G(X_n) - \frac{2}{3}qn} < \infty.$$

Note that there exists $K > 0$ such that

$$\mathbf{E}e^{G(\eta_1 - K)} \le e^{\frac{1}{3}q}.$$

Since $G(x) = o(x)$ as $x \to \infty$, there exists $\tilde{x} > 0$ such that $G(Kx) \le qx/3$ for all $x \ge \tilde{x}$. Take $n \ge \tilde{x}$. Then

$$\begin{aligned} \mathbf{E}e^{G(X_n) - \frac{2}{3}qn} &\le \mathbf{E}e^{G(X_n) - g(Kn) - \frac{1}{3}qn} \le \mathbf{E}e^{G(X_n - Kn) - \frac{1}{3}qn} \\ &\le \mathbf{E}e^{\sum_{i=1}^{n} G(\eta_i - K) - \frac{1}{3}qn} = \left(\mathbf{E}e^{G(\eta_1 - K)}\right)^n e^{-\frac{1}{3}qn} \le 1. \end{aligned}$$

Finally, note that $\mathbf{E}e^{G(\sum_{i=1}^{n} \eta_i)} \le \left(\mathbf{E}e^{G(\eta_1)}\right)^n < \infty$ for all n, which completes the proof. □

Appendix B: Construction of $\Psi(y, \xi_1)$

Let $y = \{\Gamma^{i,j}(t),\ 1 \le i \le r,\ 1 \le j \le r+1,\ t \ge 0\}$, where $\Gamma^{i,j} : [0,\infty) \to \mathbb{Z}_+$ are right-continuous non-increasing step functions such that $\max_{i,j} \inf\{t \ge$

$0 \ : \ \Gamma^{i,j}(t) = 0\}$ is finite, and let $\xi_1 = (\sigma(1), v(1))$, where $\sigma(1) = \{\sigma_k^i(1),\ 1 \leq k \leq d^i(1),\ 1 \leq i \leq r\}$, $v(1) = \{v_k^i(1),\ 1 \leq k \leq d^i(1),\ 1 \leq i \leq r\}$. To define the function $\Psi(y, \xi_1)$ we will need two constructions from [3].

Construction of the network given y.

We will use the following notation:

$$\Gamma^i(t) = \sum_{j=1}^{r+1} \Gamma^{i,j}(t), \quad q^i(t) = \Gamma^i(t) - \sum_{j=1}^{r} \Gamma^{j,i}(t), \quad 1 \leq i \leq r,\ t \geq 0.$$

We also define moments of the jumps of $\Gamma^{i,j}$: for $i = 1, \ldots, r$, $j = 1, \ldots, r+1$,

$$D_k^{i,j} = \inf\{t \geq 0 \ : \ \Gamma^{i,j}(t) < \Gamma^{i,j}(0) - k + 1\}, \quad D^{i,j} = \{D_k^{i,j},\ 1 \leq k \leq \Gamma^{i,j}(0)\}.$$

Hereafter, $m \leq k \leq n$ or $k = m, \ldots, n$ means the empty set of possible values of k, if $n < m$.

The main difficulty in the construction of the sequence of service times and routing decisions given y comes from the possibility of simultaneous departures from a station (in the case of zero-valued service times at this station). For $t > 0$, let

$$N(t) = \{(i, j, k) \ : \ D_k^{i,j} = t\}, \quad J(t) = \sharp N(t).$$

For $t > 0$, we take an arbitrary numbering $\{(i_s, j_s, k_s),\ 1 \leq s \leq J(t)\}$ of the elements of $N(t)$ satisfying the following constraints: if $i_{s_1} = i_{s_2}$ and $s_1 < s_2$, then $k_{s_1} < k_{s_2}$. For such a numbering, let $\{q_s^i(t),\ i = 1, \ldots, r,\ s = 0, \ldots, J(t)\}$ be the sequence defined by: $q_0^i(t) = q^i(t-)$,

$$q_s^i(t) = q_{s-1}^i(t) + \mathbf{I}(j_s = i) - \mathbf{I}(i_s = i), \quad \text{for } s = 1, \ldots, J(t).$$

Note that $q_{J(t)}^i(t) = q^i(t)$ for all i; in particular, if $J(t) = 0$, then q^i is continuous at t for all i.

We will assume that y satisfies the following

Assumption 1: $q^i(t) \geq 0$ for all $i = 1, \ldots, r$ and $t \geq 0$,

and

Assumption 2: for each $t > 0$, there exists a numbering of the elements of $N(t)$ satisfying the above constraints and such that $q_{s-1}^i(t) \geq \mathbf{I}(i_s = i)$ for all $i = 1, \ldots, r$, $s = 1, \ldots, J(t)$. This assumption trivially holds for $t > 0$ such that $J(t) = 0$, and it follows from assumption 1 for $t > 0$ such that $J(t) = 1$.

Since the assumptions above hold for Γ-processes associated with networks (that count the number of transitions from some station to another), and we are only interested in such processes, it is not a restriction to assume that y satisfies Assumptions 1 and 2. In particular, we assume that, for any t, the chosen numbering

of $N(t)$ is such that Assumption 2 holds.
For $1 \le i \le r$ and $t > 0$, let

$$\begin{aligned} n_i(t) &= \Gamma^i(0) - \Gamma^i(t-), \\ a(t) &= \sup\{0 \le u < t \ : \ \Gamma^i(u) < \Gamma^i(u-)\}, \\ b(t) &= \sup\{0 \le u < t \ : \ q^i(u) = 0\} \end{aligned}$$

(here $a(t) = 0$ and $b(t) = 0$ if these sets are empty).
To construct a network with Γ-processes y at time $\widehat{t}$ (i.e. $\Gamma^{i,j}(t)$ is a number of departures from i to j on the time interval $(\widehat{t} + t, \infty)$), we consider an open generalized Jackson network with no exogeneous input such that, at time $\widehat{t}$, there are $q^i(0)$ customers at station i, $i = 1, \dots, r$. At time $\widehat{t}$, with each station i we associate sequences of service times σ^i_k and routing decisions ν^i_k for $k = 1, \dots, \Gamma^i(0)$ such that
if, for some $t > 0$ and $i = 1, \dots, r$, there is a jump $n = \Gamma^i(t-) - \Gamma^i(t) > 0$ and the numbers $j_1, \dots, j_n$ are such that

$$D^{i,j_1}_{n_i(t)+1} = t, \dots, D^{i,j_n}_{n_i(t)+n} = t$$

(note that we assume that the chosen numbering of $N(t)$ is such that Assumption 2 is satisfied), then

$$\sigma^i_{n_i(t)+1} = t - \max(a(t), b(t)), \ \sigma^i_{n_i(t)+k} = 0 \ \text{ for } k = 2, \dots, n$$

and

$$v^i_{n_i(t)+k} = j_k \ \text{ for } k = 1, \dots, n.$$

By this construction we associate a family of networks (depending of a chosen numbering) with y. All these networks are equivalent in that the sequences of service times and routing decisions associated with any given station coinside for all the networks. For our further purposes we can choose any network from this family.

Composition of networks.

Let Σ be an open generalized Jackson network with no exogeneous input. At time $\widehat{t} \ge 0$, with any station i we associate sequences of service times $\{\sigma^{(i)}_j\}$, routing decisions $\{v^{(i)}_j\}$, and a queue length $q^{(i)}$. Let d^i be a total number of departures from i after time $\widehat{t}$. We assume that $\max_i d^i < \infty$. Let $\widetilde{\Sigma}$ be an initially empty open generalized Jackson network fed by one customer with arrival time $\widehat{t}$. With each station i we associate sequences of service times $\widetilde{\sigma}^{(i)}_j$ and routing decisions $\widetilde{v}^{(i)}_j$. Let $\widetilde{d^i}$ be the corresponding total number of departures from i and assume $\max_i \widetilde{d^i} < \infty$.

The composition of Σ *and* $\widetilde{\Sigma}$ *at time* $\widehat{t}$ is the open generalized Jackson network $\widehat{\Sigma}$ with no exogeneous input defined by the following relations: at time $\widehat{t}$, the queue length at station i is $\widehat{q}^{(i)} = q^{(i)} + \mathbf{I}(\widetilde{\nu}_1^{(0)} = i)$, the number of departures from i after time $\widehat{t}$ is $\widehat{d^i} = d^i + \widetilde{d^i}$, and the sequence of service times and routing decisions associated with station i at time $\widehat{t}$ is

$$(\widehat{\sigma}_k^{(i)}, \widehat{v}_k^{(i)}) = \begin{cases} (\sigma_k^{(i)}, v_k^{(i)}), & \text{if } \ 1 \le k \le d^i, \\ (\widetilde{\sigma}_k^{(i)}, \widetilde{v}_k^{(i)}), & \text{if } \ d^i + 1 \le k \le \widehat{d^i}. \end{cases}$$

We do not need to define $(\widehat{\sigma}_k^{(i)}, \widehat{v}_k^{(i)})$ for all k, since there are no customers served at station i after the $\widehat{d^i}$-th departure.

We define $\Psi(y, \xi_1)$ according to the following algorithm.

1. At time T_1 construct the network Σ from y;
2. Construct the composition of the networks Σ and Σ_1 at time T_1 (see Section 4 for the definition of Σ_1);
3. Define Γ-processes $\Gamma_1^{i,j}(t)$ for the new network as the number of departures from i to j in the time interval $(T_1 + t, \infty)$. Then $\Psi(y, \xi_1) = \{\Gamma_1^{i,j}(t),\ 1 \le i \le r,\ 1 \le j \le r+1,\ t \ge 0\}$. Note that Ψ does not depend on T_1 and we can replace T_1 with any $\widehat{t} \ge 0$ in the procedure above if we let a customer in the network Σ_1 arrive at time $\widehat{t}$. For further details on the pathwise construction of Jackson-type queueing networks under stationary and ergodic assumptions on the driving sequences we refer the reader to [3].

Appendix C: Proof of Lemma 4.1

Step 1. Let $\pi^{i,j} = \pi^i p^{i,j}$ be the mean number of transitions from i to j. Then

$$\sum_{i=1}^{r+1} \pi^{0,i} = \sum_{i=0}^{r} \pi^{i,r+1} = 1, \tag{C.1}$$

$$\sum_{i=0}^{r} \pi^{i,j} = \sum_{i=1}^{r+1} \pi^{j,i} = \pi^j \ \text{ for all } j \ne 0, r+1. \tag{C.2}$$

Note that we can study a deterministic network with source 0, sink $r+1$, and flows $\pi^{i,j}$ from i to j, $0 \le i, j \le r+1$ instead of an MC. It is sufficient to find a network with flows $\{\widetilde{\pi}^{i,j}\}$ and a renumbering of nodes (states) such that (B.1) and (B.2) hold and

1. $\widetilde{\pi}^{i,j} = 0$ for all $0 \le j \le i \le r+1$;

2. for all i, j, if $\widetilde{\pi}^{i,j} > 0$ then $\pi^{i,j} > 0$;

3. $\widetilde{\pi}^i = \sum_{j=1}^{r+1} \widetilde{\pi}^{i,j} \le \pi^i$ for all i.

Indeed, assume we find such flows. Introduce a Markov chain $\{\widetilde{V}(m)\}$ with transition probabilities

$$\widetilde{p}^{i,j} = \frac{\widetilde{\pi}^{i,j}}{\widetilde{\pi}^i}.$$

(For i such that $\widetilde{\pi}^i = 0$, we define transition probabilities from i arbitrarily to satisfy the conditions of lemma.)
For this chain, $\widetilde{\pi}^i$ is the mean number of visits of i, $0 \le i \le r+1$. It is clear, that this new MC satisfies the conditions of Lemma 4.1.
Step 2. Assume that the network with flows $\{\pi^{i,j}\}$ contains a cycle, say C, i.e. there exists a sequence of nodes $k_0, k_1, \dots, k_s$ such that $k_0 = k_s$, all $k_0, \dots, k_{s-1}$ are different, and $\pi^{k_t,k_{t+1}} > 0$ for $t = 0, 1, \dots, s-1$.

We write $\{k \to l\} \in C$ if $(k, l) = (k_t, k_{t+1})$ for some t. Let

$$\pi^C = \min_{\{k \to l\} \in C} \pi^{k,l}$$

and let

$$\widehat{\pi}^{k,l} = \begin{cases} \pi^{k,l} - \pi^C, & \text{for all } \{k \to l\} \in C, \\ \pi^{k,l}, & \text{otherwise;} \end{cases}$$

In particular, $\widehat{\pi}^i = \pi^i - \pi^C$ if $i = k_t$ for some t, and $\widehat{\pi}^i = \pi^i$ otherwise. Note that (C.1) and (C.2) still hold for $\{\widehat{\pi}^{i,j}\}$, and that at least one of $\widehat{\pi}^{k,l} = 0$ for some $\{k \to l\} \in C$.
Thus, we have found a family of flows with a smaller number of cycles than that in the original one, and such that $\widehat{\pi}^i \le \pi^i$ for all i.
Step 3. Repeat the procedure from step 2 until the last cycle disappears (note that the number of cycles is finite). Let $\{\widetilde{\pi}^{i,j}\}$ be the final family of flows. Then $\widetilde{\pi}^i \le \pi^i$ for all i, and the result follows. □

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