A Sphere Theorem for non-reversible Finsler Metrics*

Hans-Bert Rademacher †

2002-09-16

Abstract

For a non-reversible Finsler metric F on a compact smooth manifold M we introduce the *reversibility* $\lambda = \max\{F(-X)|F(X) = 1\} \geq 1$. Then we show the following generalization of the classical sphere theorem in Riemannian geometry: A simply-connected and compact Finsler manifold of dimension $n \geq 2$ with reversibility λ and with flag curvature $\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1$ is homotopy equivalent to the n-sphere.

Mathematics Subject Classification (2000): 53C60, 53C20, 53C22

1 Introduction

The classical sphere theorem states that a simply-connected and compact manifold of dimension n with a Riemannian metric whose sectional curvature K satisfies $1/4 < K \leq 1$ is homeomorphic to the n-sphere, cf. [Kl2], [AM]. In the proof the homeomorphism is constructed using the estimate for the injectivity radius $\mathrm{inj} \geq \pi$ and the Toponogov comparison theorem. In [Kl1] W. Klingenberg shows that one can give a different proof without using the Toponogov comparison theorem: The injectivity radius estimate gives as lower bound for the length of a closed geodesic the value 2π. Then a Rauch comparison argument shows that the Morse index of a closed geodesic is at least $n-1$. From the Morse theory of the energy functional on the free loop space one can conclude, that the free loop space is $(n-2)$-connected. This implies that the manifold is homotopy equivalent to the n-sphere. P. Dazord remarked that this proof extends to the case of a *reversible Finsler metric,* i. e. a Finsler metric F for which $F(-X) = F(X)$ for all tangent vectors, cf. [Da1], [Da2].

*Math. Annalen 328 (2004) 373-387

†Universität Leipzig, Mathematisches Institut, Augustusplatz 10/11, D-04109 Leipzig `rademacher@mathematik.uni-leipzig.de`

The *flag curvature,* which depends on a flag (V, σ) consisting of a non-zero tangent vector V and a 2-plane σ in which V lies, generalizes the sectional curvature.
In this paper we consider also *non-reversible Finsler metrics,* we introduce the *reversibility* $\lambda = \lambda(M, F)$ of a Finsler metric F on a compact manifold M :

$$\lambda := \max\{F(-X) | X \in T_* M, F(X) = 1\}.$$

Obviously $\lambda \geq 1$ and $\lambda = 1$ if and only if F is reversible. The reversibility enters in the following generalization of the injectivity radius estimate for Riemannian metrics:

Theorem 1 *Let (M, F) be a simply-connected, compact Finsler manifold of dimension $n \geq 2$ with reversibility λ and flag curvature $\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1$. Then the length of a closed geodesic is at least $\pi \left(1 + \frac{1}{\lambda}\right)$.*

Using a hamiltonian description A. Katok defined in [Ka] a 1-parameter family $F_\epsilon; \epsilon \in [0, 1)$ of Finsler metrics on the 2-sphere. For $\epsilon = 0$ this is the standard Riemannian metric, for $\epsilon \in (0, 1)$ these metrics are non-reversible and for irrational parameter ϵ these metrics have exactly two geometrically distinct closed geodesics, cf. [Zi], [Ra1]. These two geodesics differ by orientation. We remark in Section 4 using the Legendre transformation that these Katok examples coincide with the Finsler metrics of constant flag curvature 1, constructed by Z. Shen in [Sh1]. These examples show that the estimate for the length of a closed geodesic in Theorem 1 is sharp.
Using a Rauch comparison argument and the Morse theory of the energy functional on the free loop space one concludes from Theorem 1 the following *Sphere Theorem:*

Theorem 2 *A simply-connected and compact Finsler manifold of dimension $n \geq 3$ with reversibility λ and with flag curvature $\left(1 - \frac{1}{1+\lambda}\right)^2 < K \leq 1$ is homotopy equivalent to the n-sphere.*

2 Finsler geometry

On a manifold M with C^∞ differentiable structure and with tangent bundle TM a *Finsler metric* is a continuous mapping $F : TM \to \mathbb{R}^{\geq 0}$, which is smooth outside the zero section $T^0 M$ of the tangent bundle and satisfies the following conditions:

1. F is *positively homogeneous,* i.e. $F(\mu X) = \mu F(X)$ for all $\mu > 0$ and $X \in TM$.
2. $F(X) = 0$ if and only if $X \in T^0 M$.

3. The second derivative $D^2_{fibre}F^2(q)$ in fibre direction is positive definite in every point $q \in M$.

The last condition is called *Legendre condition.* Let

$$g_V(X,Y) := \frac{1}{2} D^2_{fibre} F^2_V(X,Y) = \frac{1}{2} \left. \frac{\partial^2}{\partial s \partial t} \right|_{(s,t)=(0,0)} F^2(V + sX + tY).$$

For every tangent vector $V \in T_qM$ we obtain an inner product g_V on T_qM with $g_V = <.,.>_V = D^2_{fibre}F^2_V$, resp. a Riemannian metric on the induced bundle $\tau^*_M(TM)$.
In coordinates the Legendre condition has the following form: If $(q_1, \dots, q_n)$ are coordinates on an open subset U of M, and $(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$ are the induced coordinates on $TU \subset TM$, then the matrix

$$D^2_{fibre} F^2_{(q,\dot{q})} = \frac{1}{2} \left(\frac{\partial^2 F^2}{\partial \dot{q}_i \partial \dot{q}_j}(q, \dot{q}) \right)_{1 \le i,j \le n}$$

is positive definite at every point $(q, \dot{q}) \in TU$.
We define a trilinear form

$$< X_1, X_2, X_3 >_V := \frac{1}{4} \left. \frac{\partial^3}{\partial s_1 \partial s_2 \partial s_3} \right|_{(s_1,s_2,s_3)=(0,0,0)} F^2 \left(V + \sum_{i=1}^{3} s_i X_i \right)$$

for a tangent vector $V \neq 0, X \in T_qM$ and vector fields X_1, X_2, X_3 defined nearby q. It is a symmetric tensor on the bundle $\tau^*_M(TM)$, this *Cartan tensor* vanishes, if the Finsler metric comes from a Riemannian metric, i. e. $F(X) = \sqrt{g(X,X)}$ for a Riemannian metric g. If V is a non-vanishing vector field on an open subset $U \subset M$ of a Finsler manifold one can introduce a connection ∇^V on the tangent bundle over U as follows:

1. ∇^V is an affine connection
2. ∇^V is torsionfree, i.e.
$$\nabla^V_X Y - \nabla^V_Y X = [X,Y]$$
for all vector fields X, Y defined over U.
3. For all vector fields X, Y, Z over U :

$$\begin{aligned} 2 < \nabla^V_X Y, Z >_V = {} & Y. < X, Z >_V + X. < Y, Z >_V - Z. < X, Y >_V \\ & + < [X,Y], Z >_V + < [Z,Y], X >_V + < [Z,X], Y >_V \\ & -2 < Y, Z, \nabla^V_X V >_V -2 < X, Z, \nabla^V_X V >_V +2 < X, Y, \nabla^V_X V >_V \end{aligned} \quad (1)$$

One can show, that these conditions determine the connection ∇^V uniquely. In the Riemannian case the Cartan tensor vanishes, then this formula is the so-called *Koszul formula* determining the *Levi-Civita connection,* which does not depend on V. By adding Equation 1 twice we obtain:

$$X. < Y, Z >_V = < \nabla^V_X Y, Z >_V + < Y, \nabla^V_X Z >_V + 2 < Y, Z, \nabla^V_X V >_V \ . \qquad (2)$$

This connection is the *Chern-connection,* it can be viewed as a linear connection on the pull-back bundle $\tau_M^*(TM)$, cf. [BCS, ch. 2.4]. Let V be a non-zero vector field on the open subset $U \subset M$, then the *Chern curvature* $R^V(X,Y)Z$ for vector fields X, Y, Z defined on U is defined by the equation:

$$R^V(X,Y)Z := \nabla^V_X \nabla^V_Y Z - \nabla^V_Y \nabla^V_X Z - \nabla^V_{[X,Y]} Z.$$

In the Riemannian case this curvature does not depend on V and coincides with the Riemannian curvature tensor. For a *flag* $(V;\sigma)$ consisting of a non-zero tangent vector $V \in T_qM$ and a 2-plane σ with $V \in \sigma$ the *flag curvature* $K(V;\sigma)$ is defined as follows:

$$K(V;\sigma) = \frac{< R^V(V,W)W, V >_V}{< V, V >_V < W, W >_V - < V, W >_V^2}.$$

Here W is a tangent vector, such that V, W span the 2-plane σ. In the Riemannian case the flag curvature is the sectional curvature of the 2-plane σ and does not depend on V. In the literature there are several connections used in Finsler geometry. This is due to the fact, that there is no connection which is torsionfree and metric. The Chern connection is torsionfree and *almost metric* (cf. Equation 2), the Cartan connection is metric but has torsion. But for the definition of the flag curvature it does not make a difference whether one uses the Chern, the Cartan or the Berwald connection. If $c : I \to M$ is a smooth curve with $\dot{c} \neq 0$, we can define the *covariant derivative* $\frac{\nabla}{dt}$ of vector fields along this curve using the Chern connection: Extend the tangent vector field $\dot{c}$ onto an open neighborhood $U \subset M$ and extend the vector fields X, Y along c to vector fields X, Y on U. Then let

$$\frac{\nabla}{dt} X(t) := \nabla^{\dot{c}}_{\dot{c}} X(t)\,.$$

A smooth curve $c : I \to M$ is a *geodesic,* if $\frac{\nabla}{dt}\dot{c}(t) = 0$.

One can give the following geometric interpretation of the flag curvature in terms of a Riemannian metric, cf. [Sh, 6.2]: Assume that V is a non-vanishing geodesic vector field in an open subset $U \subset M$.. Then using the Finsler metric we obtain a Riemannian metric $g := g_V$ on U. Then the covariant derivative of the Riemannian metric in direction of V coincides with the covariant derivative ∇^V of the Finsler metric in direction of the geodesic field. In particular V is also a geodesic field for the Riemannian metric g. In addition the flag curvature $K^V(V,W)$ of the Finsler metric coincides with the sectional curvature $K_g(V,W)$ of the Riemannian metric.

As in the Riemannian case geodesics are critical points of the *energy functional* E. For a smooth curve $c : [a, b] \to M$ the energy functional is defined by

$$E(c) = \frac{1}{2} \int_a^b F^2(\dot{c}(t))dt.$$

Then we obtain the *first variation formula:*

Lemma 1 ([Sh, ch.5.1]) *If $c_s : t \in [a, b] \to c_s(t) \in M, s \in (-\epsilon, \epsilon)$ is a smooth variation of the curve $c = c_0$ with variation vector field $V(t) = \frac{\partial}{\partial s}\big|_{s=0} c_s(t)$ then*

$$\frac{d}{ds}\bigg|_{s=0} E(c_s) = \langle \dot{c}(b), V(b) \rangle_{\dot{c}(b)} - \langle \dot{c}(a), V(a) \rangle_{\dot{c}(a)} + \int_a^b \left\langle \frac{\nabla}{dt}\dot{c}, V \right\rangle_{\dot{c}} dt.$$

One can conclude that geodesics between two fixed points are the critical points of the energy functional on the space of piecewise smooth curves between these fixed points.

Now we consider the following two cases, $c : [0, 1] \to M$ is a geodesic which is either closed (periodic), i. e. $\dot{c}(0) = \dot{c}(1)$ or it is a geodesic loop, i. e. $c(0) = c(1)$. The first case we consider is the case of a closed geodesic $c : S^1 = [0, 1]/\{0, 1\} \to M$. Let V_c be the vector space of continuous and piecewise smooth periodic vector fields $X(t)$ along $c(t)$. Then we define

$$\mathrm{V}_c^{\perp} := \{X \in \mathrm{V}_c \,|\, g_{\dot{c}}(\dot{c}, X) = 0\}.$$

We define the *index form* I_c of c as the following quadratic form on the space $\mathrm{V}_c^{\perp}$:

$$\mathrm{I}_c(X, Y) = \int_0^1 \left\{ g_{\dot{c}} \left(\frac{\nabla}{dt} X, \frac{\nabla}{dt} Y \right)(t) - g_{\dot{c}} \left(R^{\dot{c}}(\dot{c}, X)X, X \right)(t) \right\} dt\,.$$

Then the the *index* $\mathrm{ind}(c)$ is defined as the maximal dimension of a subspace of $\mathrm{V}_c^{\perp}$ on which the index form is negative definite. One can show that this number is always finite. The second variation formula shows, how one can interpret this index form geometrically: Let $c_s : S^1 \to$ be a variation of $c_0 = c$ by piecewise smooth curves with piecewise smooth variation vector field $X = X(t) = \frac{\partial}{\partial s}\big|_{s=0} c_s(t)$. Then it is a consequence of the second variation formula, that

$$\frac{d^2}{ds^2}\bigg|_{s=0} E(c_s) = \mathrm{I}_c(X, X),$$

compare [Sh, 10.2]. The index $\mathrm{ind}(c)$ is also a *Morse index,* since the index form can be seen as (restriction of) the hessian $d^2E(c)$ of the energy functional

$$E : \Lambda M \to \mathbb{R}; E(\sigma) = \frac{1}{2} \int_0^1 F^2(\dot{c}(t))dt.$$

Here ΛM is the free loop space $\Lambda = \Lambda M$:

$$\Lambda = \Lambda M = \left\{ \sigma : S^1 \to M \,|\, \sigma \text{ absolutely continuous } , E(\sigma) < \infty \right\} .$$

Now let $c : ([0,1], \{0,1\}) \to (M, \{p\})$ be a geodesic loop and let W_c be the vector space of continuous and piecewise smooth vector fields X along c with $X(0) = X(1) = 0$. We set $W_c^{\perp} := \{X \in W_c \,|\, g_{\dot{c}}(\dot{c}, X) = 0\}$ then the above defined index form defines a quadratic form $I_{c,\Omega}$ on $W_c^{\perp}$, which by the second variation formula can be seen as the restriction of hessian $d^2E(c)$ the energy functional $E : \Omega_p(M) \to \mathbb{R}$. Here

$$\Omega_p(M) := \{\sigma \in \Lambda M | \sigma(0) = p\}$$

The index of the quadratic form $I_{c,\Omega}$ is the Ω-index $\mathrm{ind}_{\Omega}(c)$ of the geodesic loop c. From the Morse theory of the energy functional $E : \Lambda \to \mathbb{R}$ one obtains the following:

Lemma 2 *[Ra1], [Sh, Thm. 17.4.3] Let (M, F) be a compact Finsler manifold of dimension $n \geq 2$. If all closed geodesics on M satisfy:* $\mathrm{ind} c \geq n - 1$ *then the manifold is homotopy equivalent to the n-dimensional sphere, i. e. for $n \neq 3$ it is homeomorphic to the n-sphere.*

As in the case of a Riemannian metric one obtains comparison results for the index of a closed geodesic:

Lemma 3 *Let (M, F) be a compact Finsler manifold of dimension n with flag curvature $\delta < K \leq 1$ for a positive $\delta \in \mathbb{R}^+$. If the length of c satisfies $L(c) \geq \pi/\sqrt{\delta}$ (resp. $L(c) \leq \pi$), then the indices* $\mathrm{ind} c; \mathrm{ind}_{\Omega} c$ *satisfy:* $\mathrm{ind}(c) \geq \mathrm{ind}_{\Omega}(c) \geq n - 1$ *(resp.* $\mathrm{ind}_{\Omega}(c) \leq \mathrm{ind}(c) \leq n - 1$*).*

As in the Riemannian case one obtains as a consequence of the second variation formula the following result, which implies *Synge's theorem:* An even-dimensional, compact and oriented Finsler manifold of positive flag curvature is simply-connected:

Lemma 4 ([Sh, ch. 10.3]) *Let (M, F) be an even-dimensional and oriented Finsler manifold with positive flag curvature $K > 0$ and let c be a closed geodesic. Then there is a periodic, smooth and parallel unit vector field W along c.*

3 The length of a shortest closed geodesic

An important step in the proof of the sphere theorem in Riemannian geometry is the injectivity radius estimate for manifolds of positive sectional curvature in the case of an even-dimensional manifold resp. for manifolds which are *quaterly pinched* for an odd-dimensional manifold, i.e. the *sectional curvature* K satisfies $1/4 < K \leq 1$. We show for non-reversible Finsler metrics with pinched positive flag curvature an estimate for the length of a shortest non-trivial closed geodesic resp. a shortest geodesic loop. Here the pinching constant depends on the reversibility λ of the Finsler metric.

If λ is the reversibility, we conclude that for all $X \in TM$:

$$\frac{1}{\lambda}F(X) \leq F(-X) \leq \lambda F(X). \tag{3}$$

We denote by

$$\theta : M \times M \to \mathbb{R}$$

the *pseudo-distance* induced by F, i.e.

$$\theta(p,q) = \inf\{L(c)\,|\, c : [0,1] \to M \text{ smooth, } c(0) = p, c(1) = q\}.$$

Then $\theta(p,q) = 0$ if and only if $p = q$ and the triangle inequality

$$\theta(p,q) \leq \theta(p,r) + \theta(r,q)$$

holds for all $p, q, r \in M$. But in general $\theta(p,q) \neq \theta(q,p)$. For a reversible Finsler metric this pseudo-distance is actually a distance. Estimate (3) implies that for all $p, q \in M$:

$$\frac{1}{\lambda}\theta(p,q) \leq \theta(q,p) \leq \lambda\theta(p,q). \tag{4}$$

We introduce the *symmetrized metric*

$$d(p,q) := \frac{1}{2}\left(\theta(p,q) + \theta(q,p)\right) . \tag{5}$$

For $\lambda = 1$ it coincides with the metric induced by the reversible Finlser metric.

We call a *(geodesic) biangle* c with *corners* $p, q \in M$ a continuous closed curve $c : [0,b] \to M$ with a point $a \in (0,b)$, such that $c(0) = p = c(b); c(a) = q$ and such that the restrictions $c|[0,a]$ and $c|[a,b]$ are geodesics. We call this biangle *minimal,* if the two geodesics $c|[0,a]; c|[a,b]$ are *minimal,* i.e. $L\left(c|[0,a]\right) = \theta(c(0), c(a)) = \theta(p,q)$ and $L\left(c|[a,b]\right) = \theta(c(a), c(b)) = \theta(q,p)$. Hence the length $L = L(c)$ of a minimal geodesic biangle with corners p, q equals $2d(p,q)$.

For $X \in T_qM$ we denote by $c_X : \mathbb{R} \to M$ the geodesic with $c'_X(0) = X$; then the exponential map $\exp_q : T_qM \to M$ is given by $\exp_q(X) = c_X(1)$. For a unit tangent vector $X \in T_q^1M$, i.e. a tangent vector X with $F(X) = 1$ we denote by $t(X) > 0$ the positive number

$$t(X) := \sup\left\{s > 0\,|\, \theta(\exp_q(sX), q) = s\right\}.$$

Then we call $\exp_q(t(X)X)$ a *cut point.* The *cut locus*

$$\operatorname{Cut}(q) := \{\exp_q(t(X)X) \mid F(X) = 1\}$$

is the union of the cut points on geodesics emanating from q. For a non-reversible Finsler metric in general $r \in \operatorname{Cut}(q)$ does not imply $q \in \operatorname{Cut}(r)$. For a compact Finsler manifold (M, F) and a point $q \in M$ we denote by $d(q) := \inf\{d(r, q); r \in \operatorname{Cut}(q)\}$ the distance (with respect to the symmetrized metric) of the cut locus $\operatorname{Cut}(q)$ and the point q. The *injectivity radius* $\operatorname{inj}(q)$ is given by $\operatorname{inj}(q) := \inf\{\theta(r, q); r \in \operatorname{Cut}(q)\}$. If the flag curvature satisfies $K \leq 1$ and if q is a point conjugate to p along a geodesic c, then it follows that $\theta(p, q) \geq \pi$, resp. $d(p, q) \geq (\pi/2)(1 + \lambda^{-1})$. If $d(p) < (\pi/2)(1 + \lambda^{-1})$ then there is a cut point $q \in \operatorname{Cut}(p)$ with $d(p, q) = d(p)$ and there is no conjugate point to p along any minimal geodesic segment starting from p and ending in q. As an immediate consequence of the first variation formula 1 we obtain for the variation of a geodesic biangle:

Lemma 5 *Let (M, F) be a Finsler manifold and let $c_s : [0, b] \to M; s \in (-\epsilon, \epsilon)$ be a variation with fixed endpoints $c_s(0) = c_0(0); c_s(b) = c_0(b)$ of the geodesic biangle $c = c_0$ with corners $p = c(0) = c(b); q = c(a); a \in (0, b)$. Denote $V := \frac{\partial c_s}{\partial s}(a)\big|_{s=0} \in T_qM$. Then we obtain for the energy $E(c_s) = 1/2 \int_a^b F^2(c_s'(t))dt$:*

$$dE(c).V = \left.\frac{d}{ds}\right|_{s=0} E(c_s) = g_{c'(a+)}\left(V, c'(a+)\right) - g_{c'(a-)}\left(V, c'(a-)\right)$$

Here $g_W(W, .) = L_F(W)$ is the Legendre transformation $L_F : TM \to T^*M$ defined by the Finsler metric F applied to the tangent vector W. If follows from the definition of the Finsler metric, that for distinct $U, W \in T_pM$ the difference $g_U(U, .) - g_W(W, .) = L_F(U) - L_F(W)$ does not vanish, since L_F is a diffeomorphism.

Lemma 6 *Let (M, F) be a compact Finsler manifold with reversibility λ and with flag curvature $K \leq 1$. If there is a point $p \in M$ with $d(p) < (\pi/2)(1 + \lambda^{-1})$ then there is a cut point $q \in \operatorname{Cut}(p)$ and a minimal geodesic biangle c with corners p, q being smooth at q, i.e. c forms a* geodesic loop *starting from p. The length $L(c)$ of this geodesic loop equals $2d(p) = 2d(p, q)$.*

Proof. Let $(q_k)_k \subset \operatorname{Cut}(p)$ be a minimal sequence, i.e.

$$d(p) = \inf\{d(p, q) | q \in \operatorname{Cut}(p)\} = \lim_{k\to\infty} d(p, q_k) < \frac{\pi}{2}\left(1 + \frac{1}{\lambda}\right).$$

Since the cut locus $\operatorname{Cut}(p)$ is a compact subset of M the sequence $(q_k)_k$ has a convergent subsequence with limit point $q \in \operatorname{Cut}(p)$. As already remarked above, q is not conjugate to p along any minimal geodesic from p to q. Therefore there are two distinct minimal geodesics $c_1, c_2 : [0, d(p,q)] \to M$ parametrized by arc length with $c_1(0) = c_2(0) = p$, $c_1(\theta(p,q)) = c_2(\theta(p,q)) = q$ and $c_1'(0) \neq c_2'(0)$. We choose a minimal geodesic $c_3 : [0, \theta(q,p)] \to M$ parametrized by arc length with $c_3(0) = q$ and $c_3(\theta(q,p)) = p$.

We want to prove that one of the minimal geodesic biangles formed by c_1, c_3 and c_2, c_3 is smooth at q, i.e. is a geodesic loop starting from p. We assume that our claim does not hold, i.e. we assume that $c_1'(\theta(p,q)) \neq c_3'(0)$ and $c_2'(\theta(p,q)) \neq c_3'(0)$. Since q is not conjugate to p along $c_j, j = 1,2$ we can choose an open neighborhood $U \subset M$ of q and open disjoint neighborhoods $U_j \subset T_pM, j = 1,2$ of $c_j'(0) \in T_pM$ such that the restrictions

$$\exp_p |U_j : U_j \to U, j = 1,2$$

of the exponential mapping $\exp_p : T_pM \to M$ are diffeomorphisms. We define two functions

$$f_j : U \to \mathbb{R}; f_j(v) = F\left((\exp_p |U_j)^{-1}(v)\right); j = 1,2.$$

Then f_j is differentiable and of maximal rank with

$$(\exp_p |U_j)^{-1}(q) = \theta(p,q) c_j'(0); f_j(q) = \theta(p,q)$$

and $\mathrm{grad} f_j(q) = c_j'(\theta(p,q))$ resp. $df_j(q).X = < c_j'(\theta(p,q)), X >_{c_j'(\theta(p,q))}$ for all X, which follows from the Gauß lemma [Sh, 11.2.1]. Since

$$\mathrm{grad} f_1(q) = c_1'(\theta(p,q)) \neq c_2'(\theta(p,q)) = \mathrm{grad} f_2(q)$$

it follows that the function

$$f := f_1 - f_2 : U \to \mathbb{R}; v \mapsto f_1(v) - f_2(v)$$

has maximal rank in an open neighborhood of q which we again denote by U, i. e. $V = f^{-1}(0)$ is a smooth hypersurface with $q \in V$. Since by assumption the tangent vectors $c_1'(\theta(p,q)); c_2'(\theta(p,q)); c_3'(0) \in T_qM$ are pairwise disjoint there is a tangent vector $v \in T_qV \subset T_qM$ with

$$g_{c_1'(\theta(p,q))}(c_1'(\theta(p,q), v) - g_{c_2'(\theta(p,q))}(c_2'(\theta(p,q)), v) \neq 0.$$

This implies that we can assume without loss of generality that for the geodesic biangle c formed by c_1, c_3 we have with the notation from Lemma 5:

$$dE(c)(v) \neq 0.$$

Hence by eventually using $-v$ instead of v we conclude: For a sufficiently small open neighborhood V' of q in V there are three pairwise distinct geodesics $\gamma_{v,j} : [0, l_v] \to M, j = 1,2; \gamma_{v,3} : [0, l_v'] \to M$ parametrized by arc length with $\gamma_{v,1}(0) = \gamma_{v,2}(0) = \gamma_{v,3}(l_v') = p; \gamma_{v,j}'(0) \in U_j; j = 1,2; L(\gamma_{v,1}) = L(\gamma_{v,2}); l_v = f_1(v) = f_2(v) : \gamma_{v,1}(l_v) = \gamma_{v,2}(l_v) = \gamma_{v,3}(0)$ and

$$L(\gamma_{v,1}) + L(\gamma_{v,3}) = L(\gamma_{v,2}) + L(\gamma_{v,3}) = l_v + l_v' < L(c).$$

Since $\gamma_{v,1}, \gamma_{v,2}$ are not minimal on any interval $[0, l_v + \epsilon]$ for $\epsilon > 0$, there is $t_v \in (0, l_v]$, such that $\gamma_{v,1}(t_v) = q_v$ is the first cut point on $\gamma_{v,1}|[0,\infty)$. Since

$$2d(p, q_v) < l_v + l'_v < L(c) = 2d(p, q)$$

we obtain a contradiction to the definition of q. Hence there is $j \in \{1, 2\} : c'_j(\theta(p, q)) = c'_3(0)$, which means that the geodesic biangle formed by c_j and c_3 is a geodesic loop starting from p.

Lemma 7 *Let (M, F) be a compact Finsler manifold with reversibilty λ and flag curvature $K \le 1$. We define the positive number $d := d(M, F) = \inf\{d(p) \,|\, p \in M\} = \inf\{d(p, q)|p \in M; q \in \mathrm{Cut}(p)\}$. If $d < \pi(1 + \lambda^{-1})/2$ then there is a shortest geodesic loop c with initial point p and a point $q \in \mathrm{Cut}(p)$ on this loop with $L(c) = 2d = 2d(p, q)$.*

Proof. Let $q \in \mathrm{Cut}(p)$ be a point with $d = d(p, q)$. By the preceding Lemma there is a geodesic loop c with $c(0) = p$ and $L(c) = 2d$. It remains to show that this curve is a shortest geodesic loop. Let γ be a shortest geodesic loop with initial point $\gamma(0) = p$. Then denote by $q = c(t_0)$ the cut point, i. e. $c|[0, t_0]$ is minimal and $L(\gamma) \ge 2d(p, q)$. The preceding Lemma implies that there is a geodesic loop c from p with $L(c) = d = d(p)$, hence $L(\gamma) = d$.

Remark 1 Guided by the Riemannian case one could expect, that d is the length of a shortest closed geodesic. But if one uses the same arguments as in the Proof of Lemma 6 one can only show the following: There are two minimal geodesics c_1, c_2 from p to $q \in \mathrm{Cut}(p)$ and a minimal geodesic c_3 from q to p, such that c_1 and c_3 form a geodesic loop from p and such that either c_3 and c_1 or c_3 and c_2 form a geodesic loop from q. Only in the first case one would obtain a closed geodesic. Therefore at least this Proof does not show whether a shortest closed geodesic has length d. But in any case we obtain a lower bound for the length of a shortest closed geodesic.

Lemma 8 *Let (M, F) be a compact Finsler manifold with reversibility λ and flag curvature $K \le 1$. Assume that $c_s : [0, 1] \to M,\ ,\ s \in [0, 1]$ is a homotopy of closed curves (i.e. $c_s(1) = c_s(0)$ for all $s \in [0, 1]$) between a point curve c_0, i.e. $c_0(t) = p$ for all t and some $p \in M$ and the geodesic loop $c = c_1$. Then*

$$\max_{s\in[0,1]} L(c_s) \ge \pi\left(1 + \frac{1}{\lambda}\right) .$$

Proof. We use the following notation: Let

$$c : (s, t) \in [0, 1] \times [0, 1] \mapsto c(s, t) = c_s(t) \in M,$$

by assumption $p = c(0,t) = c_0(t)$ for all $t \in [0,1]$, $c(s,1) = c(s,0)$ for all $s \in [0,1]$ and c_1 is a geodesic loop. Assume that for all $s \in [0,1]$ we have: $L(c_s) < \pi\left(1+\frac{1}{\lambda}\right)$. It follows that there is a $\delta > 0$ such that for all $s \in [0,1]$ we have $L(c_s) \le (\pi - \delta)\left(1+\frac{1}{\lambda}\right)$. Let $p_s := c_s(0)$, if for some $t,s \in [0,1] : \theta(p_s, c_s(t)) > \pi - \delta$, then $L(c_s) \ge \theta(p_s, c_s(t)) + \theta(c_s(t), p_s) > (\pi - \delta)\left(1+\frac{1}{\lambda}\right)$ in contradiction to our assumption. Hence we have for all $s,t \in [0,1] : c_s(t) \in \overline{B}_{\pi-\delta}(M) = \exp(\overline{B}_{\pi-\delta}(T_{p_s}M))$, here $\overline{B}_{\pi-\delta}(TM) = \{X \in TM \mid F(X) \le \pi - \delta\}$. The mapping

$$F := \tau_M \times \exp : \overline{B}_{\pi-\delta}(TM) \to \{(x_1,x_2) \in M \times M | \theta(x_1,x_2) \le \pi - \delta\} \subset M \times M,$$

with $F(X) = (p, \exp(X)), X \in T_pM$ has everywhere maximal rank, since the flag curvature satisfies $K \le 1$. Therefore there is a uniquely determined lift

$$\tilde{c} : (s,t) \in [0,1] \times [0,1] \mapsto \tilde{c}(s,t) = \tilde{c}_s(t) \in \overline{B}_{\pi-\delta}(TM)$$

with $c_s(t) = \exp_{p_s}(\tilde{c}_s(t))$ for all $s,t \in [0,1]$. Since $\tilde{c}(0,t) = \tilde{c}_0(t) = p$ for all $t \in [0,1]$ one concludes that $\tilde{c}_s(1) = \tilde{c}_s(0)$ for all $s \in [0,1]$. Since c_1 is geodesic loop with $c_1(0) = p_1$ we conclude $\tilde{c}_1(t) = tc_1'(0)$ for all $t \in [0, \pi/L(c_1))$ which contradicts $\tilde{c}_1([0,1]) \subset B_{\pi-\delta}(T_{c_1(0)}M)$.

Theorem 3 *Let (M,F) be a simply-connected compact Finsler manifold of even dimension $n \ge 2$ with reversibilty λ and with flag curvature $0 < K \le 1$. Then every non-constant closed geodesic c has length $L(c) \ge \pi\left(1+\frac{1}{\lambda}\right)$.*

PROOF. Let $c : S^1 = [0,1]/\{0,1\} \to M$ be a shortest closed geodesic with $0 < L(c) < \pi\left(1+\frac{1}{\lambda}\right)$. There exists a parallel unit vector field W along c, (cf. Lemma 4), it follows that the index form I_c on the vector space $V_c^{\perp}$ satisfies: $I_c(W,W) < 0$. Let $c_s, s \in (-\epsilon, \epsilon)$ be a variation of $c = c_0$ with variation vector field W. Then it follows from the second variation formula that $E(c_s) < E(c_0)$ for all $s \in (-\epsilon, 0) \cup (0, \epsilon)$. Since there are no critical values of E in the interval $(0, E(c_0))$ there is a mapping $h_s : S^1 \to M; s \in [-1,1]$ with $c = h_0; L(h_1) = L(h_{-1}) = 0$ and $L(h_s) < L(c) = L(h_0)$ for all $s \in (-1,1), s \ne 0$. But this contradicts the *Long Homotopy Lemma* 8.

Theorem 4 *Let (M,F) be a simply-connected and compact Finsler manifold of dimension $n \ge 3$ with reversibilty λ and with flag curvature $\left(1-\frac{1}{1+\lambda}\right)^2 < K \le 1$. Then every non-constant geodesic loop c has length $L(c) \ge \pi\left(1+\frac{1}{\lambda}\right)$ and the injectivity radius satisfies* $\mathrm{inj} \ge \pi/\lambda$.

PROOF. For every geodesic loop c^* with $L(c^*) \ge \pi\left(1+\frac{1}{\lambda}\right)$ we obtain from Lemma 3 that $\mathrm{ind}_{\Omega} c^* \ge n-1 \ge 2$. By a standard argument in Morse theory this implies that the relative homotopy group

$$\pi_1((\Omega_pM, \Omega_p^{\kappa-}M) = 0. \qquad (6)$$

vanishes, here $\Omega_p^{\kappa-}M := \{\gamma \in \Omega_p M \,|\, E(\gamma) < \kappa\}$ and $\kappa := \pi^2(1+\lambda^{-1})^2/2$.
We assume that there is a shortest geodesic loop c of length $L = L(c) < \left(1+\frac{1}{\lambda}\right)\pi$, hence then $L(c) = 2d$, cf. Lemma 7. Since the manifold is simply-connected there is a path $s \in [0,1] \mapsto c_s \in \Omega_p(M)$ such that $c_0 = p$ is the point curve and $c_1 = c$. Then it follows from Equation 6 that there is a homotopy $s \in [0,1] \mapsto \tilde{c}_s \in \Omega_p^{\kappa-}(M)$ with $\tilde{c}_0 = p, \tilde{c}_1 = c$. But this contradicts the Long Homotopy Lemma 8. The estimate for the injectivity radius follows since $\mathrm{inj} \geq 2d/(1+\lambda)$. With these estimates we can prove the *Sphere Theorem* 2 stated in the Introduction:

PROOF. *of Theorem 2*: We conclude from Theorem 4 that the length of a non-trivial closed geodesic c satisfies $L(c) \geq \pi\left(1+\frac{1}{\lambda}\right)$. Since $K > \left(1-\frac{1}{1+\lambda}\right)^2$ it follows from Lemma 3 that the index $\mathrm{ind}c$ is bounded from below by $n-1$, i. e. $\mathrm{ind}(c) \geq n-1$ for every non-constant closed geodesic. Then Lemma 2 implies that M is homotopy-equivalent to the n-sphere.

4 Example

We consider the following Finsler metric on S^2 : Let V be the Killing field which belongs to the 1-parameter subgroup $t \in \mathbb{R} \mapsto R(2\pi t) \in \mathbb{SO}(3)$ of rotations $R(2\pi t)$ around the axis through the north and south pole with angle $2\pi t$ and let g be the standard Riemannian metric on S^2 resp. the tangent bundle TS^2. Denote by g^* the dual metric on the cotangent bundle T^*M. For every $\epsilon \in (0,1)$ the function

$$H_\epsilon : T^*S^2 \to \mathbb{R}; H_\epsilon(y) := \sqrt{g^*(y,y)} + \epsilon y(V)$$

defines a quadratic Hamiltonian $\frac{1}{2}H_\epsilon^2$, whose corresponding Finsler metric we denote by F_ϵ. These Finsler metrics were introduced by A. Katok (cf. [Ka]), their geometry was investigated by W. Ziller [Zi]. In the following Theorem we add the observation, that these Katok examples coincide with the examples of constant flag curvature given by Z. Shen in [Sh1]. They provide metrics which show that our estimates for the length of a shortest closed geodesic is sharp.

Theorem 5 *For every $\epsilon \in (0,1)$ the Finsler metric F_ϵ is a non-reversible Finsler metric on S^2 with constant flag curvature $K \equiv 1$. The reversibility is $\lambda = (1+\epsilon)/(1-\epsilon)$. If ϵ is irrational then there are exactly two geometrically distinct closed geodesics $c_\pm$ of length $L(c_\pm) = 2\pi(1\pm\epsilon)^{-1}$. In particular the shortest closed geodesic c_+ satisfies $L(c_+) = 2\pi(1+\lambda^{-1}) = \pi/(1+\epsilon)$. The injectivity radius equals π.*

PROOF. In geodesic polar coordinates $(r,\phi) \in (0,\pi) \times [0,2\pi]$ the standard metric on the sphere is of the form $g = dr^2 + \sin^2(r)d\phi^2$. It is shown in [Sh1, Rem. 3.1] that the Finsler metric

$$F_\epsilon = \frac{\sqrt{(1-\epsilon^2\sin^2 r)dr^2 + \sin^2(r)d\phi^2} - \epsilon\sin^2 r d\phi}{1-\epsilon^2\sin^2 r} \tag{7}$$

for every $\epsilon \in [0,1)$ has constant flag curvature 1. For $\epsilon = 0$ it is the standard Riemannian metric. The Finsler metric F_ϵ is a *Randers metric,* i.e. there is a Riemannian metric γ and a 1-form β with $F(Y) = \sqrt{\gamma(Y,Y)} + \beta(Y)$. We denote the coefficents a_{ij} of the metric γ with respect to the coordinates $y^1 = r, y^2 = \phi$, i.e. $\gamma = a_{ij}y^iy^j$ and we denote the coefficents of the 1-form β with b_i, i.e. $\beta = b_iy^i$. Then also the corresponding Hamilton function is of Randers type, which follows by direct computation or from the result [HS, Theorem 5.8], [Sh, Example 3.1.1]. By this result

$$H(x,p) = \frac{1}{2}\left(\sqrt{\overline{a}^{ij}p_ip_j} \pm \overline{b}^i p_i\right)^2$$

provided $\|\beta\|^2 = \gamma_{ij}b^jb^j \neq 1$. Here we used the following expressions:

$$\overline{a}^{ij} = \frac{1}{1-\|b\|^2}a^{ij} + \frac{1}{(1-\|b\|^2)^2}b^ib^j\,;\;\overline{b}^i = \frac{1}{1-\|b\|^2}b^i$$

and $p_i = 1/2(\partial F^2/\partial y^i)$. For the Finsler metric F_ϵ given in Equation 7 we obtain:

$$a_{11} = \frac{1}{1-\epsilon^2\sin^2 r}; a_{12} = 0; a_{22} = \frac{\sin^2 r}{1-\epsilon^2\sin^2 r}; b_1 = 0; b_2 = -\frac{\epsilon\sin^2 r}{1-\epsilon^2\sin^2 r}$$

and therefore we obtain the corresponding Hamilton function $\frac{1}{2}H_\epsilon^2$ with

$$H_\epsilon(y,p) = \sqrt{p_1^2 + \frac{1}{\sin^2 r}p_2^2} + \epsilon p_2\,. \tag{8}$$

These coincide with the Katok example.
The statements about the closed geodesics are explained in [Zi] and [Ra1], the closed geodesics $c_\pm(t) = c(\pm t)$ are the equator in both directions, which is invariant under the rotation.

Remark 2 (a) This examples show that the estimate for the length of a shortest closed geodesic given in Theorem 1 is sharp. But we do not know whether there is a non-reversible Finsler metric with flag curvature $1/4 < K \leq 1$ on a manifold which is not homotopy equivalent to the n-sphere.

(b) In a forthcoming paper we will show how one can obtain existence result for closed geodesis on Finsler manifolds with positive flag curvature using the results of this paper.

References

[AM] U. Abresch, W. Meyer: *Injectivity radius estimates and sphere theorems.* In: *Comparison Geometry,* MSRI Publications, Vol. **30** (1997) 1-47

[BCS] D. Bao, S.- S. Chern, Z. Shen: *An Introduction to Riemann-Finsler Geometry.* Grad. Texts Mathem. **200** Springer Verlag New York 2000

[Da1] P. Dazord: *Variétés finslériennes de dimension $\delta-$pincées.* C. R. Acad. Sc. Paris, **266** (1968) 496-498

[Da2] P. Dazord: *Variétés finslériennes en forme des sphères.* C. R. Acad. Sc. Paris, **267** (1968) 353-355

[HS] D. Hrimiuc, H. Shimada: *On the $\mathcal{L}$-duality between Lagrange and Hamilton manifolds.* Nonlin. World **3** (1996) 613-641

[Ka] A. Katok: *Ergodic properties of degenerate integrable Hamiltonian systems.* Izs. Akad. Nauk SSSR Ser. Mat. **37** (1973) 539-576 = (engl. transl.) Math. USSR Izv. **7** (1973) 535-572

[Kl1] W. Klingenberg: *Manifolds with restricted conjugate locus.* Ann. Math. **78** (1963) 527-547

[Kl2] W. Klingenberg: *Riemannian geometry.* de Gruyter Studies Math. **1**, 2nd rev. ed., de Gruyter Berlin New York 1995

[Ma] H. H. Matthias: *Zwei Verallgemeinerungen eines Satzes von Gromoll und Meyer.* Bonner Math. Schr. **126** (1980)

[Ra1] H.B. Rademacher: *Morse-Theorie und Geschlossene Geodätische.* Bonner Math. Schr. **229** (1992)

[Sh] Z. Shen: *Lectures on Finsler geometry.* World Scientific, Singapore 2001

[Sh1] Z. Shen: *Two-dimensional finsler metrics with constant curvature.* To appear in: Manuscr. Math.

[Zi] W. Ziller: *Geometry of the Katok examples.* Ergod. Th. & Dynam. Sys. (1982) 135-157