The Fadell–Rabinowitz index and closed geodesics *

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Dedicated to Wilhelm Klingenberg with best wishes on his 70th birthday

1 Introduction

In critical point theory several topological *index theories* are introduced for functionals which are invariant under the action of a compact Lie group G. The *index* of a topological space with a G–action is a non–negative integer. The index satisfies the axioms of subadditivity, monotony and continuity. It is used to derive lower bounds for the number of critical orbits of a G–invariant functional on this space.

We are interested in the index introduced by Fadell–Rabinowitz [4] using the *equivariant (Borel-) cohomology.* Via a classifying map the cohomology ring $H^*(BG)$ of a classifying space is a subring of the equivariant cohomology $H^*_G(X)$ of the G–space X. For a *characteristic class* $\eta \in H^*(BG)$ the *Fadell–Rabinowitz index* $\mathrm{index}_\eta X$ of X is the smallest non–negative integer k with $\eta^k = 0$ in $H^*_G(X)$. Hence the classes η^j , $j = 0, \ldots, \mathrm{index}_\eta X - 1$ define a sequence of *subordinate cohomology classes.* Fadell–Rabinowitz use this index to study the bifurcation of time periodic solutions from an equilibrium solution for Hamiltonian systems of ordinary differential equations.

Ekeland–Hofer [3] use this index for the group S^1 and for rational coefficients to derive relations between symplectic invariants of periodic orbits of convex Hamiltonian energy surfaces in $\mathbb{R}^{2n}$. In particular they can show that there are at least two geometrically distinct periodic orbits. If there are only finitely many geometrically distinct periodic orbits then there are two geometrically distinct ones for which certain symplectic invariants coincide. This phenomenon can be interpreted as a *resonance relation.*

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In this paper we are concerned with the existence of closed geodesics on compact Riemannian manifolds. They can be characterized as the critical points of positive energy of the energy functional E on the *free loop space* $\Lambda = \Lambda M$ of M. Λ carries a canonical S^1– resp. $\mathbb{O}(2)$–action leaving E invariant. For the theory of closed geodesics we refer to Klingenberg's book [6] and to Bangert's survey article [2].

Let $\operatorname{ind}(c)$ be the morse–theoretic index of the closed geodesic c. If c is a closed geodesic then the iterates c^m , $m \geq 1$ are closed geodesics, too. We say c is *prime* if it is not an iterate of a shorter closed geodesic. Two closed geodesics $c_1, c_2 : S^1 \to M$ are *geometrically equivalent* if $c_1(S^1) = c_2(S^1)$. If c is prime, then the set $\mathbb{O}(2) \cdot c^m, m \geq 1$ is the set of geometrically equivalent closed geodesics. We associate to c the *average index* $\alpha_c := \lim_{m\to\infty}(\operatorname{ind} c^m/m)$ and the *mean average index* $\overline{\alpha}_c := \alpha_c/L(c)$, where $L(c)$ is the length of c. If $\pm\exp(\pi i\lambda), \lambda \in [0,1]$ is an eigenvalue of the linearized Poincaré map, then the average index depends linearly on λ.

In the case of a *bumpy* Riemannian metric (i.e. all closed geodesics are non–degenerate) with only finitely many geometrically distinct prime closed geodesics $c_1, \ldots, c_k$ the author shows in [9] the following: The sum $\sum_{i=1}^k \gamma_i \alpha_i^{-1}$ is a topological invariant of the manifold. Here $\gamma_i \in \{\pm 1/2, \pm 1\}$ is another invariant of c_k . It controls the orientability of the negative normal bundle of the orbits $S^1 \cdot c_i^m, m \geq 1$ and the parity of the sequence $\operatorname{ind}(c^m)$. We will derive here a lower bound for the sum $\sum \alpha_c^{-1}$ for arbitrary metrics.

The Fadell–Rabinowitz index of the sublevel sets $\Lambda^a := \{\gamma \in \Lambda \,|\, E(\gamma) \leq a^2/2\}$ is infinite since the fixed point set Λ^0 is contained in Λ^a. Therefore we modify the Fadell–Rabinowitz index. We use that the relative cohomology $H^*_{S^1}(\Lambda, \Lambda^0)$ with the rationals as coefficient field is also a module over $H^*(BS^1) = \mathbb{Q}[\eta]$ where $\dim \eta = 2$.

For simplicity we restrict ourselves in this introduction to the following case: Let $M = M_{d,m}$ be the compact simply–connected rank one symmetric space (CROSS) whose cohomology ring is generated by a d–dimensional class x with the relation $x^{m+1} = 0$. Hence $M_{d,1} = S^d$ is the d–dimensional sphere, $M_{2,m} = \mathbb{C}P^m$ (resp. $M_{4,m} = \mathbb{H}P^m$) is the m–dimensional complex (resp. quaternionic) projective space and $M_{8,2}$ is the Cayley plane.

Hingston remarks in [5] that $H^*_{S^1}(\Lambda, \Lambda^0)$ is not a torsion module for an odd–dimensional sphere. Here arguments carry over to the manifolds $M_{d,m}$, i.e. there is a class $z \in H^*_{S^1}(\Lambda, \Lambda^0)$ with $\eta^i \cdot z \neq 0$ for all i. For $a > 0$ we define the number $d_z(a)$ as the smallest integer k such that $\eta^k \cdot z$ restricted to $H^*_{S^1}(\Lambda^a, \Lambda^0)$ is trivial. The function $d_z(a)$ grows almost linearly, hence we can introduce the *global indices*

$$\underline{\sigma}_z := \liminf_{a\to\infty} \frac{d_z(a)}{a} \, , \, \overline{\sigma}_z := \limsup_{a\to\infty} \frac{d_z(a)}{a}$$

of z and obtain $0 < \underline{\sigma}_z \leq \overline{\sigma}_z < \infty$, see remarks 4.3 and 5.4.

Analogous to results in [3] we can show in theorem 6.2 and theorem 6.4:

Theorem 1.1 *Let g be any Riemannian metric on $M_{d,m}$ and let $[\underline{\sigma}_z, \overline{\sigma}_z]$ be the global index interval of the class z which we introduced above.*

a) If $t \in [\underline{\sigma}_z, \overline{\sigma}_z]$ then there is a sequence c_i of prime closed geodesics whose mean average indices $\overline{\alpha}_i$ converge to $2t$.

b) For every $\epsilon > 0$ we have the following estimate:

$$\sum_c \frac{1}{\alpha_c} \geq \frac{1}{2},$$

where we sum over all prime geometrically distinct closed geodesics c whose mean average index $\overline{\alpha}_c$ satisfies: $\overline{\alpha}_c \in (2\underline{\sigma}_z - \epsilon\,,\, 2\overline{\sigma}_z + \epsilon)$.

In the proof – in particular in the proof of theorem 5.9 – we approximate the given metric by bumpy metrics. In contrast to [3] we cannot use a finite–dimensional subspace of Λ since the finite–dimensional approximation of Λ^a by spaces of broken geodesics which already Morse used carries only a $\mathbb{Z}_k$–but not a S^1–action. Theorem 5.9 deserves independent interest since no assumption on the critical set is needed.

Corollary 1.2 *If there are only finitely many geometrically distinct closed geodesics for a metric on $M_{d,m}$, then $\underline{\sigma}_z = \overline{\sigma}_z = \sigma$ and $\sum \alpha_c^{-1} \geq 1/2$, where we sum over all prime geometrically distinct closed geodesics with $\overline{\alpha}_c = 2\sigma$.*

For metrics with positive sectional curvature one can estimate the average index by standard comparison arguments and by Klingenberg's injectivity radius estimate. In the following corollary we obtain a sufficient condition for the existence of at least two distinct closed geodesics whose mean average indices are arbitrarily close in the general case resp. coincide if there are only finitely many geometrically distinct closed geodesics:

Corollary 1.3 *If on $M = M_{d,m}$ with even dimension $\dim M = dm$ we have a metric with sectional curvature K satisfying $k^2/(dm-1)^2 < K \leq 1$ for $k \in \{1, 2, \ldots, dm-2\}$ then the following holds:*

For every $\epsilon > 0$ there are $k+1$ geometrically distinct closed geodesics $c_1, \ldots, c_{k+1}$ with mean average indices $\overline{\alpha}_1, \ldots, \overline{\alpha}_{k+1}$ satisfying $|\overline{\alpha}_i - \overline{\alpha}_j| < \epsilon$ for every pair i, j.

In particular if there are only finitely many closed geodesics on M, then there are $k+1$ geometrically distinct closed geodesics $c_1, \ldots, c_{k+1}$ with $\overline{\alpha}_1 = \ldots = \overline{\alpha}_{k+1} = 2\sigma$.

The last statement of the corollary can be interpreted as a *resonace relation* between the $k+1$ closed geodesics. In [1, 3.4] the existence of $k+1$ geometrically distinct closed geodesics on a homology $\mathbb{Z}_2$–sphere under the above pinching condition together with bounds for their lengths is shown.

Using local pertubation arguments due to Klingenberg–Takens (cf. [8] and [10]) one can show that for a C^2–generic metric two geometrically distinct closed geodesics have distinct mean average indices. Hence a C^2–generic metric on $M_{d,m}$ with sectional curvature $(dm-1)^{-2} < K \le 1$ has infinitely many geometrically distinct closed geodesics. In [10] the author shows that a C^2–generic Riemannian metric on a compact simply–connected manifold has infinitely many geometrically distinct closed geodesics. In the proof the relation between the average indices of the closed geodesics of a bumpy metric with only finitely many geometrically distinct closed geodesics derived in [9] is used.

As in [3] it is an open question, whether there are metrics with $\underline{\sigma}_z \neq \overline{\sigma}_z$. If there is a metric on $M_{d,m}$ with $\underline{\sigma}_z \neq \overline{\sigma}_z$, then there is an open neighborhood of this metric with infinitely many geometrically distinct closed geodesics.

The theorem also holds for Finsler metrics. This is interesting since there are examples of non–symmetric Finsler metrics with only finitely many geometrically distinct closed geodesics due to Katok, cf. [12] and example 6.7. In these examples there are geometrically distinct closed geodesics satisfying the resonance relation.

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2 A relative version of the Fadell–Rabinowitz index

We collect some facts about classifying spaces and equivariant (Borel) cohomology. Let G be a compact Lie group. A topological space X is a *G–space*, if X is Hausdorff and paracompact and if there is a continuous G–action on X from the left. Then the *isotropy subgroup* $I(x) := \{g \in G \,|\, g \cdot x = x\}$ of every $x \in X$ is compact, the orbit $G \cdot x$ of x in X is closed and the quotient space X/G is Hausdorff.

An *universal G–bundle* is a (locally trivial) principal G–bundle $p_G : EG \to BG$ whose total space EG is contractible. Here G acts on the right on EG. The base space $BG = EG/G$ is a *classifying space* for G. These bundles always exist, BG is unique in the category of CW complexes up to homotopy equivalence. If $p : X \to B$ is a (locally trivial) principal G–bundle with paracompact base space B then there is a map $f : B \to BG$, called the *classifying map*, such that the induced bundle f^*p_G is equivalent to the bundle p. f is unique up to homotopy. The quotient space $X_G = EG \times_G X$ is the *G–homotopy quotient*. Here G acts by $(g,(e,x)) \in G \times EG \times X \mapsto (eg^{-1}, gx) \in EG \times X$.

We denote by H^* the *Alexander–Čech cohomology*. For a locally contractible space the Alexander–Čech cohomology and the singular cohomology are isomorphic, cf. [11, ch.6.9]. The *equivariant cohomology* $H^*_G(X)$ is defined as

$$H^*_G(X) := H^*(X_G) \quad .$$

H^*_G satisfies the exactness, homotopy and excisison axioms for cohomology, but not the dimension axiom. If a one point space $\{p\}$ is considered as G–space with the trivial G–action we have

$$H^*_G(\{p\}) \cong H^*(BG) \quad .$$

The homotopy quotient is the base of the principal G–bundle $EG \times X \to X_G$. Via a classifying map $f : X_G \to BG$ of this bundle the equivariant cohomology has the structure of an $H^*(BG)$–module. H^*_G is a functor from the category of G–spaces to $H^*(BG)$–modules.

Now we fix a cohomology class $\eta \in H^*(BG)$ where we take any coefficient field, i.e. η is a *characteristic class.* Then the *Fadell–Rabinowitz index* $\mathrm{index}_\eta X$ of a G–space X is defined as

$$\mathrm{index}_\eta X := \inf\{k \in \mathbb{N}_0 \,|\, f^*(\eta^k) = 0\}\,,$$

cf. [4]. Here we allow $\mathrm{index}_\eta X = \infty$, i.e. $\mathrm{index}_\eta X \in \mathbb{N}_0 \cup \{\infty\}$.

$H^*_G(X)$ and $H^*(BG)$ are rings with the cup product as multiplicative structure and f^* is a ring homomorphism. We set $x^0 = 1$, so if $X \neq \emptyset$ then $f^*(1) = 1$ i.e. $\mathrm{index}_\eta X \geq 1$. Let $\mathrm{index}_\eta \emptyset = 0$. Then we list the following properties, which are proved in [4].

Proposition 2.1 *The Fadell–Rabinowitz index* $\mathrm{index}_\eta X \in \mathbb{N}_0 \cup \{\infty\}$ *of a G–space X satisfies the following properties:*

a) $\mathrm{index}_\eta X = 0$ *iff* $X = \emptyset$.

*b) (*normalization*)* $\mathrm{index}_\eta G = 1$.

*c) (*monotonicity*) If $g : X \to Y$ is a G–map, then* $\mathrm{index}_\eta X \leq \mathrm{index}_\eta Y$.

*d) (*continuity*) If A is an invariant subset of the G–space X then there is a closed invariant neighborhood V of A such that* $\mathrm{index}_\eta V = \mathrm{index}_\eta A$.

*e)(*subadditivity*) Let X be a G–space and A, B be closed invariant subsets of X with $X = A \cup B$. Then*

$$\mathrm{index}_\eta (A \cup B) \leq \mathrm{index}_\eta A + \mathrm{index}_\eta B\,.$$

*f) (*dimension*) Let $F = \mathbb{Q}$ be the field of rational numbers. If all isotropy subgroups $I(x), x \in X$ are finite, then the covering dimension of X/G, cf. [11, p.152] satisfies:*

$$(\mathrm{index}_\eta X - 1)\dim \eta \leq \dim X/G\,.$$

æ We introduce a relative version of the Fadell–Rabinowitz index. We make the following assumptions: Let X be a G–space, $A \subset X$ a closed invariant subset. We fix a coefficient field F and a characteristic class $\eta \in H^*(BG)$. Let $f_* : H^*(BG) \to H^*_G(X)$ be the homomorphism induced by a classifying map $f : X_G \to BG$. Then the cohomology ring $H^*_G(X,A)$ has the structure of an $H^*(BG)$–module as follows:

The cup product defines a homomorphism

$$H^*_G(X) \otimes H^*_G(X,A) \to H^*_G(X,A)\,,\ (\zeta, z) \mapsto \zeta \cup z\,.$$

For $\gamma \in H^*(BG)$, $z \in H^*_G(X,A)$ let $\gamma \cdot z := f^*(\gamma) \cup z$. Then for $z \in H^*_G(X,A)$ the *order* $\mathrm{ord}_\eta z$ with respect to η is defined by

$$\mathrm{ord}_\eta z := \inf\{k \in \mathbb{N} \cup \{\infty\} \mid \eta^k \cdot z = 0\}\,.$$

If $z \neq 0$ then $\eta^0 \cdot z = z$, hence $\mathrm{ord}_\eta z \geq 1$. If $z = 0$ we set $\mathrm{ord}_\eta z = 0$. The Fadell–Rabinowitz index $\mathrm{index}_\eta X$ is a special case of this order, since $\mathrm{index}_\eta X = \mathrm{ord}_\eta 1 = \mathrm{ord}_\eta(f^*(\eta)) - 1$, where $1 \in H^*_G(X)$ is an unit element ($A = \emptyset$).

Now we list properties of the order:

Proposition 2.2 *Let X resp. Y be a G–space, $A \subset X$ resp. $B \subset Y$ be closed invariant, $z \in H^*_G(X,B)$.*

a) If $\phi : (Y,B) \to (X,A)$ a G–map with induced homomorphism $\phi_* : H^*_G(X,A) \to H^*_G(Y,B)$, then $\mathrm{ord}_\eta \phi_*(z) \leq \mathrm{ord}_\eta z$.

b) If $A \subset B \subset X$ are closed invariant subsets, then there exists a closed invariant neighborhood N of B with the following property: If $j_B : (B,A) \to (X,A)$ and $j_N : (N,A) \to (X,A)$ are the inclusions, then $\mathrm{ord}_\eta j_N^*(z) = \mathrm{ord}_\eta j_B^*(z)$.

c) Let A, B, C be closed invariant subsets of X with $A \subset B$ and $X = B \cup C$. Let $j_B : (B,A) \to (X,A)$ be the inclusion and $z \in H^*_G(X,A)$. Then

$$\mathrm{ord}_\eta z \leq \mathrm{ord}_\eta j_B^*(z) + \mathrm{index}_\eta C\,.$$

The proofs are analogous to [4, (3.3)], [4, (3.5)] and [4, (3.6)] and are therefore omitted.

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3 On the equivariant cohomology of the free loop space

We consider compact simply–connected manifolds M whose rational cohomology algebra $H^*(M;\mathbb{Q})$ is generated by exactly one element. Hence there are two numbers d, m such that $H^*(M) = T_{d,m+1}(x)$, here $T_{d,m+1}(x) = \mathbb{Q}[x]/(x^{m+1})$ is the truncated polynomial algebra generated by an element x of degree d and height $m+1$. If d is odd then $m = 1$, M is rational homotopy equivalent to the odd–dimensional sphere S^d. The compact rank one symmetric spaces (*CROSS'es*) $M_{d,m}$ are also examples. Here $M_{d,1} = S^d$ is the d–dimensional sphere, $M_{2,m} = \mathbb{C}P^m$ is the m–dimensional complex projective space, $M_{4,m} = \mathbb{H}P^m$ is the m–dimensional quaternionic projective space and $M_{8,2} = \mathbb{C}aP^2$ is the Cayley plane. Due to Sullivan there are infinitely many other rational homotopy types of simply–connected compact manifolds in this class besides these examples.

Let (M, g) be a compact Riemannian manifold. Then we consider the *free loop space*

$$\Lambda = \Lambda M = \left\{ \gamma : S^1 \to M \,|\, \gamma \, \text{absolutely continuous} \, , \int_0^1 g(\dot\gamma, \dot\gamma) < \infty \right\}$$

with the canonical S^1 resp. $\mathbb{O}(2)$–action, cf. [6], [5].

$$E : \Lambda \to M \, , \, E(\gamma) := \frac{1}{2} \int_0^1 g(\dot\gamma, \dot\gamma)$$

is the $\mathbb{O}(2)$–invariant *energy functional*. We denote for $a > 0$ the sublevel sets

$$\Lambda^a := \left\{ \gamma \in \Lambda \,|\, E(\gamma) \leq \frac{1}{2} a^2 \right\} \, .$$

In particular the set Λ^0 of point curves which can be identified with M is the fixed point set of the S^1– resp. $\mathbb{O}(2)$–action. Λ is an infinite–dimensional Riemannian manifold modelled after the the Hilbert space $H^1(S^1, \mathbb{R}^n)$. Hence Λ is paracompact and Hausdorff, so we can apply the results of section 2. It also follows that for the free loop space Λ and for the sublevel sets Λ^a the singular and the Alexander–Čech cohomology are isomorphic, since every point has a contractible neighborhood.

Throughout this paper we will always use the field of the rationals as coefficient field for cohomology. Let $\eta \in H^2(BS^1)$ be a generator, i.e. $H^*(BS^1) = \mathbb{Q}[\eta]$. Since $\Lambda^0 \cong M$ is the fixed point set of the S^1–action, we have

$$H^*_{S^1}(\Lambda^0) \cong H^*(BS^1) \otimes H^*(M) \, .$$

Proposition 3.1 *(cf.* [5, p.105],[9, 2.4]*) Let M be simply–connected and compact, $H^*(M) \cong T_{d,m+1}(x)$, $H^*(BS^1) = \mathbb{Q}[\eta]$. Then there is an element $z \in H^{d+1}_{S^1}(\Lambda, \Lambda^0)$ of infinite order, i.e.* $\mathrm{ord}_\eta(z) = \infty$.

Proof. We consider the long exact cohomolgy sequence

$$\ldots \to H^*_{S^1}(\Lambda) \xrightarrow{i^*} H^*_{S^1}(\Lambda^0) \xrightarrow{\partial^*} H^{*+1}_{S^1}(\Lambda, \Lambda^0) \to \ldots$$

Since i^* is an $H^*(BS^1)$ map and $H^*_{S^1}(\Lambda^0)$ as a $H^*(BS^1)$–module is generated by $H^*(M) \cong T_{d,m+1}(x)$ it follows that $H^*(BS^1) \cong \mathbb{Q}[\eta]$ lies in the image of i^*.

If d is even it follows from a model for the homotopy quotient Λ_{S^1} given by Haefliger that the Poincaré polynomial of $H^*_{S^1}(\Lambda)$ is given by

$$P_{S^1}(\Lambda)[t] = \frac{1}{1-t^2} + \frac{t^{d-1}}{1-t^{d(m+1)-2}} \cdot \frac{1-t^{dm}}{1-t^d}\,,$$

see [9, thm. 2.4]. In particular $\dim H^{2k}_{S^1}(\Lambda) = 1$ for all $k \geq 1$. Hence Image $i^* = \mathbb{Q}[\eta]$ and the elements $\eta^i \otimes x^l \in H^*(BS^1) \otimes H^*(M)$ for $1 \leq l \leq m$ and $i \geq 0$ do not lie in the kernel of ∂^*. Hence in particular $\mathrm{ord}_\eta(\partial^*(x)) = \infty$.

If d is odd then M is rational homotopy equivalent to S^d. Using Morse theory for the energy functional E with the standard metric one can show [5] that the Poincaré polynomial of $H^*_{S^1}(\Lambda, \Lambda^0)$ is given by

$$P_{S^1}(\Lambda, \Lambda^0)[t] = t^{d-1} \left(\frac{1}{1-t^2} + \frac{t^{d-1}}{1-t^{d-1}} \right).$$

Using again Haefliger's model for Λ_{S^1} one can show [5, p.105]

$$P_{S^1}(\Lambda)[t] = \frac{1}{1-t^2} + \frac{t^{d-1}}{1-t^{d-1}}\,,$$

in particular $\dim H^{2k-1}(\Lambda) = 0$ for all $k \geq 1$. Hence $\partial^* : H^*_{S^1}(\Lambda^0) \to H^{*+1}_{S^1}(\Lambda, \Lambda^0)$ is injective for odd $*$, i.e. $\partial^*(\eta^i x) \neq 0$ for all $i \geq 0$. So we proved $\mathrm{ord}_\eta(\partial^*(x)) = \infty$ $\square$

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4 The mean average index of a closed geodesic

Closed geodesics on a compact Riemannian manifold M with metric g can be characterized as the critical points of positive energy of the energy functional $E : \Lambda \to \mathbb{R}$. E is a differentiable function. The metric g induces a metric g_1 on Λ. Then E satisfies the Palais–Smale condition. The canonical $\mathbb{O}(2)$–action on Λ leaves E invariant. It is an isometric action with respect to the metric g_1.

If c is a closed geodesic, then also the iterates $c^m, m \in \mathbb{N}$ are closed geodesics, here $c^m(t) := c(mt)$. If there is no curve $c_0 \in \Lambda$ with $c = c_0^m$ for some $m > 1$ we call c *prime*. If $c = c_0^m$ for a prime curve c_0 then we call $m = \mathrm{mul}(c)$ the *multiplicity* of c.

At a closed geodesic c one considers the *Hessian* (or *index form*)

$$d^2E(c)(X,Y) = \int_0^1 \{g(\nabla X, \nabla Y) - g(R(X,\dot c)\dot c, Y)\}\, dt\,.$$

Here ∇ denotes the covariant derivative along c and R is the Riemannian curvature tensor. X, Y are piecewise smooth vector fields along c, i.e. elements of the tangent space $T_c\Lambda$ of Λ at c. Let A_c be the self–adjoint operator on the tangent space $T_c\Lambda$ of Λ at c with $g_1(A_c X, Y) = d^2E(c)(X, Y)$. Since $A_c = \mathrm{id} + k_c$ where k_c is a compact operator the dimension of the subspace of $T_c\Lambda$ with negative eigenvalues is finite. This number $\mathrm{ind}(c)$ we call the *index* of the closed geodesic c. Also the dimension d_c of the kernel of A_c is finite, we call $\mathrm{null}(c) := d_c - 1$ the *nullity* of c. c is *non–degenerate* if $\mathrm{null}(c) = 0$. The nullity equals the dimension of periodic Jacobi fields along c, i.e. $\mathrm{null}(c) \leq 2n - 2$. The sequence $(\mathrm{ind}(c^m))_{m\geq 1}$ plays an important role in our further studies. Using a formula of Bott one can show

Proposition 4.1 [9] *If c is a closed geodesic on an n–dimensional Riemannian manifold then the* average index

$$\alpha_c := \lim_{m\to\infty} \frac{\mathrm{ind}(c^m)}{m}$$

exists. $\alpha_c = 0$ *iff* $\mathrm{ind}(c^m) = 0$ *for all m and the inequality*

$$|\mathrm{ind}(c^m) - m\alpha_c| \leq n - 1$$

holds for all $m \geq 1$.

Definition 4.2 *If c is a closed geodesic with average index α_c and length $L(c)$ then we call $\overline{\alpha}_c := \alpha_c / L(c)$ the* mean average index *of c.*

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Remark 4.3 One can give estimates for the index $\mathrm{ind}(c)$ of a closed geodesic c if bounds for the length $L(c)$ and for the sectional curvature K of the manifold are given. These estimates follow from the comparison theorem by Morse–Schoenberg, cf. [7, 2.6.2]. If the Ricci curvature *Ric* is positive one can give another estimate using the proof of Myers' theorem. As an immediate consequence one obtains the following estimates for the *mean average index* $\overline{\alpha}_c = \alpha_c/L(c)$ of a closed geodesic c on an n–dimensional manifold:

a) If $K \leq \Delta^2$ for some $\Delta > 0$ then $\overline{\alpha}_c \leq \Delta(n-1)/\pi$.
b) If $K \geq \delta^2$ for some $\delta > 0$ then $\overline{\alpha}_c \geq \delta(n-1)/\pi$.
c) If $Ric \geq \delta^2(n-1)$ for some $\delta > 0$ then $\overline{\alpha}_c \geq \delta/\pi$.

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5 The global index interval of a cohomology class and the local homology of critical sets

Let (M, g) be a simply–connected compact Riemannian manifold and let $z \in H^*_{S^1}(\Lambda, \Lambda^0)$ be a class with $\mathrm{ord}_\eta z = \infty$. Here $H^*(BS^1) = \mathbb{Q}[\eta]$. We denote by $j_a : (\Lambda^a, \Lambda^0) \to (\Lambda, \Lambda^0)$ the inclusion and define the function $d_z : \mathbb{R}^{\geq 0} \to \mathbb{N} \cup \{\infty\}$:

$$d_z(a) := \mathrm{ord}_\eta(j_a^*(z)) .$$

Definition 5.1 *For $z \in H^*_{S^1}(\Lambda, \Lambda^0)$ with $\mathrm{ord}_\eta z = \infty$ the* upper *(resp.* lower*) index $\underline{\sigma}_z$ (resp. $\overline{\sigma}_z$) is given by:*

$$\underline{\sigma}_z := \liminf_{a\to\infty} \frac{d_z(a)}{a} \left(\mathit{resp.} \overline{\sigma}_z := \limsup_{a\to\infty} \frac{d_z(a)}{a} \right) .$$

We call $[\underline{\sigma}_z, \overline{\sigma}_z]$ the global index interval *of z.*

æ Another consequence of the Morse–Schoenberg comparison theorem resp. the proof of Myers' theorem are the following estimates for the global index interval in terms of the curvature, cf. remark 4.3:

Proposition 5.2 *Let $z \in H^*_{S^1}(\Lambda, \Lambda^0)$ be a class with $\mathrm{ord}_\eta(z) = \infty$.*
a) If $K \leq \Delta^2$ then $\overline{\sigma}_z \leq \Delta(n-1)/(2\pi)$.
b) If $K \geq \delta^2$ then $\overline{\sigma}_z \geq \delta(n-1)/(2\pi)$.
c) If $Ric \geq \delta^2(n-1)$ then $\underline{\sigma}_z \geq \delta/(2\pi)$.

Proof. We give the proof of the first estimate, the other proofs are analogous. We can choose a sequence (g_i) of bumpy metrics converging in the strong C^2–topology to g due to the bumpy metrics theorem. Here a metric is *bumpy* if all closed geodesics are non–degenerate. Let $E_i(\gamma) = 1/2 \int_0^1 g_i(\dot\gamma, \dot\gamma)$ be the energy functional and ${\Lambda^b}_i := E_i^{-1}[0, b^2/2]$.

Choose a sequence $(a_i) \subset \mathbb{R}$ with $a = \lim_{i\to\infty} a_i$ and $\Lambda^a \subset \Lambda_i^{a_i}$ for all i. We can choose a sequence $(\Delta_i) \subset \mathbb{R}$ such that the sectional curvature of g_i is bounded from above by Δ_i^2 and $\Delta = \lim_{i\to\infty} \Delta_i$. It follows from the Morse–Schoenberg comparison theorem that the index $\mathrm{ind}(c)$ of a closed geodesic c of the metric g_i with length $\leq a_i$ satisfies

$$\mathrm{ind}(c) \leq \left(\frac{a_i}{\pi}\Delta_i + 1\right)(n-1) =: k_i .$$

Hence it follows from the Morse–lemma [6, ch.2.4] that $H^k_{S^1}(\Lambda_i^{a_i}, \Lambda^0) = 0$ for all $k > k_i$. Since $\Lambda^a \subset \Lambda_i^{a_i}$ it follows from the composition

$$H^k_{S^1}(\Lambda, \Lambda^0) \to H^k_{S^1}(\Lambda_i^{a_i}, \Lambda^0) \to H^k_{S^1}(\Lambda^a, \Lambda^0)$$

of restriction homomorphisms that $\eta^k \cdot j_a^*(z) = 0$ for all k with $k > (k_i - \dim z)/2$. Hence $d_z(a) \leq (\lim_{i\to\infty} k_i - \dim z)/2$ □

Example 5.3 Let $(M_{d,m}, g_S)$ be a simply connected compact rank one symmetric space with the following normalization. In section 3 we introduced the notation $M_{d,m}$. For $M_{d,1} = S^d$ g_S is the metric of constant sectional curvature $K \equiv 1$. For $M_{2,m} = \mathbb{C}P^m$, $M_{4,m} = \mathbb{H}P^m$ or $M_{8,2} = \mathbb{C}aP^2$ g_S is the metric with sectional curvature in the interval $[1/4, 1]$.

It follows from proposition 3.1 that there is a class $z \in H^*_{S^1}(\Lambda, \Lambda^0)$ with $\mathrm{ord}_\eta(z) = \infty$. The critical set Cr of Λ is the disjoint union $\bigcup_{j\geq 1} B_j$, where $B_j := \{c^j \mid c \in B\}$ and B is the submanifold of great circles in Λ, i.e. the set of prime closed geodesics. If $c \in B$ then $L(c) = 2\pi$, $\mathrm{ind}(c^j) = j \cdot \alpha_c + 1 - md$ with $\alpha_c = d(m+1) - 2$ and $\dim B/S^1 = 2md - 2$. It follows that

$$\underline{\sigma}_z = \overline{\sigma}_z = \frac{1}{2}\overline{\alpha} = \frac{d(m+1)-2}{4\pi}\,.$$

Remark 5.4 Let g, g^* be two metrics on a compact manifold with

$$\frac{g^*(X,Y)}{D^2} \leq g(X,Y) \leq D^2 g^*(X,Y)$$

for all tangent vectors X, Y and some $D > 1$. Let $[\underline{\sigma}_z, \overline{\sigma}_z]$, $[\underline{\sigma}^*_z, \overline{\sigma}^*_z]$ be the global index intervals of the cohomology class z with respect to the metrics g, g^*. Then one can show that $D^{-1}\underline{\sigma}_z \leq \underline{\sigma}^*_z \leq D\underline{\sigma}_z$ resp. $D^{-1}\overline{\sigma}_z \leq \overline{\sigma}^*_z \leq D\overline{\sigma}_z$. This shows that the functions $\underline{\sigma}_z$, $\overline{\sigma}_z : \mathcal{G}^0 \to \mathbb{R}$ are continuous on the space $\mathcal{G}^0$ of metrics with the strong C^0–topology. It also shows that if $\underline{\sigma}_z > 0$ for some metric, then $\underline{\sigma}_z$ is positive for all metrics on M. It follows from remark 4.3 c) that $\underline{\sigma}_z > 0$ if there is a metric with positive Ricci curvature on M. Another consequence is the following observation: If there is a metric g with $\underline{\sigma}_z < \overline{\sigma}_z$ then there is a neighborhood of g in $\mathcal{G}^0$ with $\underline{\sigma}_z < \overline{\sigma}_z$.

æ We study the discontinuity points of the function $d_z(a)$ in detail.

Lemma 5.5 *a) The function* $d_z : \mathbb{R}^{\geq 0} \to \mathbb{N}_0$ *is non–decreasing.*

b) $\lim_{a \searrow b} d_z(a) = d_z(b)$.

Proof. a) $d_z(a)$ is finite by the Morse–Schoenberg comparison theorem. The monotony of $d_z(a)$ follows from the definition.

b) Let V be a closed $\mathbb{O}(2)$–invariant neighborhood of Λ^b with $d_z(b) = \mathrm{ord}_\eta(j^*_b(z)) = \mathrm{ord}_\eta(j^*_V(z))$ where $j_b : (\Lambda^b, \Lambda^0) \to (\Lambda, \Lambda^0)$ resp. $j_V : (V, \Lambda^0) \to (\Lambda, \Lambda^0)$ are the inclusions, cf. proposition 2.2. Then there exist an $\epsilon > 0$ and $\bar{\epsilon} > 0$ such that for all $\gamma \in \Lambda^{b+\epsilon} - (\Lambda^{b-\epsilon} \cup V)$ the gradient grad$E$ of the energy functional E satisfies $\|\mathrm{grad}E(\gamma)\|_1 \geq \bar{\epsilon}$, cf. [6, 1.4.8]. Hence there is a $T > 0$ such that $\Phi_T(\Lambda^{b+\epsilon}) \subset V \cup \Lambda^{b-\epsilon}$ where $\Phi_t : \Lambda \to \Lambda$ is the flow of the vector field $-\mathrm{grad}E$ on the free loop space Λ. From the properties of ord_η it follows that

$$d_z(b) = \mathrm{ord}_\eta(j^*_b(z)) = \mathrm{ord}_\eta(j^*_V(z)) \geq \mathrm{ord}_\eta(j^*_{b+\epsilon}(z)) = d_z(b+\epsilon)\,,$$

cf. proposition 2.2, hence $d_z(b) = d_z(b+\epsilon)$ □

Let Cr be the critical set of E with positive energy, i.e the set of closed geodesics. For $a > 0$ let $\mathrm{Cr}(a) := \mathrm{Cr} \cap E^{-1}(a^2/2) = \mathrm{Cr} \cap L^{-1}(a)$ i.e. $\mathrm{Cr}(a)$ is the set of closed geodesics of length a. We use the notation

$$d_z(a-) := \lim_{\epsilon \searrow 0} d_z(a-\epsilon)\,,$$

hence at discontinuity point a of d_z we have $d_z(a) - d_z(a-) \geq 1$. Since $d_z(a) < \infty$ for all a, there are only finitely many discontinuity points in a given interval $[0, a]$.

Lemma 5.6 *For $a > 0$ we have:* $\mathrm{index}_\eta \mathrm{Cr}(a) \geq d_z(a) - d_z(a-)$.

Proof. Assume $d_z(a) - d_z(a-) \geq 1$, let V be a closed invariant neighborhood of $\mathrm{Cr}(a)$ with $\mathrm{index}_\eta(V) = \mathrm{index}_\eta \mathrm{Cr}(a)$ which exists by proposition 2.1 e). Then there exist an $\epsilon > 0$ such that $\Lambda^{a-\epsilon} \cup V$ is a S^1–deformation retract of $\Lambda^{a+\epsilon}$, i.e.

$$H^*_{S^1}(V \cup \Lambda^{a-\epsilon}, \Lambda^0) \cong H^*_{S^1}(\Lambda^{a+\epsilon}, \Lambda^0)\,.$$

and $d_z(a) = d_z(a+\epsilon)$, $d_z(a-) = d_z(a-\epsilon)$. Let $j_1 : (V \cup \Lambda^{a-\epsilon}, \Lambda^0) \to (\Lambda, \Lambda^0)$ be the inclusion. Then it follows from proposition 2.2 that

$$\begin{aligned} d_z(a) &= d_z(a+\epsilon) = \mathrm{ord}_\eta(j_1^*(z)) \leq \mathrm{ord}_\eta(j^*_{a-\epsilon}(z)) + \mathrm{index}_\eta(V) \\ &= d_z(a-) + \mathrm{index}_\eta \mathrm{Cr}(a) \end{aligned}$$

□

Corollary 5.7 *For $a > 0$ we have for the covering dimension of the set $\mathrm{Cr}(a)$ of closed geodesics of length a the following estimate:*

$$\dim \mathrm{Cr}(a) \geq 2\,(d_z(a) - d_z(a-) - 1)\,.$$

In particular, if $d_z(a) - d_z(a-) \geq 2$ then there are infinitely many distinct $\mathrm{O}(2)$–orbits $\mathrm{O}(2) \cdot c$ of closed geodesics c with length a.

Proof. Let $\rho > 0$ be the injectivity radius of M, hence we have for a closed geodesic c : $L(c) \geq 2\rho$. Hence if $L(c) = a$ then $\mathrm{mul}(c) \leq a/(2\rho)$, so we can apply proposition 2.1 f). Since a finite set has dimension 0, it follows from $d_z(a) - d_z(a-) \geq 2$ that there are infinitely many $\mathrm{O}(2)$–orbits of closed geodesics in $\mathrm{Cr}(a)$ □

Lemma 5.8 *Let $k := d_z(a) - d_z(a-) \geq 1$.*

a) If all closed geodesics of length a are non–degenerate, then $k = 1$ and there is a closed geodesic $c \in \mathrm{Cr}(a)$ with $\mathrm{ind}(c) = 2d_z(a) + \dim z - 2$.

b) If all closed geodesics of length a are isolated (i.e. their S^1–orbits are isolated in Λ) then $k = 1$ and there is a closed geodesic c with

$$0 \leq 2d_z(a) + \dim(z) - 2 - \mathrm{ind}(c) \leq 2n - 2\,.$$

Here n is the dimension of the manifold.

Proof. Non–degenerate closed geodesics are isolated, hence in a) and b) we have $\dim \mathrm{Cr}(a) = 0$, hence $k = 1$ by corollary 5.7.

Let $\epsilon > 0$ be sufficiently small such that a is the only critical value of E in $[a-\epsilon, a+\epsilon]$. Hence $l := d_z(a-\epsilon) = d_z(a-) = d_z(a) - 1 = d_z(a+\epsilon) - 1$. It follows that $\eta^{l-1} j^*_{a+\epsilon}(z) \neq 0$ but $\eta^{l-1} j^*_{a-\epsilon}(z) = 0$. Here $j_b : (\Lambda^b, \Lambda^0) \rightarrow (\Lambda, \Lambda^0)$ is the inclusion. Therefore it follows from the exactness of

$$H^*_{S^1}(\Lambda^{a+\epsilon}, \Lambda^{a-\epsilon}) \rightarrow H^*_{S^1}(\Lambda^{a+\epsilon}, \Lambda^0) \rightarrow H^*_{S^1}(\Lambda^{a-\epsilon}, \Lambda^0)$$

that $H^q_{S^1}(\Lambda^{a+\epsilon}, \Lambda^{a-\epsilon}) \neq 0$ for $q := 2(l-1) + \dim z = 2d_z(a) - 2 + \dim z$.

a) If there is no closed geodesic $c \in \mathrm{Cr}(a)$ with $\mathrm{ind}(c) = q$ then $H^q_{S^1}(\Lambda^{a+\epsilon}, \Lambda^{a-\epsilon}) = 0$ by the Morse inequalities [6, ch.2.4].

b) If there is no closed geodesic $c \in \mathrm{Cr}(a)$ with $0 \leq q - \mathrm{ind}(c) \leq 2n-2$ then $H^q_{S^1}(\Lambda^{a+\epsilon}, \Lambda^{a-\epsilon}) = 0$ follows from the *generalized Morse lemma*, cf. [6, 4.2] □

Now we are going to study the case that $\mathrm{Cr}(a)$ contains orbits which are not isolated in $\mathrm{Cr}(a)$.

Theorem 5.9 *If $d = d_z(a-)$ and $d_z(a) = d+k$, $k \geq 1$ then there is a closed geodesic c with length a and index* $\mathrm{ind}(c)$ *satisfying*

$$0 \leq q - \mathrm{ind}(c) \leq 2n - 2$$

with $q := 2d - 2 + \dim z$.

If $k \geq 2$ then c can be chosen such that the orbit $\mathbb{O}(2) \cdot c$ is not isolated in $\mathrm{Cr}(a)$.

Proof. By corollary 5.7 the subset C_1 of critical orbits in $\mathrm{Cr}(a)$ which are not isolated in $\mathrm{Cr}(a)$ is non–empty if $k \geq 2$. C_1 is a closed subset of Λ. Choose a closed invariant neighborhood V_1 of C_1. Then $C_2 := \mathrm{Cr}(a) - V_1$ contains only finitely many orbits since $\mathrm{Cr}(a)$ is compact. Choose pairwise distinct closed invariant neighborhoods of these orbits as in proposition 2.1 d) which do not intersect V_1. We denote the union of these neighborhoods by V_2. Then $V_0 = V_1 \cup V_2$ is a closed invariant neighborhood of $\mathrm{Cr}(a)$ and $\mathrm{index}_\eta V_2 = 1$.

There is a sufficiently small $\epsilon > 0$ such that the following holds: If

$$j_0 : (V_0 \cup \Lambda^{a-\epsilon}, \Lambda^0) \longrightarrow (\Lambda, \Lambda^0)$$

is the inclusion, let $h_0' := j_0^*(\eta^{d-1} \cdot z) \in H^q_{S^1}(V_0 \cup \Lambda^{a-\epsilon}, \Lambda^0)$. Then $\eta \cdot h_0' \neq 0$ since $k \geq 1$. Since $d = d_z(a-)$, $d_z(a) = d + k \geq d + 1$ there is a class h_0 with $j_1^* h_0 = h_0'$. Here

$$j_1 : (V_0 \cup \Lambda^{a-\epsilon}, \Lambda^0) \longrightarrow (V_0 \cup \Lambda^{a-\epsilon}, \Lambda^{a-\epsilon}) .$$

Since $V_0 = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and V_1, V_2 are closed the homomorphism

$$j_2^* : H^q_{S^1}(V_0 \cup \Lambda^{a-\epsilon}, \Lambda^{a-\epsilon}) \to H^q_{S^1}(V_1 \cup \Lambda^{a-\epsilon}, \Lambda^{a-\epsilon}) \oplus H^q_{S^1}(V_2 \cup \Lambda^{a-\epsilon}, \Lambda^{a-\epsilon})$$

is an isomorphism. We denote by p_1 resp. p_2 the projection onto the first resp. second factor. Let $h_1 := p_1(h_0)$ and $h_2 := p_2(h_0)$. If $k \geq 2$ then $h_1 \neq 0$ since $\mathrm{index}_\eta V_2 = \mathrm{index}_\eta C_2$.

Due to the bumpy metrics theorem we can choose a sequence g_i of bumpy Riemannian metrics converging to the given metric g. Let E_i be the corresponding energy functionals and let $\Lambda_i^b := E_i^{-1}[0, b^2/2]$. If $k \geq 2$ we set $V := V_1$, $h := h_1$ otherwise $V := V_0$, $h := h_0$. Let $j_b : \left((\Lambda_i^b \cap V) \cup \Lambda^{a-\epsilon}, \Lambda^{a-\epsilon}\right) \to \left(V \cup \Lambda^{a-\epsilon}, \Lambda^{a-\epsilon}\right)$ be the inclusion and let $a_i := \sup\{b > 0 \,|\, j_b^*(h) \neq 0\}$. Then $a = \lim_{i\to\infty} a_i$ and for sufficiently large i the set $V \cap \mathrm{Cr}_i(a_i)$ of closed geodesics in V of length a_i of the metric g_i lies in the interior of V. Let $\Lambda_i^{b-} := E_i^{-1}[0, b^2/2)$. Then it follows from the definition of a_i that

$$H^q_{S^1}(\Lambda_i^{a_i} \cap V, \Lambda_i^{a_i -} \cap V) \neq 0\,.$$

Hence there is a sequence (c_i) of closed geodesics in V of the metric g_i with length a_i and index q, cf. lemma 5.8. A subsequence (c_i) converges to a closed geodesic c in V of the metric g of length a and with index:

$$q - (2n-2) \leq \mathrm{ind}(c) \leq q\,.$$

Hence we are done if $k = 1$.

If $k \geq 2$ then $V = V_2$. Now the orbit $\mathbb{O}(2) \cdot c$ of c could be isolated in $\mathrm{Cr}(a)$. But by choosing V_1 as small as we want we obtain a closed geodesic $\tilde{c}$ of the metric g with a non–isolated orbit in $\mathrm{Cr}(a)$ and with $q - (2n-2) \leq \mathrm{ind}(c) \leq q$ $\square$

Now we can argue as in [3].

The *discontinuity sequence*$(a_k)_{k\geq 0}$ of d_z is the non–decreasing sequence of discontinuity points a repeated according to the *jump* $d_z(a) - d_z(a-)$. We denote by $\mathcal{P} := \{\mathbb{O}(2) \cdot c \,|\, c \text{ prime closed geodesic}\}$ the set of critical orbits of prime closed geodesics.

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Definition 5.10 An orbit $\mathbb{O}(2) \cdot c \in \mathcal{P}$ is called *k–essential* if there is an $l \in \mathbb{N}$ with $L(c^l) = lL(c) = a_k$ and if

$$\left|\,\mathrm{ind}(c^l) - (\dim z + 2k)\,\right| \leq 2n - 2\,.$$

Theorem 5.11 *There is a sequence $(\mathbb{O}(2)\cdot c_k)_{k\geq 1} \subset \mathcal{P}$ of k–essential closed geodesics. If $a_j = a_{j+1} = \ldots = a_{j+k}$ for some $k \geq 1$ then the orbits $\mathbb{O}(2) \cdot c_j, \ldots, \mathbb{O}(2) \cdot c_{j+k}$ can be chosen pairwise distinct.*

Proof. By theorem 5.9 we find for all j an orbit $\mathbb{O}(2) \cdot c_j \in \mathcal{P}$ and $l \in \mathbb{N}$ with $L(c_j^l) = a_j$ and $0 \leq \dim z + 2j - \mathrm{ind}(c) \leq 2n - 2$. If $a_j = a_{j+1} = \ldots = a_{j+k}$ with $k \geq 1$ then by theorem 5.9 we find a critical orbit $\mathbb{O}(2) \cdot \tilde{c}_j \in \mathcal{P}$ and $l \in \mathbb{N}$ whose critical orbit $\mathbb{O}(2) \cdot \tilde{c}_j^l$ is not isolated in $\mathrm{Cr}(a_j)$ and $0 \leq \dim z + 2j - \mathrm{ind}(c_j^l) \leq 2n-2$. Hence there are in particular orbits $\mathbb{O}(2) \cdot c_j, \ldots, \mathbb{O}(2) \cdot c_{j+k} \in \mathcal{P}$ and numbers $l_j, \ldots, l_{j+k} \in \mathbb{N}$ such that $\mathbb{O}(2) \cdot c_{j+1}^{l_{j+1}}, \ldots, \mathbb{O}(2) \cdot c_{j+k}^{l_{j+k}}$ lie in an arbitrarily small neighborhood of $\mathbb{O}(2) \cdot c_j^{l_j}$ in $\mathrm{Cr}(a_j)$ and such that $\mathbb{O}(2) \cdot c_j, \ldots, \mathbb{O}(2) \cdot c_{j+k}$ are pairwise distinct. For a sufficiently small neighborhood the assumptions of definition 5.10 can be satisfied □

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Lemma 5.12 *If $\mathbb{O}(2) \cdot c \in \mathcal{P}$ is k–essential then we have the following estimate for the mean average index $\overline{\alpha}_c$:*

$$\left|\overline{\alpha}_c - \frac{2k}{a_k}\right| \leq \frac{3n - 3 + \dim z}{a_k}\,.$$

Proof. By definition 5.10 there is an $l \geq 1$ with $L(c) = a_k/l$ and $|\mathrm{ind}(c^l) - 2k| \leq 2n - 2 + \dim z$. Since $|\mathrm{ind}(c^l) - l\alpha_c| \leq n - 1$, cf. proposition 4.1 the claim follows □

Lemma 5.13 *For all $a > 0$ we have the following upper bound for the jump of d_z:*

$$d_z(a) - d_z(a-) \leq n\,.$$

Proof. Assume that there is an $a \in (0, \infty)$ with $d_z(a) - d_z(a-) > n$. Then $\mathrm{index}_\eta \mathrm{Cr}(a) \geq n + 1$. Since $2\,(\mathrm{index}_\eta(\mathrm{Cr}(a) - 1) \leq \dim \mathrm{Cr}(a)/S^1$ by proposition 2.1 f) it follows that $\dim \mathrm{Cr}(a)/S^1 \geq 2n$. But we can consider $\mathrm{Cr}(a)$ as a closed subset of the compact $(2n-1)$–dimensional unit tangent bundle, hence $\dim \mathrm{Cr}(a) \leq 2n - 2$ □

6 The mean average index and the global index interval

We have the following relation between the discontinuity sequence $(a_k)_k$ of $d_z(a)$ and the global indices of z:

Lemma 6.1 $\underline{\sigma}_z = \liminf_{k\to\infty} \frac{k}{a_k}$, $\overline{\sigma}_z = \limsup_{k\to\infty} \frac{k}{a_k}$.

Proof. Let $a_k \leq a \leq a_{k+1}$ then by lemma 5.13

$$\frac{k + 1 - n}{a_{k+1}} = \frac{d_z(a_{k+1}) - n}{a_{k+1}} \leq \frac{d_z(a)}{a} \leq \frac{d_z(a_k) + n}{a_k} = \frac{k + n}{a_k}\,.$$

From this estimate the claim follows □

Theorem 6.2 *If $t \in [\underline{\sigma}_z, \overline{\sigma}_z]$ then there is a sequence $\mathbb{O}(2) \cdot c_k \in \mathcal{P}$ with $t = 1/2 \lim_{k\to\infty} \overline{\alpha}_{c_k}$.*

Proof. (cf.[3, thm.2 ii)]) If $\underline{\sigma}_z = \overline{\sigma}_z$ it follows from lemma 6.1 that $t = \lim_{k\to\infty} k/a_k$. From lemma 5.12 the claim follows.

So assume $\underline{\sigma}_z < \overline{\sigma}_z$. We assume that t is not in the closure of the set $A := \{\overline{\alpha}_{c_k}/2 \,|\, k \in \mathbb{N}\}$. Then there is $\epsilon > 0$ such that $|\overline{\alpha}_{c_k}/2 - t| \geq \epsilon$ for all $k > 0$. Choose $j \geq 1$ with $j/a_j \leq t - \epsilon\, ;\, a_j > \epsilon^{-1}$. Then it follows from

$$\frac{k+1}{a_{k+1}} - \frac{k}{a_k} \leq \frac{1}{a_{k+1}}$$

that $k/a_k \leq t - \epsilon$ for all $k \geq j$. This contradicts the definition of $\overline{\sigma}_z$ □

Corollary 6.3 *If there are only finitely many geometrically distinct closed geodesics (i.e. $\#\mathcal{P} < \infty$) then $\underline{\sigma}_z = \overline{\sigma}_z$.*

Theorem 6.4 *Let M be a simply–connected compact Riemannian manifold with a cohomology class $z \in H^*_{S^1}(\Lambda, \Lambda^0)$ of infinite order and let $\underline{\sigma}_z > 0$. We define for $\epsilon > 0$: $\mathcal{P}_\epsilon := \{\mathbb{O}(2) \cdot c \in \mathcal{P} \,|\, \overline{\alpha}_c \in (2\underline{\sigma}_z - \epsilon, 2\overline{\sigma}_z + \epsilon)\}$. Then for every $\epsilon > 0$:*

$$\sum_{\mathbb{O}(2)\cdot c \in \mathcal{P}_\epsilon} \frac{1}{\alpha_c} \geq \frac{1}{2}\,.$$

Proof.(cf.[3, thm.2 ii)]) By theorem 5.11 there is a sequence $\mathbb{O}(2) \cdot c_k \in \mathcal{P}$ of k–essential closed geodesics. If $a_k = a_{k+1} = \ldots = a_{k+j}$ then $c_k, c_{k+1}, \ldots, c_{k+j}$ are geometrically distinct. By lemma 5.12 there is a $k_0 = k_0(\epsilon) > 0$ such that for $k \geq k_0$: $|\overline{\alpha}_k - (2k/a_k)| < \epsilon/2$ with $\overline{\alpha}_k := \overline{\alpha}_{c_k}$. Hence

$$k - k_0 \leq \sum_{\mathbb{O}(2)\cdot c \in \mathcal{P}_\epsilon} \frac{a_k}{L(c)}$$

resp.

$$1 - \frac{k_0}{k} \leq \sum_{\mathbb{O}(2)\cdot c \in \mathcal{P}_\epsilon} \frac{a_k}{k} \frac{1}{L(c)}\,.$$

Choose $\delta \in (0, \overline{\sigma}_z)$ and let k_l be a monotonic sequence with $k_l/a_{k_l} \geq \overline{\sigma}_z - \delta$, then

$$1 - \frac{k_0}{k_l} \leq \sum_{\mathbb{O}(2)\cdot c \in \mathcal{P}_\epsilon} \frac{1}{L(c)} \frac{1}{\overline{\sigma}_z - \delta}\,.$$

For $l \to \infty$ it follows that

$$1 \leq \frac{1}{\overline{\sigma}_z} \sum_{\mathbb{O}(2)\cdot c \in \mathcal{P}_\epsilon} \frac{1}{L(c)}\,,$$

since $\delta > 0$ is arbitrary. For $c \in \mathcal{P}_\epsilon$ we have $\overline{\alpha}_c \leq 2\overline{\sigma}_z + \epsilon$, hence

$$\frac{1}{2} \leq \sum_{\mathbb{O}(2)\cdot c \in \mathcal{P}_\epsilon} \frac{1}{L(c)} \frac{1}{\overline{\alpha}_c - \epsilon}\,.$$

Since $\overline{\alpha}_c - \epsilon \geq (1 - \frac{\epsilon}{2\underline{\sigma}_z - \epsilon})\overline{\alpha}_c$ we obtain

$$\frac{1}{2}\left(1 - \frac{\epsilon}{2\underline{\sigma}_z - \epsilon}\right) \leq \sum_{\mathbb{O}(2)\cdot c \in \mathcal{P}_\epsilon} \frac{1}{\alpha_c}\,.$$

Choose $\epsilon_1 \in (0, \epsilon)$ then

$$\frac{1}{2}\left(1 - \frac{\epsilon_1}{2\underline{\sigma}_z - \epsilon_1}\right) \leq \sum_{\mathbb{O}(2)\cdot c \in \mathcal{P}_{\epsilon_1}} \frac{1}{\alpha_c} \leq \sum_{\mathbb{O}(2)\cdot c \in \mathcal{P}_\epsilon} \frac{1}{\alpha_c}\,.$$

For $\epsilon_1 \to 0$ the claim follows $\square$

Corollary 6.5 *If there are only finitely many geometrically distinct closed geodesics (i.e. $\mathcal{P} < \infty$) then: $\underline{\sigma}_z = \overline{\sigma}_z =: \sigma$, cf. corollary 5.7. Let $\mathcal{P}_\sigma := \{\mathbb{O}(2) \cdot c \in \mathcal{P} \,|\, \overline{\alpha}_c = 2\sigma\}$, then*

$$\sum_{\mathbb{O}(2)\cdot c \in \mathcal{P}_\sigma} \frac{1}{\alpha_c} \geq \frac{1}{2}\,.$$

In particular, if for every c we know $\alpha_c > 2$ then there are two geometrically distinct closed geodesics c_1, c_2 which satisfy the *resonance relation* $\overline{\alpha}_1 = \overline{\alpha}_2$.

From the estimates in remark 4.3 and from Klingenberg's injectivity radius estimate [7, ch.2.6] one can give a sufficient pinching condition for the existence of at least two distinct closed geodesics c_1, c_2 which are almost *resonant*. If there are only finitely many closed geodesics on M then they are resonant.

Proposition 6.6 *Let M be a n–dimensional compact simply–connected manifold with a cohomology class $z \in H^*_{S^1}(\Lambda, \Lambda^0)$ of infinite order. Let M carry a Riemannian metric whose sectional curvature K satisfies $k^2/(n-1)^2 \leq K \leq 1$ for some $k = 1, \ldots, n-2$ and let either n be even or $k \geq (n-1)/2$.*

Then for every $\epsilon > 0$ there are $k+1$ closed geodesics $c_1, \ldots, c_{k+1}$ with $|\overline{\alpha}_i - \overline{\alpha}_j| < \epsilon$ for all pairs $i, j \in \{1, \ldots, k+1\}$.

In particular if there are only finitely many closed geodesics on M, then there are $k+1$ geometrically distinct closed geodesics $c_1, \ldots, c_{k+1}$ with $\overline{\alpha}_1 = \ldots = \overline{\alpha}_{k+1} = 2\sigma$.

The theorem also holds for *Finsler metrics*, cf. [12] [9]. If the Finsler metric is non–symmetric one has to replace $\mathbb{O}(2)$ in the definition of $\mathcal{P}_\epsilon$

by S^1, but everything else remains the same. An analogous statememt as proposition 6.6 also holds for Finsler metrics with the following changes: the sectional curvature is the sectional curvature of an *osculating metric* along a geodesic. One has also to add the assumption that the length of a closed geodesic is bounded from below by 2π.

Example 6.7 In every C^∞–neighborhood of the standard metric on the sphere S^n there is a non–symmetric Finsler metric F with only finitely many geometrically distinct closed geodesics. If n is even then there are exactly n closed geodesics, if n is odd then there are $n+1$. This examples were first discovered by Katok, their geometry was investigated by Ziller in [12].

Hence for every $\epsilon > 0$ there is a Finsler metric C^∞–near the standard metric such that the average index of a closed geodesic is bounded from below by $2n-2-\epsilon$, cf. example 5.3. Then by the theorem there are for $\epsilon < 1$ at least $n-1$ geometrically distinct closed geodesics $c_1, \ldots, c_{n-1}$ whose mean average indices satisfy the resonance relations $\overline{\alpha}_1 = \ldots = \overline{\alpha}_{n-1} = \sigma$. By choosing F sufficiently near the standard metric we can get σ arbitrarily close to $(2n-2)/(4\pi)$, cf. example 5.3 and remark 5.4.

Remark 6.8 Let M be a compact manifold. We define the following subset $\tilde{\mathcal{G}}$ of the set $\mathcal{G}$ of Riemannian metrics with the strong C^2–topology:

$$\tilde{\mathcal{G}} := \{g \in \mathcal{G} \,|\, \Phi(2) \cdot c_1 \,,\, \Phi(2) \cdot c_2 \in \mathcal{P}, \overline{\alpha}_1 = \overline{\alpha}_2 \Rightarrow \Phi(2) \cdot c_1 = \Phi(2) \cdot c_2\}\,.$$

Then one can show that $\tilde{\mathcal{G}}$ is a residual subset in $\mathcal{G}$ using a local pertubation argument due to Klingenberg–Takens [8] as in [10].

Then it follows from proposition 6.6 and from corollary 6.3 that a C^2–generic metric on an even–dimensional simply–connected compact manifold and with sectional curvature $(n-1)^{-1} < K \leq 1$ there are infinitely many geometrically distinct closed geodesics.

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