Contents

1 Potential theory  .................................................. 7
  1.1 Preliminaries .................................................. 9
  1.2 Dipole potential .............................................. 11
  1.3 Single layer potential ......................................... 20
  1.4 Integral equations ............................................ 26
  1.5 Volume potential .............................................. 37
  1.6 Exercises ..................................................... 42

2 The Perron method .................................................. 47
  2.1 A maximum principle .......................................... 47
  2.2 Subharmonic, superharmonic functions ........................ 50
  2.3 Boundary behavior ............................................ 56
    2.3.1 Examples for local barriers ............................... 59
  2.4 Generalizations .............................................. 63
  2.5 Exercises ..................................................... 64

3 Maximum principles .................................................. 65
  3.1 Basic maximum principles .................................... 65
    3.1.1 Directional derivative boundary value problem .......... 70
    3.1.2 Behavior near a corner ................................... 71
    3.1.3 An a priori estimate ..................................... 74
  3.2 A discrete maximum principle ................................ 76
  3.3 Exercises ..................................................... 81
Preface

These notes are a corrected version with minor changes of the version from December 2013. In the following we present an introduction to linear second order elliptic partial differential equations. It can be considered as a continuation of a chapter on elliptic equations of the lecture notes [17] on partial differential equations. In [17] we focused our attention mainly on explicit solutions for standard problems for elliptic, parabolic and hyperbolic equations.

The first chapter concerns integral equation methods for boundary value problems of the Laplace equation. This method can be extended to a large class of linear elliptic equations and systems. In the following chapter we consider the Perron method for the Dirichlet problem for the Laplace equation. This method is based on the maximum principle and on estimates of derivatives of solutions of the Laplace equation.

For additional reading we recommend following books: W. I. Smirnov [22], I. G. Petrowski [21], D. Gilbarg and N. S. Trudinger [10], S. G. Michlin [14], P. R. Garabedian [9], W. A. Strauss [23], F. John [13], L. C. Evans [5] and R. Courant and D. Hilbert [4]. Some material of these lecture notes was taken from some of these books.
Chapter 1

Potential theory

The notation *potential* has its origin in the Newton attraction rule

\[ K(x, y) = -G \frac{Mm}{|y - x|} \frac{y - x}{|y - x|}, \]

where \( G = 6.67 \cdot 10^{-11} \text{ m}^3/(\text{kg} \cdot \text{s}^2) \), and \( K \) is the force acting between two mass points \( M \) and \( m \) located at \( x, y \in \mathbb{R}^3 \), respectively. Since \( \text{rot} \ K = 0 \), there is a scalar function \( Q(x, y) \), called *potential*, such that \( \nabla_x Q(x, y) = K(x, y) \). Thus, \( Q(x, y) = -GMm|y - x|^{-1} \) is a Newton potential. The function \( Q(x, y) \) defines the work which has to be done to move one of the mass points to infinity if the other one is fixed.

Let \( \Omega \subset \mathbb{R}^n \) be a bounded, connected and sufficiently regular domain. Consider for given \( f \) and \( h \) the boundary value problem

\[ -\Delta v = f \text{ in } \Omega, \]
\[ v = h \text{ on } \partial \Omega. \]

We can transform this problem into a boundary value problem for the Laplace equation by setting \( v = u - w \), where

\[ w(x) = \int_{\Omega} s(|x - y|) f(y) \, dy. \]

Here \( s(r) \) denotes the *singularity function*, see also [17],

\[ s(r) := \begin{cases} \frac{1}{2\pi} \ln r & : n = 2, \\ \frac{r^{n-2}}{(n-2)\omega_n} & : n \geq 3. \end{cases} \]

We recall that \( \omega_n = |\partial B_1(0)| \). Since \( w \in C^2(\Omega) \) and \( -\Delta w = f \text{ in } \Omega \) if \( f \) is sufficiently regular, see Sec. 7.5 in [17], we arrive at the problem \( \Delta u = 0 \)
in $\Omega$ and $v = h - w$ on $\partial\Omega$. Consequently, it is sufficient to consider the boundary value problem for the Laplace equation, which is a problem with a homogeneous differential equation.

The *Dirichlet problem* (first boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ of

\begin{align*}
\triangle u &= 0 \text{ in } \Omega, \quad (1.1) \\
u &= \Phi \text{ on } \partial\Omega, \quad (1.2)
\end{align*}

where $\Phi$ is given and continuous on $\partial\Omega$.

The *Neumann problem* (second boundary value problem) is to find a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of

\begin{align*}
\triangle u &= 0 \text{ in } \Omega, \quad (1.3) \\
\frac{\partial u}{\partial n} &= \Psi \text{ on } \partial\Omega, \quad (1.4)
\end{align*}

where $\Psi$ is given and continuous on $\partial\Omega$.

In [17], Chap. 7, we derived an explicit formula for the solution of (1.1), (1.2) if $\Omega$ is a ball. In general, one gets explicit solutions, provided the Green function is known for the domain $\Omega$ considered.

We denote (1.1), (1.2) by $(D_i)$ and (1.3), (1.4) by $(N_i)$ to indicate that the problems considered concerns the interior of $\Omega$. Then $(D_e)$ and $(N_e)$ denote the associated exterior problems, that is, we have to replace in (1.1) and (1.3) the domain $\Omega$ by its complement $\mathbb{R}^n \setminus \Omega$.

For the Dirichlet problems we make an ansatz with a dipole potential

$$W(z) = \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z - y|^{n-2}} \right) dS_y$$

(1.5)

if $n \geq 3$. In the case that $n = 2$ we have to replace $|z - y|^{2-n}$ by $-\ln(|z - y|)$. In the formula above $\nu(y)$ denotes the exterior unit normal at $y \in \partial\Omega$ and $\sigma(y)$ the *dipole density*.

For the Neumann problem we make an ansatz with a single layer potential

$$V(z) = \int_{\partial\Omega} \frac{\sigma(y)}{|z - y|^{n-2}} dS_y$$

(1.6)

if $n \geq 3$. In the case that $n = 2$ we have to replace $|z - y|^{2-n}$ by $-\ln(|z - y|)$. 

Both potentials solve the Laplace equation in $\mathbb{R}^n \setminus \partial \Omega$.

In the rest of this chapter we assume that $n \geq 3$.

We will see that discontinuous properties of these surface potentials lead to integral equations which can be studied by using the Fredholm results on integral equations. Thus, the method of surface potentials provide a beautiful example for the Fredholm theory.

1.1 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a bounded and connected domain with a sufficiently regular boundary $\partial \Omega$.

Definition. We say that $\partial \Omega \in C^{1,\lambda}, 0 < \lambda \leq 1$, if:

(i) for each given $x \in \partial \Omega$ there exists a $\rho > 0$ and $N = N(x, \rho)$ balls $B_{2\rho}(x_i) \subset \mathbb{R}^n$, $i = 1, \ldots, N$, with centers $x_i \in \partial \Omega$, where $x_1 = x$, such that

$$\partial \Omega \subset \bigcup_{i=1}^{N} B_{\rho}(x_i).$$

(ii) Let $T_{x_i}$ be a plane which contains $x_i$ and denote by $Z_{2\rho}(x_i)$ a circular cylinder parallel to the normal on $T_{x_i}$ such that its intersection with the plane $T_{x_i}$ is a ball in $\mathbb{R}^{n-1}$ with radius $2\rho$ and the center at $x_i$. We assume that the intersection $\partial \Omega \cap Z_{2\rho}(x_i)$ has a local representation $\tau = f(\xi)$, $f \equiv f_i$, where $\xi$ is in an $(n - 1)$-dimensional ball $D_{2\rho} = D_{2\rho}(0)$ with radius $2\rho$ and the center at $0 \in \mathbb{R}^{n-1}$. Moreover, we assume that

$$f \in C^{1,\lambda}(\overline{D_{2\rho}}), \quad f(0) = 0, \quad \nabla f(0) = 0.$$

Lemma 1.1.1 (Partition of unity). There exists $\eta_i \in C_0^\infty(B_{2\rho}(x_i)), 0 \leq \eta_i \leq 1$, such that

$$\sum_{i=1}^{N} \eta_i(x) = 1 \quad \text{if} \quad x \in \bigcup_{i=1}^{N} B_{\rho}(x_i).$$

Proof. For given $B_{2\rho}(x_i)$ there exists $\phi_i \in C_0^\infty(B_{2\rho}(x_i))$ with the properties
that $\phi_i = 1$ in $B_\rho(x_i)$ and $0 \leq \phi_i(x) \leq 1$, see an exercise. Set

$$
\eta_1 = \phi_1 = 1 - (1 - \phi_1),
$$
$$
\eta_i = \phi_1(1 - \phi_1) \cdot \ldots \cdot (1 - \phi_{i-1}) = (1 - (1 - \phi_1))(1 - \phi_1) \cdot \ldots \cdot (1 - \phi_{i-1}).
$$

Then

$$
\sum_{i=1}^{N} \eta_i(x) = 1 - (1 - \phi_1) \cdot \ldots \cdot (1 - \phi_N),
$$

which implies that

$$
\sum_{i=1}^{N} \eta_i(x) = 1, \quad \text{if} \quad x \in \bigcup_{i=1}^{N} B_\rho(x_i)
$$

since at least one of the factors is zero.

Assume that $\partial \Omega \in C^{1,\lambda}$, then we define the area integral by

$$
\int_{\partial \Omega} g(y) \, dS_y = \int_{\partial \Omega} \sum_{i=1}^{N} \eta_i(y) \, g(y) \, dS_y = \sum_{i=1}^{N} \int_{\partial \Omega} \eta_i(y) \, g(y) \, dS_y,
$$
1.2. DIPOLE POTENTIAL

where

$$\int_{\partial \Omega} \eta_i(y) \ g(y) \ dS_y$$

$$= \int_{D_2} \eta_i(\xi, f_i(\xi)) \ g(\xi, f_i(\xi)) \sqrt{1 + |\nabla f_i(\xi)|^2} \ d\xi.$$ 

Here we suppose that $\partial \Omega \cap Z_{2\rho}(x_i)$ has the parametric representation $y = (\xi, f_i(\xi))$.

1.2 Dipole potential

Dipole density. The following consideration leads to the formula (1.5) for the dipole potential in the case of three dimensions.

Consider two parallel surfaces, one inside of $\Omega$ and the other one outside, of distance $\epsilon/2$, $\epsilon > 0$ small, from $\partial \Omega$, see Fig. 1.2.

Assume that there is a charge of power $\epsilon^{-1}$ at $y^+ = y + (\epsilon/2)\nu(y)$ and a charge of power $-\epsilon^{-1}$ at $y^- = y - (\epsilon/2)\nu(y)$. Set $z^+ = y^+ - x$ and $z^- = y^- - x$, then the potential at $x$ of the dipole is

$$u = \frac{1}{\epsilon} \left( \frac{1}{|z^+|} - \frac{1}{|z^-|} \right)$$

$$= \frac{1}{\epsilon} \left( \frac{|z^-|^2 - |z^+|^2}{(|z^+| + |z^-|)|z^+||z^-|} \right).$$

Since $z^- = z^+ - \epsilon \nu(y)$, we have

$$|z^-|^2 = \langle z^+ - \epsilon \nu(y), z^+ - \epsilon \nu(y) \rangle$$

$$= |z^+|^2 - 2\epsilon z^+ \cdot \nu(y) + \epsilon^2.$$
Thus
\[ u = -\frac{1}{\epsilon} \frac{2\epsilon z^+ \cdot \nu(y) + \epsilon^2}{(|z^+| + |z^-|)|z^+||z^-|}. \]

Set \( z = y - x \), then
\[
\lim_{\epsilon \to 0} u = -z \cdot \nu(y) |z|^{-3} = \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|y - x|} \right)
\]
is the potential of a single dipole with density \( \sigma(y) = 1 \) at \( y \). Multiplication with a density \( \sigma(y) \) and integration over \( \partial \Omega \) leads to formula (1.5).

The right hand side of (1.5) is called dipole potential or potential of a double layer with density \( \sigma \). The dipole potential is in \( C^\infty(\mathbb{R}^n \setminus \partial \Omega) \) and a solution of the Laplace equation in \( \mathbb{R}^n \setminus \partial \Omega \). In fact, see the following proposition, the right hand side of (1.5) is defined and continuous on \( \partial \Omega \) provided the boundary \( \partial \Omega \) is sufficiently smooth, but \( W(x) \) makes a jump across \( \partial \Omega \).

Some of the following calculations are based on the formula for the directional derivative in direction \( \nu(y) \)
\[
\frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z - y|^{n-2}} \right) = \frac{n - 2}{|z - y|^{n}} \sum_{i=1}^{n} (z_i - y_i)(\nu(y))_i.
\] (1.7)

**Lemma 1.2.1.** Assume that \( \partial \Omega \in C^{1,\lambda} \) and \( \sigma \in C(\partial \Omega) \). Then the right hand side of (1.5) is defined and is continuous on \( \partial \Omega \).

**Proof.** Consider the case \( n \geq 3 \). Let \( x \) be the center of a local coordinate system and \( z \in \partial \Omega \cap Z_{2\rho}(x) \), see Fig. 1.3.

We have to show that, see Sec. 1.1 for the definition of the surface integral and formula (1.7),
\[
q(\zeta) := \int_{D_{2\rho}} \eta(\xi, f(\xi)) \sigma(\xi, f(\xi)) \frac{-(\zeta - \xi) \cdot \nabla f(\xi) + f(\xi) - f(\xi)}{(|\zeta - \xi|^2 + |f(\xi) - f(\xi)|^2)^{n/2}} \, d\xi
\]
is continuous in a neighborhood of \( 0 \in \mathbb{R}^{n-1} \). Here is \( D_{2\rho} = D_{2\rho}(0) \), \( z = (\zeta, f(\zeta)) \) and \( y = (\xi, f(\xi)) \) in local coordinates, and \( \eta \in C^\infty \) in its arguments.

Because of \( f \in C^{1,\lambda}(D_{2\rho}) \), \( f(0) = 0 \) and \( \nabla f(0) = 0 \) we have
\[
q(\zeta) = \int_{D_{2\rho}} \frac{A(\xi, \zeta)}{|\xi - \zeta|^{n-1-\lambda}} \, d\xi,
\]
1.2. DIPOLE POTENTIAL

where $A(\xi, \zeta)$ is bounded on $D_{2\rho} \times D_{2\rho}$ and continuous if $\xi \neq \zeta$. Since the integrand is weakly singular, it follows that $q(\zeta)$ is continuous on $D_{2\rho}$, see an exercise.

Let $x_0 \in \partial \Omega$ and $x \in \mathbb{R}^n$. Set

$$W(x) = W_1(x) + \sigma(x_0)W_0(x),$$

where

$$W_1(x) = \int_{\partial \Omega} (\sigma(y) - \sigma(x_0)) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y,$$

$$W_0(x) = \int_{\partial \Omega} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y.$$

The integral $W_0(x)$ is called Gauss integral.

**Lemma 1.2.2.** Suppose that $\partial \Omega \in C^{1,\lambda}$. Then

$$W_0(x) = \begin{cases} -(n - 2)\omega_n : & x \in \Omega, \\ 0 : & x \notin \overline{\Omega}, \\ -\frac{n-2}{2}\omega_n : & x \in \partial \Omega. \end{cases}$$

**Proof.** (i) If $x \in \mathbb{R}^n \setminus \overline{\Omega}$ is fixed, then there is a domain $\Omega_0 \supset \Omega$ where $|x - y|^{2-n} \in C^\infty(\Omega_0)$ and satisfies the Laplace equation. Then

$$0 = \int_\Omega \Delta_y \left( \frac{1}{|x - y|^{n-2}} \right) dy$$

$$= \int_{\partial \Omega} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y.$$
(ii) Let \( x \in \Omega \) be fixed, then there is a ball \( B_\rho(x) \subset \Omega \). Then

\[
0 = \int_{\Omega \setminus B_\rho(x)} \Delta y \left( \frac{1}{|x - y|^{n-2}} \right) \, dy
\]

\[
= \int_{\partial \Omega} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS_y - \int_{\partial B_\rho(x)} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS_y,
\]

where in the second integral \( \nu(y) \) denotes the exterior unit normal at the boundary of \( B_\rho(x) \). Using polar coordinates with center at \( x \), we find for the second integral

\[
\int_{\partial B_\rho(x)} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS_y = \int_{\partial B_1(x)} \frac{\partial}{\partial \rho} \left( \rho^{2-n} \right) \rho^{n-1} \, dS
\]

\[
= (2 - n) \omega_n.
\]

(iii) Let \( x \in \partial \Omega \) and set for a sufficiently small \( \rho > 0 \),

\[
S_\rho = \Omega \cap \partial B_\rho(x), \quad C_\rho = \partial \Omega \setminus B_\rho(x),
\]

see Fig. 1.4.

![Figure 1.4: Figure to the proof of Lemma 1.2.2](image)

Then

\[
0 = \int_{\Omega \setminus B_\rho(x)} \Delta y \left( \frac{1}{|x - y|^{n-2}} \right) \, dy
\]

\[
= \int_{C_\rho} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS_y - \int_{S_\rho} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS_y.
\]
1.2. DIPOLE POTENTIAL

Since, see an exercise,

\[
\lim_{\rho \to 0} \int_{C_{\rho}} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS = \int_{\partial \Omega} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS,
\]

it follows that

\[
\int_{\partial \Omega} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS = \lim_{\rho \to 0} \int_{S_{\rho}} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS.
\]

We have

\[
\int_{S_{\rho}} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS = -(n - 2) \rho^{1-n} \int_{S_{\rho}} dS
\]

and

\[
\int_{S_{\rho}} dS = \frac{\omega_n}{2} \rho^{n-1} \left( 1 + O(\rho^{2 \lambda}) \right).
\]

The previous formula follows by introducing local coordinates at \(x\). Let \((\xi, h(\xi)) \in \partial B_{\rho}(x) \cap \partial \Omega\), then

\[
|h(\xi)| \leq c \rho^{1+\lambda}.
\]

Let \(F\) be a layer of a sphere with radius \(\rho\) of height \(c \rho^{1+\lambda}\), see Fig. 1.5, then

\[
\frac{\omega_n}{2} \rho^{n-1} - |F| \leq |S_{\rho}| \leq \frac{\omega_n}{2} \rho^{n-1} + |F|.
\]

We have

\[
|F| = \frac{1}{2} \rho^{n-1} \omega_n (1 - \cos \theta)
\]

\[
= \frac{1}{2} \rho^{n-1} \omega_n \left( 1 - (1 - c^2 \rho^{2 \lambda})^{1/2} \right)
\]

\[
= \frac{1}{2} \rho^{n-1} \omega_n O(\rho^{2 \lambda})
\]
as \( \rho \to 0 \).

\[ \square \]

**Lemma 1.2.3.** Let \( \partial \Omega \in C^{1,\lambda} \). Then

\[
\int_{\partial \Omega} \left| \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| dS_y
\]

is uniformly bounded for \( x \in \mathbb{R}^n \).

**Proof.** (i) For fixed \( d > 0 \) consider \( x \) such that \( \text{dist}(x, \partial \Omega) \geq d/2 \). Then, see formula (1.7),

\[
\left| \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| \leq \frac{(n-2)2^{n-1}}{d^{n-1}},
\]

which implies that

\[
\int_{\partial \Omega} \left| \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| dS_y \leq \frac{(n-2)2^{n-1}}{d^{n-1}} |\partial \Omega|.
\]

(ii) Consider \( x \in \mathbb{R}^n \) such that \( \text{dist}(x, \partial \Omega) < d/2 \) for a \( d > 0 \) and let

\[
x_0 \in \partial \Omega : \quad |x-x_0| = \min_{y \in \partial \Omega} |x-y|.
\]

Set \( S_d = \partial \Omega \cap B_d(x_0) \). Then for \( y \in \partial \Omega \setminus S_d \) we have

\[
|y-x| \geq |y-x_0| - |x-x_0| > d/2,
\]

which implies that

\[
\int_{\partial \Omega \setminus S_d} \left| \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| dS_y \leq \frac{(n-2)2^{n-1}}{d^{n-1}} |\partial \Omega \setminus S_d| \leq \frac{(n-2)2^{n-1}}{d^{n-1}} |\partial \Omega|.
\]

(iii) Consider

\[
I_d := \int_{S_d} \left| \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \right| dS_y.
\]

In local coordinates, see Fig. 1.6, we have, since \( x-x_0 \) is perpendicular on the tangent plane \( T_{x_0} \), that \( x = (0, \ldots, 0, \delta) \) and \( y = (\xi, f(\xi)) \), where \( f \in C^{1,\lambda}(\overline{D}_\rho) \), \( f(0) = 0 \), \( \nabla f(0) = 0 \) and \( D_\rho \) is a ball in \( \mathbb{R}^{n-1} \) with the center
1.2. DIPOLE POTENTIAL

at 0 ∈ ℝ^{n-1} and the radius ρ. We choose d > 0 such that ρ > d. Then, see formula (1.7),

\[ \sqrt{1 + |\nabla f(\xi)|^2} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) = \frac{\xi \cdot \nabla f(\xi) + (\delta - f(\xi))}{(|\xi|^2 + (\delta - f(\xi))^2)^{n/2}} \]

\[ = \frac{h_1(\xi) + \delta}{(|\xi|^2 + (\delta + h_2(\xi))^2)^{n/2}}, \]

where \( h_i \in C(D(\rho(0))), |h_i(\xi)| \leq c|\xi|^{1+\lambda}. \)

Since

\[ |\xi|^2 + |\delta + h_2(\xi)|^2 = |\xi|^2 + \delta^2 + 2\delta h_2 + h_2^2 \]

\[ \geq |\xi|^2 + \frac{1}{2}\delta^2 - 7h_2^2 \]

\[ \geq \frac{1}{2}|\xi|^2 + \frac{1}{2}\delta^2, \]

provided that ρ satisfies \( 7cp^{2\lambda} < 1/2. \) It follows that

\[ I_d \leq 2^{n/2} \int_{D_\rho} \frac{c|\xi|^{1+\lambda} + |\delta|}{(|\xi|^2 + \delta^2)^{n/2}} \, d\xi. \] (1.10)

The right hand side of (1.10) is uniformly bounded with respect to |δ| < d.

More precisely, we have

\[ I_d \leq 2^{n/2} \omega_{n-1} \max\{c\lambda^{-1} \rho^{\lambda}, \pi/2\}, \]

see an exercise.

Lemma 1.2.4. Assume that \( \sigma \in C(\partial \Omega) \) and \( x_0 \in \partial \Omega. \) Then \( W_1(x), \) see definition (1.8), is continuous at \( x_0. \)
Proof. Set \( S_\rho = \partial \Omega \cap B_\rho(x_0) \) and \( W_1(x) = I_1 + I_2 \), where
\[
I_1(x) = \int_{S_\rho} (\sigma(y) - \sigma(x_0)) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y,
\]
\[
I_2(x) = \int_{\partial \Omega \setminus S_\rho} (\sigma(y) - \sigma(x_0)) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y.
\]
We have
\[
|W_1(x) - W_1(x_0)| \leq |I_1(x)| + |I_1(x_0)| + |I_2(x) - I_2(x_0)|
\]
and
\[
|I_1(x)| \leq \int_{S_\rho} |\sigma(y) - \sigma(x_0)| \left| \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \right| dS_y.
\]
Set, see Lemma 1.2.3,
\[
C = \sup_{x \in \mathbb{R}^n} \int_{\partial \Omega} \left| \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \right| dS_y
\]
and choose for given \( \epsilon > 0 \) a \( \rho = \rho(\epsilon) \) such that
\[
|\sigma(y) - \sigma(x_0)| < \frac{\epsilon}{3C}
\]
if \( y \in S_\rho \). Then \( |I_1(x)| < \epsilon/3 \) and \( |I_1(x_0)| < \epsilon/3 \).

Consider \( x \in \mathbb{R}^n \) such that \( |x - x_0| < \rho/2 \), then
\[
|y - x| \geq |y - x_0| - |x - x_0| \geq \rho/2,
\]
provided that \( y \in \partial \Omega \setminus S_\rho \). Since \( I_2 \) is continuous in \( B_{\rho/2}(x_0) \), there is a \( \delta = \delta(\epsilon) \) such that
\[
|I_2(x) - I_2(x_0)| < \epsilon/3
\]
if \( |x - x_0| < \delta(\epsilon) \). Summarizing, we have
\[
|W_1(x) - W_1(x_0)| < \epsilon
\]
if \( |x - x_0| < \min\{\rho(\epsilon)/2, \delta(\epsilon)\} \).

Let \( x_0 \in \partial \Omega \) and denote by \( W_i(x_0) \) the limit of \( W(x) \) from interior to \( x_0 \) and by \( W_e(x_0) \) the limit of \( W(x) \) from exterior to \( x_0 \).
Proposition 1.2.1. Suppose that $\sigma \in C(\partial \Omega)$ and $x_0 \in \partial \Omega$. The limits $W_i(x_0)$ and $W_e(x_0)$ exist and satisfy the jump relations

$$W_i(x_0) = -\frac{(n-2)\omega_n}{2} \sigma(x_0) + W(x_0),$$

$$W_e(x_0) = \frac{(n-2)\omega_n}{2} \sigma(x_0) + W(x_0).$$

Proof. We will prove the first of the jump relations. For $x \in \Omega$ we set $W(x) = W_1(x) + \sigma(x_0)W_0(x)$, where $W_1(x)$ is continuous at $x_0$, see Lemma 1.2.4, and $W_0(x)$ is the Gauss integral, see Lemma 1.2.2. Thus

$$W_i(x_0) = \lim_{x \to x_0, x \in \Omega} (W_1(x) + \sigma(x_0)W_0(x))$$

$$= W_1(x_0) - (n-2)\sigma(x_0)$$

$$= \int_{\partial \Omega} (\sigma(y) - \sigma(x_0)) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x_0 - y|^{n-2}} \right) dS_y - (n-2)\sigma(x_0)$$

$$= W(x_0) - \frac{(n-2)\omega_n}{2} \sigma(x_0).$$

□

Corollary. The double layer potential

$$W(x) = \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y,$$

where $\sigma \in C(\partial \Omega)$, defines a solution of the interior Dirichlet problem $(D_i)$ if and only if $\sigma \in C(\partial \Omega)$ is a solution of the integral equation

$$\Phi(x) = -\frac{(n-2)\omega_n}{2} \sigma(x) + \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y,$$

where $x \in \partial \Omega$, and $W(x)$ is a solution of the exterior Dirichlet problem $(D_e)$ if and only if $\sigma \in C(\partial \Omega)$ satisfies the integral equation

$$\Phi(x) = \frac{(n-2)\omega_n}{2} \sigma(x) + \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y.$$

We recall that $W$ is a solution of $(D_i)$ if and only if

$$\Phi(x) = \lim_{z \to x, z \in \Omega} W(z),$$
and of \((D_e)\) if and only if
\[
\Phi(x) = \lim_{z \to x, z \in \mathbb{R}^n \setminus \Omega} W(z).
\]

### 1.3 Single layer potential

Consider the single layer potential
\[
V(x) = \int_{\partial \Omega} \frac{\sigma(y)}{|x - y|^{n-2}} dS_y,
\]
where \(\sigma \in C(\partial \Omega)\).

**Lemma 1.3.1.** \(V \in C(\mathbb{R}^n)\).

**Proof.** It remains to show that \(V(x)\) is continuous if \(x \in \partial \Omega\). Let \(x \in \partial \Omega\), set \(S_\rho = \partial \Omega \cap B_\rho(x)\), \(\rho > 0\) sufficiently small, and
\[
V(x) = V_1(x) + V_2(x),
\]
where
\[
V_1(x) = \int_{S_\rho} \frac{\sigma(y)}{|x - y|^{n-2}} dS_y,
\]
\[
V_2(x) = \int_{\partial \Omega \setminus S_\rho} \frac{\sigma(y)}{|x - y|^{n-2}} dS_y.
\]
Consider \(z \in \mathbb{R}^n\) where \(z\) is in a neighborhood of \(x\). We have
\[
|V(z) - V(x)| \leq |V_1(z)| + |V_1(x)| + |V_2(z) - V_2(x)|.
\]
In local coordinates it is \(y = (\xi, f(\xi))\), \(z = (\zeta, \delta)\), where \(\xi, \zeta \in D_\rho = D_\rho(0)\), see Fig. 1.7, and
\[
|V_1(z)| \leq \int_{D_\rho} \frac{|\sigma(\xi, f(\xi))| \sqrt{1 + |\nabla f(\xi)|^2}}{(|\zeta - \xi|^2 + (\delta - f(\xi))^2)^{(n-2)/2}} d\xi
\]
\[
\leq c \int_{D_\rho} \frac{d\xi}{|\xi - \zeta|^{n-2}}
\]
\[
\leq c \int_{D_{2\rho}} \frac{d\xi}{|\xi|^{n-2}}
\]
\[
= 2c\omega_{n-1}\rho.
\]
Let $\epsilon > 0$ be given and set $\rho = \rho(\epsilon) = \epsilon/(6\omega_{n-1})$, then $|V_1(z)| < \epsilon/3$ if $|z-x| < \rho(\epsilon)$. Consequently, for those $z$ we have

$$|V(z) - V(x)| \leq \frac{2}{3}\epsilon + |V_2(z) - V_2(x)|.$$ 

For fixed $\rho > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that

$$|V_2(z) - V_2(x)| < \frac{\epsilon}{3}$$

if $|z-x| < \delta(\epsilon)$. Summarizing, we have

$$|V(z) - V(x)| < \epsilon,$$

provided that $|z-x| < \min\{\rho(\epsilon), \delta(\epsilon)\}$. \hfill $\square$

**Definition.** Assume that $u \in C^1(\Omega)$ and $\partial \Omega \in C^{1,\lambda}$. We say that there exists a *regular interior normal derivative* of $u$ at $\partial \Omega$ if the limit

$$\left( \frac{\partial u(x)}{\partial \nu(x)} \right)_i := \lim_{z \to x} \frac{\partial u(z)}{\partial \nu(x)}$$

exists for each $x \in \partial \Omega$. Here is $z \in \Omega$ on the line defined by the exterior normal $\nu_x$ at $x$, see Fig. 1.8. This limit is a continuous function on $\partial \Omega$.

Analogously, we define the *regular exterior normal derivative* of $u \in C^1(\mathbb{R}^n \setminus \overline{\Omega})$ on $\partial \Omega$ by

$$\left( \frac{\partial u(x)}{\partial \nu(x)} \right)_e := \lim_{z \to x} \frac{\partial u(z)}{\partial \nu(x)},$$
\[ \Omega \] is on the line defined by \( \nu(x) \) and \( x \).

Assume that \( z \notin \partial \Omega \), then

\[ \frac{\partial V(z)}{\partial l} = \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial l} \left( \frac{1}{|z - y|^{n-2}} \right) \, dS_y, \]

where \( l \) is any direction. Let \( x \in \partial \Omega \), then we define

\[ \frac{\partial V(x)}{\partial \nu(x)} := \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS_y. \quad (1.11) \]

In some of the following considerations we need the formula

\[ \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x - y|^{n-2}} \right) = -\frac{n - 2}{|x - y|^n} \sum_{i=1}^{n} (x_i - y_i)(\nu(x))_i. \quad (1.12) \]

**Lemma 1.3.2.** The right hand side of (1.11) exists\(^1\) if \( x \in \partial \Omega \).

**Proof.** We introduce a local coordinate system with center at \( x \) as in previous considerations and show that

\[ I_\rho(x) := \int_{S_\rho} \sigma(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS_y \]

\(^1\)i. e., this weakly singular integral exists in the sense of Riemann or as a Lebesgue integral, and it is bounded
exists, where \( S_\rho = \partial \Omega \cap B_\rho(x), \rho > 0 \) sufficiently small. In local coordinates it is \( y = (\xi, f(\xi)) \). Using formula (1.12), we obtain

\[
|I_\rho(x)| \leq c_1 \int_{D_\rho} \frac{|f(\xi)|}{|\xi|^n} \, d\xi \\
\leq c_2 \int_{D_\rho} |\xi|^{-n+1+\lambda} \, d\xi \\
= c_2 \omega_{n-1} \lambda^{-1} \rho^\lambda.
\]

Let \( x \in \partial \Omega \) and consider the sum

\[
s(z) = \frac{\partial V(z)}{\partial \nu(x)} + W(z)
\]

\[
= \int_{\partial \Omega} \sigma(y) \left( \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|z-y|^{n-2}} \right) + \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) \right) \, dS_y,
\]

where \( W \) is the dipole potential and \( z \) is on the line defined by \( \nu(x) \), see Fig. 1.8.

**Lemma 1.3.3.** The sum \( s(z) \) is continuous at \( x \).

**Proof.** Set \( S_\rho = \partial \Omega \cap B_\rho(x), \rho > 0 \) sufficiently small, and

\[
s(z) = s_1(z) + s_2(z),
\]

where

\[
s_1(z) = \int_{S_\rho} \sigma(y) \left( \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|z-y|^{n-2}} \right) + \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) \right) \, dS_y,
\]

\[
s_2(z) = \int_{\partial \Omega \setminus S_\rho} \sigma(y) \left( \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|z-y|^{n-2}} \right) + \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) \right) \, dS_y.
\]

We have

\[
|s(z) - s(x)| \leq |s_1(z)| + |s_1(x)| + |s_2(z) - s_2(x)|.
\]

The lemma is shown if for given \( \epsilon > 0 \) there exists a \( \rho = \rho(\epsilon) > 0 \) such that \( |s_1(z)| < \epsilon/3 \) if \( |x-z| < \rho(\epsilon) \), see the proof of Lemma 1.3.1. We have, see (1.7), (1.12),

\[
\frac{\partial}{\partial \nu(x)} \left( \frac{1}{|z-y|^{n-2}} \right) + \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z-y|^{n-2}} \right) \\
= (n-2) \frac{1}{|z-y|^n} \left( \sum_{i=1}^n (z_i - y_i) (\nu(y))_i - \sum_{i=1}^n (z_i - y_i) (\nu(x))_i \right),
\]
where, in local coordinates, \( x = (0, \ldots, 0, 0) \), \( z = (0, \ldots, 0, \delta) \), \( \nu(x) = (0, \ldots, 0, 1) \) and
\[
\nu(y) = \frac{1}{\sqrt{1 + |\nabla f(\xi)|^2}}(-f_{\xi_1}, \ldots, -f_{\xi_{n-1}}, 1).
\]

It follows that
\[
|s_1(z)| \leq c_1 \int_{S_y} \left| \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|z - y|^{n-2}} \right) + \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z - y|^{n-2}} \right) \right| dS_y
\]
\[
\leq c_2 \int_{D_{\nu}(0)} \left| \xi \cdot \nabla f(\xi) + (\delta - f(\xi)) - \delta \sqrt{1 + |\nabla f(\xi)|^2} \right| d\xi
\]
\[
\leq c_3 \int_{D_{\nu}(0)} \frac{|\xi|^{1+\lambda} + |\delta||\xi|^{2\lambda}}{(\|\xi\|^2 + \delta^2)^{n/2}} d\xi
\]
\[
\leq c_3 \omega_{n-1} \max\{\lambda^{-1} \rho^{\lambda}, \pi \rho^{2\lambda}/2\},
\]
where the constants \( c_i \) are independent of \( \rho \).

**Proposition 1.3.1.** Suppose that \( \partial \Omega \in C^{1,\lambda} \). Then there exists a regular interior and a regular exterior normal derivative of \( V \), and these derivatives satisfy the jump relations
\[
\left( \frac{\partial V(x)}{\partial \nu(x)} \right)_i = \frac{(n - 2)\omega_n}{2} \sigma(x) + \frac{\partial V(x)}{\partial \nu(x)} + W_i(x),
\]
\[
\left( \frac{\partial V(x)}{\partial \nu(x)} \right)_e = -\frac{(n - 2)\omega_n}{2} \sigma(x) + \frac{\partial V(x)}{\partial \nu(x)} + W_e(x),
\]
where \( x \in \partial \Omega \).

**Proof.** The existence of regular normal derivatives follow from Lemma 1.3.3 and Proposition 1.2.1 since
\[
\frac{\partial V(z)}{\partial \nu(x)} = \left( \frac{\partial V(z)}{\partial \nu_x} + W(z) \right) - W(z),
\]
where \( z \) is on the line defined by \( \nu(x) \). From Lemma 1.3.3 it follows that
\[
\left( \frac{\partial V(x)}{\partial \nu(x)} \right)_i + W_i(x) = \left( \frac{\partial V(x)}{\partial \nu(x)} \right)_e + W_e(x)
\]
\[
= \frac{\partial V(x)}{\partial \nu(x)} + W(x),
\]
1.3. SINGLE LAYER POTENTIAL

where

\[
\frac{\partial V(x)}{\partial \nu(x)} : = \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \, dS_y,
\]

\[
W(x) : = \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x-y|^{n-2}} \right) \, dS_y.
\]

Using Lemma 1.3.1, we obtain

\[
\left( \frac{\partial V(x)}{\partial \nu(x)} \right)_i = W(x) - W_i(x) + \frac{\partial V(x)}{\partial \nu(x)} \frac{(n-2)\omega_n}{2} \sigma(x) + \frac{\partial V(x)}{\partial \nu(x)}
\]

and

\[
\left( \frac{\partial V(x)}{\partial \nu(x)} \right)_e = W(x) - W_e(x) + \frac{\partial V(x)}{\partial \nu_e} \frac{(n-2)\omega_n}{2} \sigma(x) + \frac{\partial V(x)}{\partial \nu(x)}.
\]

Remark. Let \( x \in \partial \Omega \), then it follows immediately that

\[
\left( \frac{\partial V(x)}{\partial \nu(x)} \right)_i - \left( \frac{\partial V(x)}{\partial \nu_e} \right)_e = (n-2)\omega_n \sigma(x).
\]

Corollary. The single layer potential

\[
V(x) = \int_{\partial \Omega} \frac{\sigma(y)}{|x-y|^{n-2}} \, dS_y,
\]

where \( \sigma \in C(\partial \Omega) \) defines a solution of the interior Neumann problem \((N_i)\) if and only if \( \sigma \in C(\partial \Omega) \) is a solution of the integral equation

\[
\Psi(x) = \frac{(n-2)\omega_n}{2} \sigma(x) + \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x-y|^{n-2}} \right) \, dS_y,
\]

where \( x \in \partial \Omega \), and \( V(x) \) is a solution of the exterior Neumann problem \((N_e)\) if and only if \( \sigma \in C(\partial \Omega) \) satisfies the integral equation

\[
\Psi(x) = -\frac{(n-2)\omega_n}{2} \sigma(x) + \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x-y|^{n-2}} \right) \, dS_y.
\]
CHAPTER 1. POTENTIAL THEORY

We recall that $V$ is a solution of $(N_i)$ if and only if
$$\Psi(x) = \left( \frac{\partial V(x)}{\partial \nu(x)} \right)_i,$$
and of $(N_e)$ if and only if
$$\Psi(x) = \left( \frac{\partial V(x)}{\partial \nu(x)} \right)_e.$$

1.4 Integral equations

Denote by $H = L^2(\partial \Omega)$ the Hilbert space with the inner product
$$\langle \sigma, \mu \rangle = \int_{\partial \Omega} \sigma(x) \mu(x) \, dS_x,$$
and, if $n \geq 3$, we define the linear operator $T$ from $H$ into $H$ by
$$(T \sigma)(x) = \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS_y.$$
Set
$$(T^* \sigma)(x) = \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS_y,$$
then
$$\langle T \sigma, \mu \rangle = \langle \sigma, T^* \mu \rangle$$
for all $\sigma, \mu \in H$. Below we will show that $T$ is bounded. Then it follows that $T^*$ is the adjoint operator to $T$.

According to the above corollaries to Proposition 1.2.1 and Proposition 1.3.1, the potentials $W$ and $V$ are solutions of the boundary value problems $(D_i), (D_e), (N_i)$ and $(N_e)$ if the density $\sigma$ is continuous on $\partial \Omega$ and satisfies the integral equations, respectively,

$$\sigma - \frac{2}{(n-2)\omega_n} T \sigma = -\frac{2}{(n-2)\omega_n} \Phi, \quad (D_i)_I$$
$$\sigma + \frac{2}{(n-2)\omega_n} T \sigma = \frac{2}{(n-2)\omega_n} \Phi, \quad (D_e)_I$$
$$\sigma + \frac{2}{(n-2)\omega_n} T^* \sigma = \frac{2}{(n-2)\omega_n} \Psi, \quad (N_i)_I$$
$$\sigma - \frac{2}{(n-2)\omega_n} T^* \sigma = -\frac{2}{(n-2)\omega_n} \Psi, \quad (N_e)_I$$
Remark. Since we make the ansatz with the above potentials for the exterior problems, we prescribe in fact the behavior $|u(z)| \leq c|z|^{1-n}$, $|u(z)| \leq c|z|^{2-n}$, respectively, as $z \to \infty$.

The above integral equations are defined for $\sigma \in L^2(\partial \Omega)$. In the following we will discuss whether or not there exist solutions in $L^2(\partial \Omega)$. From a regularity result which says that an $L^2$-solution is in fact in $C(\partial \Omega)$, we recover that the potentials define solutions of the boundary value problem, see the corollaries to Proposition 1.2.1 and Proposition 1.3.1.

**Proposition 1.4.1.** Suppose that $\partial \Omega \in C^{1,\lambda}$. Then $T$ is a completely continuous operator from $H$ into $H$.

**Proof.** (i) $T$ is bounded. It is sufficient, see Sec. 1.1, to show that

$$(P\mu)(\zeta) := \int_{D_\rho} a(\xi)\mu(\xi)K(\xi, \zeta) \, d\xi$$

is bounded from $L^2(D_\rho)$ into $L^2(D_\rho)$. Here is $D_\rho = D_\rho(0) \subset \mathbb{R}^{n-1}$, $a \in C_0^\infty(D_\rho)$, $\mu(\xi) = \sigma(\xi, f(\xi))$ and

$$K(\xi, \zeta) = \frac{(n-2)\left(-(|\xi - \zeta|^2 + (f(\xi) - f(\zeta))^2)^{n/2}\right)}{(\xi - \zeta)^{n-1-\lambda}}.$$ 

Set $q(\zeta) := (P\mu)(\zeta)$, then

$$q(\zeta) = \int_{D_\rho} \mu(\xi) \frac{A(\xi, \zeta)}{|\xi - \zeta|^{n-1-\lambda}} \, d\xi,$$

where $A$ is bounded on $D_\rho \times D_\rho$ and continuous if $\xi \neq \zeta$. Let $\kappa = n - 1 - \lambda$, then we have, with constants $c_i$ independent of $\mu$ and $\rho$, that

$$|q(\zeta)| \leq c_1 \int_{D_\rho} \frac{|\mu(\xi)|}{|\xi - \zeta|^{\kappa/2}} \frac{1}{|\xi - \zeta|^{\kappa/2}} \, d\xi,$$

$$|q(\zeta)|^2 \leq c_2 \int_{D_\rho} \frac{|\mu(\xi)|^2}{|\xi - \zeta|^{\kappa}} \, d\xi \int_{D_\rho} \frac{d\xi}{|\xi - \zeta|^{\kappa}} \leq c_3 \rho^\lambda \int_{D_\rho} \frac{|\mu(\xi)|^2}{|\xi - \zeta|^{\kappa}} \, d\xi,$$

$$\int_{D_\rho} |q(\zeta)|^2 \, d\zeta \leq c_3 \rho^\lambda \int_{D_\rho} \frac{|\mu(\xi)|^2}{|\xi - \zeta|^{\kappa}} \left( \int_{D_\rho} \frac{d\zeta}{|\xi - \zeta|^{\kappa}} \right) \, d\xi \leq c_4 \rho^{2\lambda} \int_{D_\rho} |\mu(\xi)|^2 \, d\xi.$$
(ii) \( T \) is completely continuous. According to a lemma due to Kolmogoroff, see for example [22], pp. 246, or [1], pp. 31, \( P \) is completely continuous if for given \( \epsilon_1 > 0 \) there exists an \( h_0(\epsilon_1) > 0 \) such that

\[
\int_{D_\rho} |q(\zeta + h) - q(\zeta)|^2 \, d\zeta \leq \epsilon_1^2
\]

for all \( h \in \mathbb{R}^{n-1} \) such that \( |h| \leq h_0(\epsilon_1) \), and uniformly for \( \|\mu\|_{L^2(D_\rho)} \leq M \), where \( M < \infty \). Thus, the set \( \|\mu\|_{L^2(D_\rho)} \leq M \) is uniformly continuous in the mean. Above we set \( q(\zeta) = 0 \) if \( \zeta \notin D_\rho \). Let \( \eta \in C(\mathbb{R}_+) \), \( 0 \leq \eta \leq 1 \), such that for given \( \epsilon > 0 \)

\[
\eta(t) = \begin{cases} 
1 & : 0 \leq t \leq \epsilon/2, \\
0 & : t \geq \epsilon.
\end{cases}
\]

Set

\[
q(\zeta) = q_1(\zeta) + q_2(\zeta),
\]

where

\[
q_1(\zeta) = \int_{D_\rho(0)} \mu(\xi) \frac{A(\xi, \zeta)\eta(\xi - \zeta)}{\xi - \zeta} \, d\xi,
\]

\[
q_2(\zeta) = \int_{D_\rho(0)} \mu(\xi) \frac{A(\xi, \zeta)(1 - \eta(\xi - \zeta))}{\xi - \zeta} \, d\xi.
\]

We have

\[
|q(\zeta + h) - q(\zeta)| \leq |q_1(\zeta + h)| + |q_1(\zeta)| + |q_2(\zeta + h) - q_2(\zeta)|. \tag{1.13}
\]

Let \( \epsilon > 0 \) be fixed, then for given \( \tau > 0 \) there is an \( h_0 = h_0(\tau) > 0 \) such that

\[
|q_2(\zeta) - q_2(\zeta)| < \tau \tag{1.14}
\]
for all $|h| \leq h_0$. Concerning $q_1(\zeta)$ we have

\[
|q_1(\zeta)| \leq c \int_{D^\rho(0) \cap B^\rho(\zeta)} |\mu(\xi)| \frac{d\xi}{|\xi - \zeta|^\kappa},
\]

\[
|q_1(\zeta)|^2 \leq c^2 \int_{D^\rho(0) \cap B^\rho(\zeta)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - \zeta|^\kappa}
\]

\[
\int_{D^\rho(0)} |q_1(\zeta)|^2 \, d\zeta \leq c^2 \omega_{n-1} \epsilon^{\lambda-1} \int_{D^\rho(0) \cap B^\rho(\zeta)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - \zeta|^\kappa} \int_{D^\rho(0)} \frac{d\zeta}{|\xi - \zeta|^\kappa}
\]

\[
\leq c^2 \omega_{n-1} \epsilon^{\lambda-1} \int_{D^\rho(0) \cap B^\rho(\zeta)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - \zeta|^\kappa} \int_{D^\rho(0)} \frac{d\zeta}{|\xi - \zeta|^\kappa}
\]

\[
\leq c^2 \omega_{n-1} \epsilon^{\lambda-1} \int_{D^\rho(0) \cap B^\rho(\zeta)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - \zeta|^\kappa} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} \frac{d\xi}{|\xi - (\zeta + h)|^\kappa}
\]

\[
\int_{D^\rho(0)} |q_1(\zeta + h)|^2 \, d\zeta \leq c^2 \omega_{n-1} \epsilon^{\lambda-1} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - (\zeta + h)|^\kappa} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} \frac{d\xi}{|\xi - (\zeta + h)|^\kappa}
\]

\[
\leq c^2 \omega_{n-1} \epsilon^{\lambda-1} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - (\zeta + h)|^\kappa} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} \frac{d\xi}{|\xi - (\zeta + h)|^\kappa}
\]

\[
\leq c^2 \omega_{n-1} \epsilon^{\lambda-1} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - (\zeta + h)|^\kappa} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} \frac{d\xi}{|\xi - (\zeta + h)|^\kappa}
\]

\[
\leq c^2 \omega_{n-1} \epsilon^{\lambda-1} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - (\zeta + h)|^\kappa} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} \frac{d\xi}{|\xi - (\zeta + h)|^\kappa}
\]

\[
\leq c^2 \omega_{n-1} \epsilon^{\lambda-1} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} |\mu(\xi)|^2 \frac{d\xi}{|\xi - (\zeta + h)|^\kappa} \int_{D^\rho(0) \cap B^\rho(\zeta+h)} \frac{d\xi}{|\xi - (\zeta + h)|^\kappa}
\]

\[
= c^2 \omega_{n-1} \epsilon^{\lambda-2} |\mu|_{L^2(D^\rho)}^2.
\]

Combining these $L^2$-estimates with inequalities (1.13) and (1.14) we obtain that the mapping $T$ is completely continuous. \hfill \Box
From a result of functional analysis we have

**Corollary.** $T^*$ is bounded with the same norm as $T$ and $T^*$ is completely continuous.

In the following we study the question of the existence of solutions $\sigma \in L^2(\partial \Omega)$ of the above integral equations. To recover that the associated surface potentials define solutions of the original boundary value problems, we need more regularity, namely $\sigma \in C(\partial \Omega)$. We obtain this property by using the integral equations.

**Proposition 1.4.2** (Regularity). Let $w \in L^2(D_\rho)$ be a solution of the integral equation

$$w(\zeta) - \int_{D_\rho} w(\xi) \frac{A(\xi, \zeta)}{|\xi - \zeta|^\kappa} d\xi = b(\zeta),$$

where $\kappa = n - 1 - \lambda$, $D_\rho = D_\rho(0) \subset \mathbb{R}^{n-1}$, $A$ is bounded in $\overline{D_\rho} \times \overline{D_\rho}$ and continuous if $\xi \neq \zeta$. The function $b \in C(\overline{D_\rho})$ is given. Then it follows that $w \in C(\overline{D_\rho})$.

**Proof.** Let $\eta \in C(\mathbb{R}_+)$, $0 \leq \eta \leq 1$, such that for given $\epsilon > 0$

$$\eta(t) = \begin{cases} 1 & : 0 \leq t \leq \epsilon/2, \\ 0 & : t \geq \epsilon. \end{cases}$$

Set

$$\frac{A(\xi, \zeta)}{|\xi - \zeta|^\kappa} = K_1(\xi, \zeta) + K_2(\xi, \zeta),$$

where

$$K_1(\xi, \zeta) = \frac{A(\xi, \zeta) \eta(|\xi - \zeta|)}{|\xi - \zeta|^\kappa},$$

$$K_2(\xi, \zeta) = \frac{A(\xi, \zeta) (1 - \eta(|\xi - \zeta|))}{|\xi - \zeta|^\kappa}.$$ Then

$$w(\zeta) - \int_{D_\rho} w(\xi) K_1(\xi, \zeta) d\xi = g(\zeta),$$

with

$$g(\zeta) = \int_{D_\rho} w(\xi) K_2(\xi, \zeta) d\xi + b(\zeta).$$
is a continuous function on $D_\rho$. Define the integral operator $T_1$ from $L^2(D_\rho)$ into $L^2(D_\rho)$ by

$$(T_1 w)(\zeta) = \int_{D_\rho} w(\xi)K_1(\xi, \zeta)\,d\xi,$$

then we can write the above integral equation as $(I - T_1)w = g$, where $I$ denotes the identity operator. The $L^2$-norm of $T_1$ satisfies the inequality $||T_1|| < 1$, provided $\epsilon > 0$ is sufficiently small, see an exercise. It follows that $w$ is given by the Neumann series

$$w = (I - T_1)^{-1}g = \sum_{n=0}^{\infty} T_1^n g,$$

which is a uniformly convergent series of continuous functions, provided $\epsilon > 0$ was chosen sufficiently small, see an exercise. \qed

**Fredholm Theorems.** Here we recall some results from functional analysis, see for example [24]. Let $H$ be a Hilbert space over $\mathbb{C}$ and $T : H \mapsto H$ a completely continuous linear operator. Consider for given $f, g \in H$ and $\lambda \in \mathbb{C}$ the equations

$$u + \lambda Tu = f, \quad (I)$$
$$v + \overline{\lambda} T^* v = g \quad (I^*)$$

and the associated homogeneous equations

$$u + \lambda Tu = 0, \quad (I_h)$$
$$v + \overline{\lambda} T^* v = 0. \quad (I^*_h)$$

Equations $(I^*_h), (I^*_h)$ are called adjoint to $(I), (I_h)$, respectively.

(i) Let $\lambda$ be an eigenvalue of $(I_h)$, then the linear space of solutions has finite dimension.

(ii) The eigenvalue problem $(I_h)$ has at most a countable set of eigenvalues with at most one limit element at infinity.

(iii) $\lambda$ is an eigenvalue of $(I_h)$ if and only if $\overline{\lambda}$ is an eigenvalue of $(I^*_h)$ and $\dim N(I + \lambda T) = \dim N(I + \overline{\lambda} T^*)$.

(iv) $(I)$ has a solution if and only if $f \perp N(I + \overline{\lambda} T^*)$ and $(I^*)$ has a solution if and only if $g \perp N(I + \lambda T)$. 

We recall that $N(A)$ denotes the null space $N(A) = \{ w \in H : Aw = 0 \}$ of a linear operator $A$.

**Proposition 1.4.3.** Suppose that $\partial \Omega \in C^2$, then $\lambda_0 = -2/((n-2)\omega_n)$ is no eigenvalue of the homogeneous integral equation to $(D_i)$.

*Proof.* Suppose that $\lambda_0$ is an eigenvalue and $\mu_0 \in L^2(\partial \Omega)$ an associated eigenvector of the adjoint problem $(N_e)_I$. From Proposition 1.4.2 we have that $\mu_0 \in C(\partial \Omega)$. Consider the single layer potential

$$V(x) := \int_{\partial \Omega} \frac{\mu_0(y)}{|x-y|^{n-2}} \, dS_y.$$ 

From a jump relation of Proposition 1.3.1 it follows that

$$\left( \frac{\partial V(x)}{\partial \nu(x)} \right)_e = 0$$

(1.15)

since $\mu_0$ is an eigenvector of $(N_e)_I$.

Set for a (small) $h > 0$

$$\Omega_h = \Omega \cup \{ y \in \mathbb{R}^n : y = x + s\nu(x), \ x \in \partial \Omega, \ 0 \leq s < h \},$$

see Fig. 1.9.

![Figure 1.9: Parallel surface](https://example.com/fig1.9.png)

The surface $\partial \Omega_h$ is called parallel surface to $\partial \Omega$. 
1.4. INTEGRAL EQUATIONS

Consider a ball $B_R = B_R(0)$ such that $\Omega_h \subset B_R$, then
\[
\int_{B_R \setminus \Omega_h} |\nabla V|^2 \, dx = \int_{\partial B_R} V(x) \frac{\partial V(x)}{\partial \nu(x)} \, dS_x - \int_{\partial \Omega_h} V(z) \frac{\partial V(z)}{\partial \nu(z)} \, dS_z. \tag{1.16}
\]
We have on $\partial \Omega_h$
\[
\nu(z) = \nu(x). \tag{1.17}
\]
To show this equation, we consider the surface $\partial \Omega$ which is given (locally) by $x = x(u)$, where $u \in U$ and $U$ is an (n–1)-dimensional parameter domain. The parallel surface $\partial \Omega_h$ is defined by $z(u) = x(u) + h\nu(x(u))$. Then we consider a $C^1$-curve $X(t)$ on $\partial \Omega$ with $X(0) = x$. Let $Z(t)$ be the associated curve on $\partial \Omega_h$. Then
\[
|X(t) - Z(t)|^2 = h^2.
\]
It follows that
\[
(X(t) - Z(t)) \cdot X'(t) - (X(t) - Z(t)) \cdot Z'(t) = 0,
\]
which proves (1.17) since the first term is zero.

Combining equation (1.16), (1.17), (1.15) and
\[
\lim_{h \to 0} \frac{\partial V(z)}{\partial \nu(x)} = \left( \frac{\partial V(x)}{\partial \nu(x)} \right)_e,
\]
we obtain
\[
\lim_{h \to 0} \int_{B_R \setminus \Omega_h} |\nabla V|^2 \, dx = \int_{\partial B_R} V(x) \frac{\partial V(x)}{\partial \nu(x)} \, dS_x
\]
since the surface element $dS_z$ converges uniformly to $dS_x$ on $U$ as $h \to 0$.\footnote{In the case $\mathbb{R}^3$ we have $dS_z = \sqrt{EG - F^2} du_1 du_2$, where $E = z_{u_1} \cdot z_{u_1}$, $G = z_{u_2} \cdot z_{u_2}$, $F = z_{u_1} \cdot z_{u_2}$ and $z(u) = x(u) + h\nu(x(u))$.}

We have $V = O(R^{2-n})$ and $\partial V/\partial \nu(x) = O(R^{1-n})$. Consequently,
\[
\lim_{R \to \infty} \left( \lim_{h \to 0} \int_{B_R \setminus \Omega_h} |\nabla V|^2 \, dx \right) = 0.
\]
Thus $V = \text{const.}$ on $\mathbb{R}^n \setminus \overline{\Omega}$. From the behavior of $V$ at infinity it follows that $V \equiv 0$ on $\mathbb{R}^n \setminus \overline{\Omega}$. Because of $V \in C(\mathbb{R}^n)$, see Lemma 1.3.1.

From the maximum principle we find that $V \equiv 0$ in $\Omega$ since $\Delta V = 0$ in $\Omega$ and $V = 0$ on $\partial \Omega$. Consequently, the interior regular normal derivative

CHAPTER 1. POTENTIAL THEORY

on \( \partial \Omega \) is zero. Finally the jump relations, see Proposition 1.3.1, imply that 
\( \mu_0(x) \equiv 0 \) on \( \partial \Omega \).

Proposition 1.4.3 and the Fredholm theorems lead to

**Theorem 1.4.1.** Let \( \Omega \) be bounded and \( \partial \Omega \in C^2 \). Then there exists for given \( \Phi, \Psi \in C(\partial \Omega) \) a unique solution of the interior Dirichlet problem \((D_i)\) and the exterior Neumann problem \((N_e)\), respectively.\(^3\)

**Proof.** \( N(I + \lambda_0 T) = N(I + \lambda_0 T^*) = \{0\} \).

**Proposition 1.4.4.** Let \( \Omega \) be bounded and \( \partial \Omega \in C^2 \). The number \( \lambda_0 = 2/(n - 2) \omega_n \) is a simple eigenvalue of \((D_e)_I\) to the eigenvector \( \sigma \equiv 1 \).

**Proof.** From, see Lemma 1.2.2,

\[
\int_{\partial \Omega} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y = -\frac{(n - 2) \omega_n}{2}
\]

we see that \( \lambda_0 \) is an eigenvalue and \( \sigma \equiv 1 \) is an associated eigenvector. From the Fredholm theorems it follows that there exists at least one eigenfunction \( \mu_0(x) \) to \( \lambda_0 \) of \((N_i)_h\). We will show that \( \dim N(I + \lambda_0 T^*) = 1 \). Set

\[
V(x) = \int_{\partial \Omega} \frac{\mu_0(y)}{|x - y|^{n-2}} dS_y.
\]

From the jump relations, see Proposition 1.3.1 and from the fact that \( \mu_0 \) is an eigenvector it follows that \( (\partial V(x)/\partial \nu(x))_i = 0 \). We obtain as in the proof of Proposition 1.4.3 that \( V(x) = \text{const.} =: c_0 \) in \( \Omega \). This constant is different from zero. If not, then \( V = 0 \) on \( \partial \Omega \). Then the maximum principle implies that \( V \equiv 0 \) in \( \mathbb{R}^n \setminus \bar{\Omega} \). We recall that \( V = O(|x|^{2-n}) \) as \( |x| \to \infty \). Consequently, we have \( (\partial V(x)/\partial \nu(x))_e = 0 \), which implies that \( \mu_0 = 0 \), see the jump relations of Proposition 1.3.1.

Let \( \mu_1 \) be another eigenvector to \( \lambda_0 \), then we can assume that \( \mu_1 \in C(\partial \Omega) \) according to the regularity result Proposition 1.4.2. Set

\[
V_1(x) = \int_{\partial \Omega} \frac{\mu_1(y)}{|x - y|^{n-2}} dS_y.
\]

\(^3\)In this and in the following two theorems it is sufficient to assume that \( \partial \Omega \in C^{1,1} \) by the *Rademacher Theorem*: a Lipschitz continuous function is differentiable almost everywhere, see for example [5], pp. 280 or [6].
As above, we conclude that $V_1(x) \equiv \text{const.} = c_1$ in $\Omega$, where $c_1 \neq 0$. The linear combination $\mu_2 := c_1\mu_0 - c_0\mu_1$ is contained in the null space $N(I + \lambda_0 T^*)$. Set

$$V_2(x) = \int_{\partial\Omega} \frac{\mu_2(y)}{|x - y|^{n-2}} \, dS_y = c_1V_0(x) - c_0V_1(x).$$

In $\Omega$ we have $V_2(x) = c_1c_0 - c_0c_1$, and from the jump relations we find as above that $\mu_2(x) \equiv 0$, then

$$\mu_1(x) = \frac{c_1}{c_0} \mu_0(x).$$

From the Fredholm theorems it follows

**Theorem 1.4.2.** Let $\Omega$ be bounded and $\partial\Omega \in C^2$. Then there exists a solution of ($N_i$) if and only if

$$\int_{\partial\Omega} \Psi(y) \, dS_y = 0.$$

In fact, we obtain also the existence of a solution of ($D_e)_1$ under the assumption that

$$\int_{\partial\Omega} \mu_0(y)\Phi(y) \, dS_y = 0,$$

where $\mu_0$ is the eigenvector from above. It turns out that there is a solution of the exterior Dirichlet problem without this restriction if we look for solutions with a weaker decay at infinity. We make the ansatz of a sum of a double layer and a single layer potential

$$u(x) = \int_{\partial\Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) \, dS_y + d \int_{\partial\Omega} \frac{\mu_0(y)}{|x - y|^{n-2}} \, dS_y,$$

where $d$ is a constant which we will determine later. The ansatz defines a solution of the exterior Dirichlet problem if and only if

$$\lim_{y \to x, \ y \in \mathbb{R}^n \setminus \Omega} u(y) = \phi(x),$$
where \( x \in \partial \Omega \). From a jump relation of Proposition 1.2.1 we see that the unknown density \( \sigma \) must satisfy the integral equation

\[
\Phi(x) = \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y \\
+ \frac{(n - 2)\omega_n}{2} \sigma(x) + d \int_{\partial \Omega} \frac{\mu_0(y)}{|x - y|^{n-2}} dS_y.
\]

Above we have shown that, if \( x \in \partial \Omega \),

\[
\int_{\partial \Omega} \frac{\mu_0(y)}{|x - y|^{n-2}} dS_y = \text{const.} = c_0
\]

with a constant \( c_0 \neq 0 \). Thus we have to consider the integral equation

\[
\frac{(n - 2)\omega_n}{2} \sigma(x) + \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y = \Phi(x) - dc_0.
\]

This equation has a solution if

\[
\int_{\partial \Omega} (\Phi(x) - dc_0) \mu_0(x) dS_x = 0
\]

is satisfied. We can find an appropriate constant \( d \) such that this equation is satisfied since

\[
\int_{\partial \Omega} \mu_0(x) dS_x \neq 0.
\]

This inequality is a consequence of a jump relation and of the fact that \( \mu_0 \) is an eigenvector:

\[
\left( \frac{\partial V_0(x)}{\partial \nu(x)} \right)_e = -\frac{(n - 2)\omega_n}{2} \mu_0(x) + \int_{\partial \Omega} \mu_0(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y,
\]

\[
0 = \frac{(n - 2)\omega_n}{2} \mu_0(x) + \int_{\partial \Omega} \mu_0(y) \frac{\partial}{\partial \nu(x)} \left( \frac{1}{|x - y|^{n-2}} \right) dS_y,
\]

which implies that

\[
\left( \frac{\partial V_0(x)}{\partial \nu(x)} \right)_e = -(n - 2)\omega_n \mu_0(x).
\]

Suppose that

\[
\int_{\partial \Omega} \mu_0(x) dS_x = 0,
\]
then
\[ \int_{\partial \Omega} \left( \frac{\partial V_0(x)}{\partial \nu(x)} \right) \, dS_x = 0, \]
which implies, see the proof of Proposition 1.4.3 for notations,
\[ \int_{B_R \setminus \Omega} |\nabla V_0|^2 \, dx = \int_{\partial B_R} V_0(x) \frac{\partial V_0(x)}{\partial \nu(x)} \, dS_x - \int_{\partial \Omega} h V_0(z) \frac{\partial V_0(z)}{\partial \nu(z)} \, dS_z, \]
Letting \( h \to 0 \) and \( R \to \infty \), it follows \( V_0 = \text{const.} \) in \( \mathbb{R}^n \setminus \overline{\Omega} \) and \( V_0 = c_0 \) since \( V_0(x) = \text{const.} = c_0 \) on \( \partial \Omega \).
From the decay behavior of \( V_0 \) at infinity and since \( V \in C(\mathbb{R}^n) \) we find that \( V_0 = 0 \) in \( \mathbb{R}^n \setminus \Omega \). Thus we have \( c_0 = 0 \), a contradiction to a consideration above.
Thus, we have shown

**Theorem 1.4.3.** Let \( \Omega \) be bounded and \( \partial \Omega \in C^2 \). Then for given \( \Phi \in C(\partial \Omega) \) there exists a unique solution \( u \) of \((D_e)\) with the property \( u = O(|x|^{2-n}) \) as \( |x| \to \infty \).

**Proof.** It remains to show that the solution is unique. This follows from the maximum principle. \( \square \)

**Remark.** There is no uniqueness without the decay assumption. Let \( \Omega = B_R(0) \) be a ball in \( \mathbb{R}^3 \). Then \( u = 1/|x| \) and \( u = 1/R \) are two solutions of the Laplace equation with the same boundary values on \( \partial \Omega \).

### 1.5 Volume potential

Set
\[ \Gamma(x,y) = s(|x-y|), \]
where \( s(r) \) is the singularity function
\[
 s(r) := \begin{cases} 
 -\frac{1}{2\pi} \ln r & : \ n = 2, \\
 \frac{r^{n-2}}{(n-2)\omega_n} & : \ n \geq 3. 
\end{cases}
\]
We recall that \( \omega_n = |\partial B_1(0)| \). Let \( \Omega \subset \mathbb{R}^n \) be a bounded and sufficiently regular domain. We define for given \( f \) the volume potential (Newton potential) through
\[ V(x) = \int_{\Omega} \Gamma(x,y) f(y) \, dy. \]
If \( f \) is bounded in \( \Omega \) and \( f \in C^1(\Omega) \), then \( V \in C^2(\Omega) \) and \(-\Delta V = f\) in \( \Omega \), see for instance [17]. This result holds under the weaker assumption that \( f \) is bounded and locally Hölder continuous in \( \Omega \), see Proposition 1.5.1. In fact, also the second derivatives are Hölder continuous (with the same Hölder exponent), see [10], for instance.

**Definition.** Let \( f \) be a real function, defined in a fixed bounded neighborhood \( D \) of \( x_0 \in \mathbb{R}^n \). Then \( f \) is called Hölder continuous at \( x_0 \) if there exists a real number \( \alpha, 0 < \alpha \leq 1 \), such that

\[
[f]_{\alpha,x_0} := \sup_{x \in D \setminus \{x_0\}} \frac{|f(x) - f(x_0)|}{|x - x_0|^{\alpha}} < \infty.
\]

The constant \([f]_{\alpha,x_0}\) is called Hölder constant and \( \alpha \) Hölder exponent.

The function \( f \) is said to be uniformly Hölder continuous with respect to \( x, y \) in \( D \) if

\[
[f]_{\alpha,D} := \sup_{x,y \in D, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty,
\]

and \( f \) is called locally Hölder continuous in a domain \( \Omega \) if \( f \) is uniformly Hölder continuous on compact subsets of \( \Omega \).

In the following we will use the abbreviations \( D_i = \partial / \partial x_i \) and \( D_{ij} = \partial^2 / \partial x_i \partial x_j \).

**Proposition 1.5.1.** (i) Let \( f \) be bounded and integrable over \( \Omega \). Then \( V \in C^1(\mathbb{R}^n) \) and for any \( x \in \Omega \)

\[
D_i V(x) = \int_\Omega D_i \Gamma(x,y) f(y) \, dy.
\]

(ii) Let \( f \) be bounded and locally Hölder continuous in \( \Omega \) with exponent \( 0 < \alpha \leq 1 \). Then \( V \in C^2(\Omega) \), \(-\Delta V = f\) in \( \Omega \), and for any \( x \in \Omega \)

\[
D_{ij} V(x) = \int_{\Omega_0} D_{ij} \Gamma(x,y)(f(y) - f(x)) \, dy + \int_{\partial \Omega_0} D_i \Gamma(x,y)(\nu(y)) j \, dS_y,
\]

where \( \Omega_0 \supset \Omega \) is any domain for which the divergence theorem holds and \( f \) is extended to vanish outside \( \Omega \).
1.5. VOLUME POTENTIAL

Proof. See [10], Chap. 4.

(i) Set for \( x \in \mathbb{R}^n \)

\[ v(x) = \int_\Omega D_i \Gamma(x, y) f(y) \, dy. \]

This function is well defined since \( |D_i \Gamma| \leq |x - y|^{1-n}/\omega_n \) holds. We will show that \( v = D_i V \) and \( v \in C(\mathbb{R}^n) \). Let \( \eta \in C^1(\mathbb{R}) \) be a fixed function satisfying \( 0 \leq \eta \leq 1, 0 \leq \eta' \leq 2, \eta(t) = 0 \) if \( t \leq 1 \) and \( \eta(t) = 1 \) if \( t \geq 2 \). For a (small) \( \epsilon > 0 \) set \( \eta_\epsilon = \eta(|x - y|/\epsilon) \) and consider the regularized potential

\[ V_\epsilon(x) = \int_\Omega \Gamma(x, y) \eta_\epsilon f(y) \, dy. \]

Then \( V_\epsilon \in C^1(\mathbb{R}^n) \) and

\[ v(x) - D_i V_\epsilon(x) = \int_{B_{2\epsilon}(x)} D_i ((1 - \eta_\epsilon) \Gamma) f(y) \, dy. \]

Then we obtain that

\[ |v(x) - D_i V_\epsilon(x)| \leq \sup_\Omega |f| \int_{B_{2\epsilon}(x)} (|D_i \Gamma| + 2|\Gamma|/\epsilon) \, dy \]

\[ \leq \sup_\Omega |f| \left\{ \begin{array}{ll}
4(1 + |\ln(2\epsilon)|)\epsilon & : n = 2, \\
2n\epsilon/(n - 2) & : n \geq 3.
\end{array} \right. \]

It follows that \( V_\epsilon \) and \( D_i V_\epsilon \) converge uniformly on compact subsets of \( \mathbb{R}^n \) to \( V \) and \( v \), resp., as \( \epsilon \to 0 \). Consequently, \( V \in C^1(\mathbb{R}^n) \) and \( D_i V = v \).

(ii) Set for \( x \in \Omega \)

\[ u(x) = \int_{\Omega_0} D_\delta \Gamma(x, y)(f(y) - f(x)) \, dy \\
- f(x) \int_{\partial \Omega_0} D_i \Gamma(x, y)(\nu(y))_j \, dS_y. \]

The right hand side is well-defined since \( f \) is locally Hölder continuous and since \( |D_\delta \Gamma| \leq (1 + n)|x - y|^n/\omega_n \) holds. Set \( v = D_i V \) and define for \( \epsilon > 0 \) the regularized function

\[ v_\epsilon(x) = \int_{\Omega_0} D_i \Gamma \eta_\epsilon f(y) \, dy. \]
Then \(^4\)
\[
D_j v_\epsilon = \int_{\Omega_0} D_j(D_i \Gamma \eta_\epsilon) f(y) \, dy \\
= \int_{\Omega_0} D_j(D_i \Gamma \eta_\epsilon)(f(y) - f(x)) \, dy \\
+ f(x) \int_{\Omega_0} D_j(D_i \Gamma \eta_\epsilon) \, dy \\
= \int_{\Omega_0} D_j(D_i \Gamma \eta_\epsilon)(f(y) - f(x)) \, dy \\
- f(x) \int_{\partial \Omega_0} D_i \Gamma(\nu(y))_j \, dS_y,
\]
provided that \(\epsilon > 0\) is small enough such that \(\eta_\epsilon = 1\) on \(\partial \Omega_0\), see Fig. 1.10.

\[\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.10.png}
\caption{Proof of Proposition 1.5.1}
\end{figure}\]

Then
\[
u(x) - D_j v_\epsilon(x) = \int_{B_{2 \epsilon}(x)} D_j((1 - \eta_\epsilon) D_i \Gamma) (f(y) - f(x)) \, dy.
\]

We suppose that \(2\epsilon < \text{dist} (x, \partial \Omega)\) if \(x \in \Omega_c, \Omega_c \subset \Omega\). Then
\[
|\nu(x) - D_j v_\epsilon(x)| \leq \int_{B_{2 \epsilon}(x)} (|D_{ij} \Gamma| + 2|D_i \Gamma|/\epsilon) |x - y|^\alpha \, dy \\
\leq |f|_{\alpha, \Omega_c} (4 + n/\alpha)(2\epsilon)^\alpha.
\]

\(^4\)We recall that \((\partial/\partial x_j) h(|x - y|) = -(\partial/\partial y_j) h(|x - y|)\), where \(h\) is a differentiable function.
It follows that $V \in C^2(\Omega)$ and $u = D_{ij}V$ since $D_i v \varepsilon$ converges to $u$ uniformly on compact subsets of $\Omega$.

Set in formula (1.18) $\Omega_0 = B_R(x)$, $R$ sufficiently large, then

$$
\Delta V = -f(x) \int_{\partial B_R(x)} \sum_{i=1}^{n} D_i \Gamma(x,y)(\nu(y))_i \ dS_y
$$

$$
= -f(x) \frac{1}{\omega_n R^{n-1}} \int_{\partial B_R(x)} \nu(y) \cdot \nu(y) \ dS_y
$$

$$
= -f(x).
$$

From Proposition 1.5.1 and Theorem 1.4.1 we obtain

**Theorem 1.5.1.** Let $\Omega$ be a bounded domain with $\partial \Omega \in C^2$, $f$ be bounded and locally Hölder continuous in $\Omega$ and $\Phi \in C(\partial \Omega)$. Then there exists a unique solution $u \in C^2(\Omega) \cap C(\Omega)$ of the Dirichlet problem $-\Delta u = f$ in $\Omega$, $u = \Phi$ on $\partial \Omega$.

**Proof.** Set $u = V + w$, where

$$
V(x) = \int_{\Omega} \Gamma(x,y) f(y) \ dy.
$$

Then $u$ is a solution of the Dirichlet problem if and only if $\Delta w = 0$ in $\Omega$ and $w = \Phi - V$ on $\partial \Omega$. The existence of a $w$ follows from Theorem 1.4.1. The uniqueness of $u$ is a consequence of the maximum principle. Moreover, $w$ is given by a dipole potential, see Theorem 1.4.1. \qed
1.6 Exercises

1. Let $S$ be a surface in $\mathbb{R}^3$ defined by $z = f(x)$, $x = (x_1, x_2)$, $x \in U$, where $U$ is a neighborhood of $x = 0$. Assume that $f \in C^{1,\lambda}(U)$, $f(0) = 0$, $\nabla f(0) = 0$. Consider the intersection $I = S \cap \partial B_{\rho_0}(0)$, $\rho_0 > 0$ sufficiently small, i. e.,

$I = \{(x, z) : z = f(x) \text{ and } x_1^2 + x_2^2 + z^2 = \rho_0^2\}$.

Show that there is a function $\epsilon(\rho, \phi)$, $0 < \rho \leq \rho_0$, $\phi \in [0, 2\pi)$, $2\pi$-periodic in $\phi$ and in $C^1$ with respect to $\phi$, such that $\epsilon(\rho, \phi) = O(\rho^\lambda)$ as $\rho \to 0$, uniformly in $\phi \in [0, 2\pi)$, and

$x_1 = \rho(1 + \epsilon(\rho, \phi)) \cos \phi$
$x_2 = \rho(1 + \epsilon(\rho, \phi)) \sin \phi$.

*Hint:* implicit function theorem.

2. Let $B_{2R}(x_0) \subset \mathbb{R}^n$ be a ball with radius $2R$ and the center at $x_0$. Show that there is a function $\eta \in C^\infty_0(B_{2R}(x_0))$ which satisfies $0 \leq \eta \leq 1$ in $B_{2R}(x_0)$ and $\eta \equiv 1$ in $B_R(x_0)$.

*Hint:* set $r = |x|$ and define

$\phi(r) = e^{-1/((3/2R) - r^2 - r^2/4)}$

if $R < r < 2R$ and set $\phi(r) = 0$ if $0 < r < R$ or $r \geq 2R$. Let

$\psi(r) = \frac{\int_0^r \phi(t) \, dt}{\int_0^\infty \phi(t) \, dt}$

and $\chi(r) = 1 - \psi(r)$. Show that $\eta(x) = \chi(|x - x_0|)$ is a function which satisfies the above properties.

3. Suppose that $\partial \Omega \in C^{1,\lambda}$ and $x \in \partial \Omega$. Show that

$$
\lim_{\rho \to 0} \int_{\partial \Omega \setminus B_\rho(x)} \frac{\partial}{\partial \nu(y)} \left( \frac{1}{|z - y|^{n-2}} \right) \, dS_y = 0
$$

*Hint:* local coordinates and formula (1.7).
4. Let \( \Omega \subset \mathbb{R}^n \) be a bounded and sufficiently regular domain and set

\[
q(\zeta) = \int_{\Omega} \frac{A(\xi, \zeta)}{|\xi - \zeta|^{n-\lambda}} \, d\xi,
\]

where \( A \) is bounded in \( \overline{\Omega} \times \overline{\Omega} \) and continuous if \( \xi \neq \zeta \) and \( 0 < \lambda \leq 1 \). Show that \( q \in C(\overline{\Omega}) \).

*Hint:* let \( \eta \in C(\mathbb{R}) \) be a fixed function satisfying \( 0 \leq \eta \leq 1 \), \( \eta(t) = 0 \) if \( t \leq 1 \) and \( \eta(t) = 1 \) if \( t \geq 2 \). For (small) \( \epsilon > 0 \) set \( \eta_\epsilon = \eta(|\zeta - \xi|/\epsilon) \).

Then consider the regularized function

\[
q_\epsilon(\zeta) = \int_{\Omega} \eta_\epsilon \frac{A(\xi, \zeta)}{|\xi - \zeta|^{n-\lambda}} \, d\xi
\]

and prove that \( q_\epsilon \) converges uniformly to \( q \) in \( \Omega \).

5. Show that

\[
||K_1 w||_{L^2(D_\rho)} \leq c\epsilon^\lambda ||w||_{L^2(D_\rho)}.
\]

For the definition of \( K_1 \) see the proof of Proposition 1.4.2.

6. Let \( g \in C(\overline{D_\rho}) \). Prove that

(i) \( |K_1^l g| \leq (c\epsilon^\lambda)^l \max_{\overline{D_\rho}} |g(\zeta)| \).

(ii) \( K_1^l g \) are continuous on \( \overline{D_\rho} \).

(iii) \( \sum_{l=1}^{\infty} K_1^l g \) is uniformly convergent on \( \overline{D_\rho} \), provided that \( \epsilon > 0 \) is small enough.

For the definition of \( K_1 \) see the proof of Proposition 1.4.2.

7. The solution \( u \) of the interior Dirichlet problem \( \triangle u = 0 \) in \( \Omega \) and \( u = \Phi \) on \( \partial \Omega \), where \( \Omega \subset \mathbb{R}^2 \) and \( \Phi \in C(\partial \Omega) \), is given by

\[
u
u(x) = -\int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \ln(|x - y|) \, dS_y.
\]

Here is \( \sigma(x) \), \( x \in \partial \Omega \), the solution of the integral equation

\[
\pi \sigma(x) + \int_{\partial \Omega} \sigma(y) \frac{\partial}{\partial \nu(y)} \ln(|x - y|) \, dS_y = -\Phi.
\]
Find the density $\sigma$ if $\Omega$ is a disk $B_R(0)$.

Hint: show that
\[
\frac{\partial}{\partial \nu(y)} \ln(|x-y|) = \frac{1}{2R}
\]
if $x, y \in \partial B_R(0)$. This formula is a consequence of
\[
\frac{\partial}{\partial \nu(y)} \ln(|x-y|) = \frac{1}{|y-x|^2} (y-x) \cdot \nu(y)
= \frac{1}{|y-x|} \cos \beta,
\]
see Fig. 1.11 for notations.

Figure 1.11: Notations to the exercise

8. Show that
\[
C^\alpha[a,b] := \{ u \in C[a,b] : ||u||_\alpha < \infty \},
\]
where $-\infty < a < b < \infty$ and $0 < \alpha \leq 1$, defines a Banach space, where
\[
||u||_\alpha := \max_{x \in [a,b]} |u(x)| + \sup_{x,y \in [a,b], x \neq y} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}}.
\]

9. Show that $C^\infty[a,b]$ is not dense in $C^\alpha[a,b]$, i.e., there is a $u \in C^\alpha[a,b]$ such that no sequence $u_n \in C^\infty[a,b]$ exists such that $||u_n - u||_\alpha$ tends to zero.

Hint: consider $u = \sqrt{x}$ and $[a,b] = [0,1]$. We have $\sqrt{x} \in C^{1/2}[0,1]$. Assume that
\[
\sup_{x,y \in [a,b], x \neq y} \frac{|u_n(x) - u(x) - (u_n(y) - u(y))|}{|x-y|^{\alpha}} \to 0
\]
if \( n \to \infty \), where \( \alpha = 1/2 \). Then
\[
\left| \frac{u_{n_0}(x) - u_{n_0}(0)}{\sqrt{x}} - 1 \right| \leq \epsilon
\]
for a given \( \epsilon > 0 \) and an integer \( n_0 = n_0(\epsilon) \).

10. Show that \( C^\infty_0(a,b) \) is not dense in \( C^\alpha[a,b] \).

Hint: consider \((a,b) = (-1,1)\) and
\[
u(x) = \begin{cases} \sqrt{x} : & 0 \leq x \leq 1 \\ 0 : & -1 \leq x \leq 0 \end{cases}.
\]
CHAPTER 1. POTENTIAL THEORY
Chapter 2

The Perron method

The Perron method is an existence theory for second order linear or quasi-linear elliptic equations based on maximum principles. In this chapter we consider the Dirichlet problem for the Laplace equation $\triangle u = 0$ in $\Omega$ and $u = \Phi$ on $\partial \Omega$, where $\Omega \subset \mathbb{R}^n$ is bounded and connected, and $\Phi$ is a given function defined on $\partial \Omega$.

In contrast to many other existence theories the Perron method provides results under rather weak assumptions on the boundary $\partial \Omega$ since the problem of existence is separated from the question of the boundary behavior.

2.1 A maximum principle

We know that a harmonic function $u$ must be a constant if $u$ achieves its supremum or infimum in a connected domain. This result is a consequence of the mean value formula for harmonic functions, see [17], Chap. 7, for instance. Fortunately, there is a related principle for functions which satisfy $\triangle u \geq 0$ or $\triangle u \leq 0$ throughout in $\Omega$.

**Lemma 2.1.1** (Mean value theorems). Suppose that $u \in C^2(\Omega)$ satisfies $\triangle u = 0$, $\triangle u \geq 0$, $\triangle u \leq 0$ in $\Omega$, resp. Then for any ball $B = B_R(x) \subset \subset \Omega$

\[
\begin{align*}
    u(x) &= (\leq, \geq) \frac{1}{|\partial B|} \int_{\partial B} u(y) \, dS_y, \\
    u(x) &= (\leq, \geq) \frac{1}{|B|} \int_B u(y) \, dy.
\end{align*}
\]

(2.1) (2.2)
CHAPTER 2. THE PERRON METHOD

Proof. See [10], Chap. 2, for instance. Let \( \rho \in (0, R) \) and \( B_\rho = B_\rho(x) \), then

\[
\int_{B_\rho} \Delta u \, dy = \int_{\partial B_\rho} \frac{\partial u}{\partial \nu(y)} \, dS_y
\]

\[
= (\geq, \leq) 0,
\]

respectively. Here \( \nu(y) \) is the exterior unit normal at \( y \) on \( \partial B_\rho \).

Set \( r = |x - y|, \omega = (y - x)/r \), then \( u(y) = u(x + r\omega) \). Thus

\[
\int_{\partial B_\rho} \frac{\partial u}{\partial \nu(y)} \, dS_y = \int_{\partial B_\rho} u_{yi}(x + \rho\omega)\omega_i \, dS_y
\]

\[
= \int_{\partial B_\rho} \frac{\partial u(x + r\omega)}{\partial r} \bigg|_{r=\rho} \, dS_y
\]

\[
= \rho^{n-1} \int_{\partial B(0)} \frac{\partial u(x + r\omega)}{\partial r} \bigg|_{r=\rho} \, d\omega
\]

\[
= \rho^{n-1} \frac{\partial}{\partial \rho} \int_{\partial B(0)} u(x + \rho\omega) \, d\omega
\]

\[
= \rho^{n-1} \frac{\partial}{\partial \rho} \left( \rho^{1-n} \int_{\partial B_\rho} u(y) \, dS_y \right)
\]

\[
= (\geq, \leq) 0.
\]

Consequently, for any \( \rho \in (0, R) \)

\[
\frac{\partial}{\partial \rho} \left( \rho^{1-n} \int_{\partial B_\rho} u(y) \, dS_y \right) = (\geq, \leq) 0.
\]

It follows that

\[
\rho^{1-n} \int_{\partial B_\rho} u(y) \, dS_y = (\leq, \geq) R^{1-n} \int_{\partial B_R} u(y) \, dS_y
\]

or

\[
\frac{1}{|\partial B_\rho|} \int_{\partial B_\rho} u(y) \, dS_y = (\leq, \geq) \frac{1}{|\partial B_R|} \int_{\partial B_R} u(y) \, dS_y.
\]

Letting \( \rho \to 0 \), we obtain

\[
u(x) = (\leq, \geq) \frac{1}{|\partial B_R|} \int_{\partial B_R} u(y) \, dS_y.
\]

Formula (2.2) follows from

\[
\rho^{n-1} \omega_n u(x) = (\leq, \geq) \int_{\partial B_\rho} u(y) \, dS_y,
\]
2.1. A MAXIMUM PRINCIPLE

where \(0 < \rho \leq R\). We recall that \(\omega_n = |\partial B_1(0)|\). Integrating over \((0, R)\), we obtain

\[
\frac{\omega_n R^n}{n} u(x) = (\leq, \geq) \int_{B_R} u(y) \, dy,
\]

which is formula (2.2) since \(|B_R| = \frac{R^n}{n\omega_n}\).

As a consequence of Lemma 2.1.1 we get the following generalization of the maximum principle for harmonic functions.

Theorem 2.1.1 (Strong maximum principle). Assume that \(\Omega \subset \mathbb{R}^n\) is a connected domain and \(u \in C^2(\Omega)\). Let \(\Delta u \geq 0\) (\(\Delta u \leq 0\)) in \(\Omega\) and suppose there exists a point \(y \in \Omega\) for which \(u(y) = \sup_{\Omega} u\) (\(u(y) = \inf_{\Omega} u\)). Then \(u\) is a constant.

Proof. Consider the case that \(\Delta u \geq 0\) in \(\Omega\). Let \(x_0 \in \Omega\) such that

\[
M := u(x_0) = \sup_{x \in \Omega} u(x).
\]

Set \(\Omega_1 = \{x \in \Omega : u(x) = M\}\) and \(\Omega_2 = \{x \in \Omega : u(x) < M\}\). The set \(\Omega_1\) is not empty and the set \(\Omega_2\) is open since \(u \in C(\Omega)\). Consequently, \(\Omega_2\) is empty if we can show that \(\Omega_1\) is an open set. Let \(y \in \Omega_1\), then there is a \(\rho_0 > 0\) such that \(B_{\rho_0}(y) \subset \Omega\) and \(u(x) = M\) for all \(x \in B_{\rho_0}(y)\). If not, then there are \(\rho > 0, z \in \Omega\) such that \(|z - y| = \rho, 0 < \rho < \rho_0\) and \(u(z) < M\). From Lemma 2.1.1 we have

\[
M \leq \frac{1}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(y)} u(x) \, dS_x < \frac{M}{\omega_n \rho^{n-1}} \int_{\partial B_\rho(y)} u(x) \, dS = M,
\]

which is a contradiction. \(\square\)

Corollary. Assume that \(\Omega\) is connected and bounded and \(u \in C^2(\Omega) \cap C(\overline{\Omega})\) satisfies \(\Delta u \geq 0\) (\(\Delta u \leq 0\)) in \(\Omega\). Then \(u\) achieves its maximum (minimum) on the boundary \(\partial \Omega\).

Corollary. Assume that \(\Omega\) is connected and bounded and \(v, w \in C^2(\Omega)\) \((C(\overline{\Omega}))\) satisfy \(\Delta v \geq \Delta w\) in \(\Omega\) and \(v \leq w\) on \(\partial \Omega\). Then \(v \leq w\) in \(\Omega\).

Proof. Exercise.
2.2 Subharmonic, superharmonic functions

A function \( u \in C^2(\Omega) \) is called subharmonic (superharmonic) in a domain \( \Omega \subset \mathbb{R}^n \) if \( \Delta u \geq 0 \) (\( \Delta u \leq 0 \)) in \( \Omega \). It turns out that we can define superharmonic and subharmonic functions if \( u \) merely is in \( C(\Omega) \).

**Definition.** A function \( u \in C(\Omega) \) is called subharmonic (superharmonic) in \( \Omega \) if for every ball \( B \subset \subset \Omega \) and every function \( h \) harmonic in \( B \), i.e., \( h \in C^2(B) \cap C(\overline{B}) \) and \( \Delta h = 0 \) in \( B \), satisfying \( u \leq h \) (\( u \geq h \)) on \( \partial B \) we have \( u \leq h \) (\( u \geq h \)) in \( B \).

**Corollary.** A harmonic function in \( \Omega \) is both a superharmonic and a subharmonic function. In particular, constants are super- and subharmonic.

**Remark.** A function \( u \) in the class \( C^2(\Omega) \) is subharmonic (superharmonic) if \( \Delta (u - h) \geq 0 \) (\( \Delta (u - h) \leq 0 \)) in \( B \) for any harmonic function \( h \) in \( B \) such that \( u \leq h \) (\( u \geq h \)) on \( \partial B \).

**Lemma 2.2.1** (Strong maximum principle for sub-, superharmonic functions). Assume \( \Omega \) is connected. If a subharmonic function \( u \) attains its supremum in \( \Omega \), then \( u \equiv \text{const.} \) in \( \Omega \), and if a superharmonic function attains its infimum in \( \Omega \), then \( u = \text{const.} \) in \( \Omega \).

**Proof.** Consider the case of a subharmonic function. Let \( x^0 \in \Omega \) and

\[
M := u(x^0) = \sup_{\Omega} u(x).
\]

We will show that

\[
\Omega_1 = \{ x \in \Omega : u(x) = M \}
\]

is an open set. It is not empty since \( x^0 \in \Omega_1 \). Let \( x^1 \in \Omega_1 \), then \( B_{\rho_0}(x^1) \subset \Omega_1 \) if \( \rho_0 > 0 \) is sufficiently small such that \( B_{\rho_0}(x^1) \subset \Omega \). If not, then there is a \( \rho \), \( 0 < \rho \leq \rho_0 \) and an \( x^2 \in \partial B_{\rho}(x^1) \) such that \( u(x^2) < M \). Consider a function \( h \) harmonic in \( B = B_{\rho}(x^1) \) and \( h = u \) on \( \partial B \). Then, since \( h \) is harmonic in \( B \) and \( u \) is subharmonic in \( \Omega \),

\[
M \geq \max_{\partial B} u = \max_{\partial B} h \geq h(x) \geq u(x),
\]

where \( x \in \overline{B} \). Consequently, \( h(x^1) = M \) since \( u(x^1) = M \). Thus \( h = \text{const.} \) in \( B \) since the harmonic function \( h \) attains its maximum in \( B \). Since \( h(x) = u(x) \) on \( \partial \Omega \) we have \( u(x) = M \) for all \( x \in \partial B \), a contradiction to \( u(x^2) < M \). \( \square \)
2.2. SUBHARMONIC, SUPERHARMONIC FUNCTIONS

The following lemma is a generalization of the comparison principle for\textcolor{red}{u, v \in C^2(\Omega) \cap C(\overline{\Omega}) which says that }\Delta v \leq \Delta u in \Omega and v \geq u on \partial\Omega imply that either v > u throughout \Omega or v \equiv u.\textcolor{red}{\)

Lemma 2.2.2 (Comparison principle) Suppose that \Omega is bounded and connected. Let \( u \in C(\Omega) \) be a subharmonic and \( v \in C(\Omega) \) a superharmonic function with \( u - v \leq 0^1 \) on \( \partial\Omega \). Then either \( v > u \) throughout \( \Omega \) or \( v \equiv u \).

Proof. We will show that \( u - v \equiv \text{const.} \) in \( \Omega \) if \( u - v \) attends a non-negative supremum in \( \Omega \). Let \( x_0 \in \Omega \) such that \( M := \sup_{\Omega} (u - v) = (u - v)(x_0) \geq 0 \).

Set \( \Omega_1 = \{ x \in \Omega : (u - v)(x) = M \} \). This set is not empty since \( x_0 \in \Omega_1 \). We will show that \( \Omega_1 \) is an open set. Let \( x_1 \in \Omega_1 \) and consider a ball \( B_{\rho_0}(x^1) \subset \subset \Omega \). Then \( B_{\rho_0}(x^1) \in \Omega_1 \). If not, then there is a ball \( B = B_{\rho}(x^1) \), \( 0 < \rho \leq \rho_0 \), and an \( x^2 \in \partial B \) such that \( (u - v)(x^2) < M \). Let \( h_1, h_2 \) harmonic in \( B \) with \( h_1 = u \) on \( \partial B \) and \( h_2 = v \) on \( \partial B \). Then, if \( x \in B \),

\[
M \geq \max_{\partial B} (u - v) = \max_{\partial B} (h_1 - h_2)
\geq h_1(x) - h_2(x) \geq u(x) - v(x).
\]

Set \( x = x^1 \), then by assumption \( u(x^1) - v(x^1) = M \) which implies that the harmonic function \( h_1 - h_2 \) attends its maximum in \( B \). Consequently, \( h_1 - h_2 = \text{const.} \) in \( B \). Thus \( u(x) - v(x) = M \) on \( \partial B \) which is a contradiction to \( u(x^2) < M \). We have seen that \( u - v \equiv M \geq 0 \) in \( \Omega \). Finally, the assumption \( u - v \leq 0 \) on \( \partial\Omega \) implies \( u(x) \equiv v(x) \) in \( \Omega \).

Let \( u \) be subharmonic in \( \Omega \), \( B \subset \subset \Omega \) a ball and \( \overline{u} \) harmonic in \( B \) such that \( \overline{u} = u \) on \( \partial B \).

Definition. The function

\[
U(x) = \begin{cases} 
\overline{u} & : x \in B, \\
u(x) & : x \in \Omega \setminus \overline{B}.
\end{cases}
\]

is called harmonic lifting of \( u \) in \( B \).

\[
^1\text{Here } u - v \leq 0 \text{ on } \partial\Omega \text{ means that }
\limsup_{y \to x, \ y \in \Omega, \ x \in \partial\Omega} (u - v) \leq 0
\]
Lemma 2.2.3. $U$ is subharmonic in $\Omega$.

Proof. Let $B' \subset \subset \Omega$ and $h$ harmonic in $B_1$ with $h \geq U$ on $\partial B_1$. We have to show that $h \geq U$ in $B_1$. For some of the following notations see Fig. 2.1.

![Figure 2.1: Proof of Lemma 2.2.3](image)

On $C_1 = \partial B_1 \setminus B$ we have $h \geq U \equiv u$. On $C_3 = \partial B_1 \cap B$ it is, according to the definition of $\pi$, $h \geq U = \pi \geq u$. Combining these inequalities, we find that $h \geq u$ on $\partial B_1$ which implies that $h \geq u$ in $B_1$. Then

$$U \leq h \quad \text{in} \quad B_1 \setminus B \quad (2.3)$$

since $U \equiv u$ in $B_1 \setminus B$. It remains to show that also $U \leq h$ in $B_1 \cap B$. On $\partial (B_1 \cap B)$ we have $h \geq U$, see (2.3) and assumption $h \geq U$ on $B_1$. Since $U \equiv \pi$ in $B \cap B_1$ and $h$ is harmonic in $B_1$ it follows that $h \geq U$ in $B \cap B_1$.

$\Box$

Lemma 2.2.4. Let $u_1, u_2, \ldots, u_N$ be subharmonic in $\Omega$. Then

$$u(x) := \max\{u_1(x), \ldots, u_N(x)\}$$

is also subharmonic in $\Omega$, and if $u_1, \ldots, u_N$ are superharmonic, then $u(x) := \min\{u_1(x), \ldots, u_N(x)\}$ is superharmonic in $\Omega$.

Proof. Exercise.

Definition. Let $\Omega$ be bounded and $\phi$ a bounded function on $\partial \Omega$. A subharmonic function $u \in C(\Omega)$ is called subfunction with respect to $\phi$ if $u \leq \phi$ on $\partial \Omega$, and a superharmonic function $u \in C(\Omega)$ is called a superfunction with respect to $\phi$ if $u \geq \phi$ on $\partial \Omega$. 

Here \( u \leq \phi \) on \( \partial \Omega \) means that
\[
\limsup_{y \to x, \ y \in \Omega, \ x \in \partial \Omega} u(y) \leq \phi(x).
\]

**Lemma 2.2.5.** Suppose \( u \) is a subfunction and \( \overline{u} \) a superfunction with respect to \( \phi \). Then \( u \leq \overline{u} \) in \( \Omega \).

**Proof.** Lemma 2.2.2 and
\[
\limsup_{y \to x, \ y \in \Omega, \ x \in \partial \Omega} (u(y) - \overline{u}(y)) = \limsup_{y \to x, \ y \in \Omega, \ x \in \partial \Omega} (u(y) - \phi(x) + \phi(x) - \overline{u}(y)) \\
\leq \limsup_{y \to x, \ y \in \Omega, \ x \in \partial \Omega} (u(y) - \phi(x)) \\
\quad + \limsup_{y \to x, \ y \in \Omega, \ x \in \partial \Omega} (\phi(x) - \overline{u}(y)) \\
\leq 0.
\]

\( \square \)

**Remark.** The set of subfunctions with respect to \( \phi \) and the set of superfunctions with respect to \( \phi \) are not empty since constants \( \leq \inf_{\partial \Omega} \phi \) are subfunctions and constants \( \geq \sup_{\partial \Omega} \phi \) are superfunctions.

Set
\[
S_\phi = \{ v \in C(\Omega) \text{ subharmonic in } \Omega : \ v \leq \phi \text{ on } \partial \Omega \}.
\]

**Theorem 2.2.1** (Perron, [20]). The function
\[
u(x) := \sup_{v \in S_\phi} v(x)
\]
is harmonic in \( \Omega \).

**Proof.** (i) We have in \( \Omega \) that
\[
\inf_{\partial \Omega} \phi \leq u(x) \leq \sup_{\partial \Omega} \phi.
\]

To show this inequality, let \( v \in S_\phi \), then \( v(x) \leq \phi(x) \leq \sup_{\partial \Omega} \phi \) on \( \partial \Omega \). Since the constant \( \sup_{\partial \Omega} \phi \) is a superfunction with respect to \( \phi \) and \( v \) is a subfunction with respect to \( \phi \) we obtain from Lemma 2.2.5 the inequality \( v(x) \leq \sup_{\partial \Omega} \phi, \ x \in \Omega \). Consequently, \( u(x) \leq \sup_{\partial \Omega} \phi, \ x \in \Omega \). The other
side of the above inequality follows since the constant $\inf_{\partial \Omega} \phi$ is an element of $S_\phi$.

(ii) Let $y \in \Omega$ be fixed. Then there is a sequence $v_n \in S_\phi$ with $\lim_{n \to \infty} v_n(y) = u(y)$. Let $B = B_R(y) \subset \subset \Omega$, $R$ sufficiently small, and let $V_n$ be the harmonic lifting of $v_n$ in $B$. Then

$$V_n \in S_\phi, \quad (2.4)$$

and

$$\lim_{n \to \infty} V_n(y) = u(y). \quad (2.5)$$

Proof of (2.4): That $V_n$ is subharmonic is the assertion of Lemma 2.2.3. Since $V_n = v_n$ on $\partial B$, we have $V_n \leq \phi$ on $\partial B$.

Proof of (2.5): We have

$$v_n(y) \leq V_n(y)$$

since $v_n = V_n$ on $\partial B$, $V_n$ is harmonic in $B$ and $v_n$ is superharmonic. Then

$$u(y) \leq \liminf_{n \to \infty} V_n(y).$$

On the other hand, since $V_n \in S_\phi$, we have

$$V_n(y) \leq \sup_{v \in S_\phi} v(y) = u(y),$$

which implies that

$$\limsup_{n \to \infty} V_n(y) \leq u(y).$$

(iii) For every function $h$ harmonic in $B$ we have,

$$\sup_{B_\rho(y)} |D^\alpha h| \leq C(\rho, R, \alpha, n) \sup_{B_R(y)} |h|, \quad (2.6)$$

where $0 < \rho < R$ and the constant $C$ is finite. This inequality is a consequence of the Poisson formula for the solution of the Dirichlet problem in a ball, see [13, 10, 17], for example. Consequently, for each fixed $\rho$, $0 \leq \rho \leq R$ there exists a sub-sequence $V_{n_k}$ which converges uniformly in $B_\rho(y)$ to a harmonic function $v$. It follows that there is a sub-sequence of $V_n$, denoted also by $V_{n_k}$, which converges uniformly on compact subsets of $B_R(0)$ to a function $v$ harmonic in $B_R(0)$. We have

$$v(x) \leq u(x), \quad x \in B_R(y) \quad (2.7)$$
since $V_{n_k}(x) \leq u(x)$ on $B_R(y)$, see the definition of $u(x)$. At the center $y$ it is, see (2.5),

$$v(y) = u(y). \quad (2.8)$$

(iv) Claim: $v(x) = u(x), \quad x \in B$.

Proof: If not, then there is a $z \in B$ such that $v(z) < u(z)$. Then there exists an $u_0 \in S_\phi$ with $v(z) < u_0(z)$. Set

$$w_k(x) := \max(u_0(x), V_{n_k}(x)).$$

Let $W_k$ be the harmonic lifting of $w_k$ in $B$. A subsequence of $W_k$ converges uniformly on each compact subset of $B$ to a function $w$ harmonic in $B$ such that

$$v(x) \leq w(x) \leq u(x), \quad x \in B = B_R(y). \quad (2.9)$$

These inequalities follow since $w_{n_k}(x) \leq W_{n_k}(x)$, $w_{n_k}(x) \geq V_{n_k}(x)$ and $W_{n_k}(x) \leq u(x)$, where $x \in B$.

Combining equation (2.8) and inequalities (2.9), we obtain

$$v(y) = w(y) = u(y). \quad (2.10)$$

Thus, the harmonic function $v - w$ is less or equal zero in $B$ and zero in the interior point $y \in B$. The strong maximum principle implies that

$$v(x) = w(x), \quad x \in B. \quad (2.11)$$

According to the assumption we have for a $z \in B$

$$v(z) < u_0(z).$$

On the other hand, see the definition of $w_n$ and $W_n$, if $x \in B$ then

$$u_0(x) \leq w_{n_k}(x) \leq W_{n_k}(x),$$

which implies that

$$u_0(x) \leq w(x), \quad x \in B.$$

Summarizing, we have for the particular $z$ under consideration the inequalities

$$v(z) < u_0(z) \leq w(z),$$

which is a contradiction to (2.11). \qed
2.3 Boundary behavior

One of the advantages of the Perron method is that the boundary behavior of solutions is separated from the existence problem.

**Definition.** A \( C(\overline{\Omega}) \)-function \( w = w_\xi \) is called a barrier at \( \xi \in \partial \Omega \) relative to \( \Omega \) if

(i) \( w \) is superharmonic in \( \Omega \),

(ii) \( w > 0 \) in \( \overline{\Omega} \setminus \{\xi\} \) and \( w(\xi) = 0 \).

\( w \) is called a local barrier at \( \xi \in \partial \Omega \) if there is a neighborhood \( \mathcal{N} \) of \( \xi \) such that \( w \) satisfies (i) and (ii) in \( \Omega \cap \mathcal{N} \) instead in \( \Omega \).

Let \( w \) be a local barrier at \( \xi \in \partial \Omega \), then we can define a barrier at \( \xi \in \partial \Omega \) relative to \( \Omega \) as follows. Let \( B = B_R(\xi) \), \( R > 0 \) sufficiently small such that \( B \subset \subset \mathcal{N} \), see Fig. 2.2.

![Figure 2.2: Definition of a local barrier](image)

Set

\[
m = \inf_{N \setminus B} w
\]

We have \( m > 0 \), see assumption (ii) in the above definition.
Lemma 2.2.6. The function
\[ w_0(x) = \begin{cases} \min(m, w(x)) & : x \in \Omega \cap \overline{B}, \\ m & : x \in \Omega \setminus B \end{cases} \]
is a barrier at $\xi$ relative to $\partial \Omega$.

Proof. The property $w_0 \in C(\overline{\Omega})$ follows since $w_0 = m$ on $\Gamma = \partial B \cap \Omega$. If not, then there is an $x_0 \in \Gamma$ with $w(x_0) < m$, which is a contradiction to the definition of $m$. Now we will show that $w_0$ is superharmonic in $\Omega$. This follows since
\[ w_0(x) = \begin{cases} w_1 & : x \in \Omega \cap \overline{B}, \\ w_2 & : x \in \Omega \setminus B, \end{cases} \]
where $w_1(x) = \min(m, w(x))$, $w_2 = m$ and $w_1$, $w_2$ are superharmonic in $\Omega \cap \overline{B}$ and $\Omega \setminus B$, respectively, and since $w_1 = w_2$ on $\Gamma$. To show this, consider a ball $B' \subset \subset \Omega$ located as shown in Fig. 2.3.

![Figure 2.3: A local barrier defines a barrier](image)

Let $h$ be harmonic in $B_1$ with $h \leq w_0$ on $\partial B_1$. We have to show that $h \leq w_0$ in $B_1$. Since $w_0 \leq m$ on $\partial_1 B_1 = \partial B_1 \cap B$ and $w_0 = m$ on $\partial_2 B_1 = \partial B_1 \setminus \partial_1 B_1$, we have $w_0 \leq m$ on $\partial B_1$. Thus, the assumption $h \leq w_0$ on $\partial B_1$ implies that $h \leq m$ on $\partial B_1$. In particular, $h \leq w_0$ in $B_1 \setminus B$ since $w_0 = m$ on $B_1 \setminus B$. Finally we have $h \leq w_0$ in $B \cap B_1$ since $h \leq w_0$ on $B_1 \cap \partial B$ and on $B \cap \partial B_1$. We recall that $w_0 = w_1$ in $\Omega \cap B$ and $w_1$ is superharmonic in $\Omega \cap B$, see Lemma 2.2.4.

Definition. A boundary point is said to be regular if there exists a barrier at that point.

Lemma 2.2.7. Let $u$ be a harmonic function defined in $\Omega$ by the Perron method with boundary data $\phi$. If $\xi$ is a regular point of $\partial \Omega$ and if $\phi$ is
continuous at $\xi$, then
\[
\lim_{x \to \xi, x \in \Omega} u(x) = \phi(\xi).
\]

**Proof.** Fix $\epsilon > 0$. Then there is a $\delta = \delta(\epsilon) > 0$ such that $|\phi(x) - \phi(\xi)| < \epsilon$ for all $x \in \partial\Omega$ satisfying $|x - \xi| < \delta$. Set $M = \sup_{\partial\Omega} |\phi|$. Let $w$ be a barrier at $\xi$. Then there is a $k = k(\epsilon) > 0$ such that $kw(x) > 2M$ if $|x - \xi| \geq \delta$.

**Step 1.** We will show that $\phi(\xi) + \epsilon + kw(x)$ is a superfunction relative to $\phi$.

We recall that $w \in C(\overline{\Omega})$, $w$ is superharmonic in $\Omega$, $w > 0$ in $\overline{\Omega} \setminus \{\xi\}$ and $w(\xi) = 0$. Then
\[
\phi(\xi) + \epsilon + kw(x) \geq \phi(x)
\]
on $\partial\Omega$, since for $x \in \partial\Omega$ with $|x - \xi| \geq \delta$ we have
\[
\phi(\xi) + \epsilon + kw(x) \geq \phi(\xi) + \epsilon + 2M \\
\geq \phi(x),
\]
and for $x \in \partial\Omega$ with $|x - \xi| < \delta$ it is
\[
\phi(x) - \phi(\xi) | \leq \epsilon
\]
since $|\phi(x) - \phi(\xi)| \leq \epsilon$ if $x \in \partial\Omega \cap B_\delta(\xi)$ and $kw(x) \geq 0$.

**Step 2.** Since
\[
u(x) = \sup_{v \in S_\phi} v(x)
\]
and $\phi(\xi) - \epsilon - kw(x)$ is a subfunction relative to $\phi$, we have
\[
\phi(\xi) - \epsilon - kw(x) \leq u(x)
\]
in $\Omega$. The function $\phi(\xi) + \epsilon + kw(x)$ is a superfunction relative to $\phi$, see Step 1. Consequently,
\[
v(x) \leq \phi(\xi) + \epsilon + kw(x)
\]
for all $v \in S_\phi$, see Lemma 2.2.5. This inequality implies that
\[
u(x) \leq \phi(\xi) + \epsilon + kw(x).
\]
Summarizing, we get
\[
|u(x) - \phi(\xi)| \leq \epsilon + kw(x).
\]
2.3. **BOUNDARY BEHAVIOR**

Since \( \lim_{x \to \xi, x \in \Omega} w(x) = 0 \), we obtain finally

\[
\lim_{x \to \xi, x \in \Omega} u(x) = \phi(\xi).
\]

\[\Box\]

**Theorem 2.2.2.** Let \( \Omega \subset \mathbb{R}^n \) be bounded and connected. Then the Dirichlet problem \( \Delta u = 0 \) in \( \Omega \), \( u = \phi \) on \( \partial \Omega \), where \( \phi \in C(\partial \Omega) \) is given, is solvable if and only if the boundary points are all regular.

**Proof.** (i) Assume all points of \( \partial \Omega \) are regular points. Then the assertion follows from previous Lemma 2.2.7.

(ii) Assume the Dirichlet problem is solvable for all continuous \( \phi \in C(\partial \Omega) \). Set \( \phi(x) := |x - \xi| \) and consider the Dirichlet problem with the boundary condition \( u(x) = \phi(x) \) on \( \partial \Omega \). Let \( u_\xi(x) \) be the solution, then \( u_\xi(x) \) is a barrier at \( \xi \). Consequently, all boundary points are regular.

\[\Box\]

### 2.3.1 Examples for local barriers

#### Slit domains in \( \mathbb{R}^2 \)

Let \( \Omega \subset \mathbb{R}^2 \) with a slit along the negative \( x \)–axis at \( x = 0 \) as indicated in Fig. 2.4.

![Figure 2.4: A slit domain](image)

Let \( \text{Ln} \ z := \ln |z| + i\phi, \ -\pi < \phi \leq \pi \). Then

\[ w := -\text{Re} \left( \frac{1}{\text{Ln} z} \right) = -\frac{\ln r}{\ln^2 r + \phi^2} \]

is a local barrier at \( \xi = 0 \). Here \( w(0) \) is defined as the limit \( w(0) := \lim_{z \to 0, z \in \Omega} w(z) \). We have \( w(0) = 0 \) and \( w(x) > 0 \) in \( \Omega \cap B_R(0) \). For higher
dimensions there are counterexamples. One of them was given by Lebesgue, see [4], Part II, p. 272, for example, which shows that sufficiently sharp cusps are not regular at the tip of the cusp, see Fig. 2.5.

**Exterior sphere condition**

We say that $\Omega$ satisfies the *exterior sphere condition* at $\xi \in \partial \Omega$ if there is a sphere $B_R(y) \subset \mathbb{R}^n \setminus \Omega$ such that

$$B_R(y) \cap \Omega = \{\xi\},$$

see Fig. 2.6. Let $\Omega$ satisfies the exterior sphere condition at $\xi \in \partial \Omega$, then

$$w(x) = \begin{cases} R^{2-n} - |x - y|^{2-n} & : n \geq 3 \\ \ln \left( \frac{|x - y|}{R} \right) & : n = 2 \end{cases}$$

is a local barrier at $\xi$. 

---

**Figure 2.5: A cusp boundary point**

**Figure 2.6: Exterior sphere condition**
2.3. BOUNDARY BEHAVIOR

Exterior cone condition

We say that $\Omega$ satisfies an exterior cone condition at $\xi \in \partial \Omega$ if there is a finite circular cone $\mathcal{C}$ with the vertex at $\xi$ such that $\overline{K} \cap \overline{\Omega} = \{\xi\}$, see Fig. 2.7. Let

\[ \mathcal{C} \subset \{(r,\theta) : r > 0, -\alpha < \theta < \alpha\} \]

see Fig. 2.8. Introducing polar coordinates $(r, \theta)$, where

\[ x_1 = r \sin \theta, \quad x_2 = r \cos \theta, \]

we have

\[ w(x) = W(r, \theta) := w(r \cos \theta, r \sin \theta), \]

\[ \Delta w = \frac{1}{r} \frac{\partial}{\partial r} (rW_r) + \frac{1}{r^2} W_{\theta\theta}. \]

Figure 2.7: Exterior cone condition

$\xi$ be the origin and assume the exterior cone property is satisfied at $\xi \in \partial \Omega$. Then we can find a positive constant $\lambda$ and a positive function $f(\theta)$, where $\theta$ is the polar angle, such that

\[ w = r^\lambda f(\theta), \]

\[ r = |x|, \]

is a local barrier at $\xi$.

Two-dimensional domains. Here we find $\lambda$ and $f(\theta)$ as follows. Let

\[ \mathcal{C} \subset \{(r,\theta) : r > 0, -\alpha < \theta < \alpha\}, \]

considering the ansatz

\[ W(r, \theta) = r^\lambda \cos(\mu \theta), \]

where $\lambda$ and $\mu$ are positive constants, then

\[ \Delta w = r^{\lambda-2}(\lambda^2 - \mu^2) \cos(\mu \theta). \]
Consequently, $\triangle w \leq 0$ on $C$ if $\lambda \leq \mu$ and $|\mu\theta| \leq \pi/2$ for all $\theta$ satisfying $\alpha < \theta < 2\pi - \alpha$ ($0 < \alpha < \pi/2$).

Then $w > 0$ if $r > 0$ and $\alpha < \theta < 2\pi - \alpha$. Then $w$ is a local barrier at the origin if we choose $\lambda = \mu$ with a sufficiently small positive $\mu$.

*Higher dimensional case.* Let $\mathcal{M} \subset \partial B_1(0)$ be the manifold as indicated in Fig. 2.9. Consider the eigenvalue problem
\[ -\triangle' v = \nu^2 v \text{ in } \mathcal{M}, \]
\[ v = 0 \text{ on } \partial \mathcal{M}, \]

where \( \triangle' \) is the Laplace-Beltrami differential operator on the unit sphere. We recall that in the two-dimensional case
\[ \triangle' = \frac{\partial^2}{\partial \theta^2} \]
and in the three-dimensional case
\[ \triangle' = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \]

For the definition of the Laplace-Beltrami operator see for example [2] and for n-dimensional polar coordinates see [7], Part III, pp. 395, for instance.

Let \( \nu_1 \) be the first eigenvalue of the above eigenvalue problem. It is known that \( \nu_1 \) is positive, a simple eigenvalue and the associated eigenfunction \( v_1 \) has no zero in \( \mathcal{M} \). Thus, we can assume that \( v_1 > 0 \) in \( \mathcal{M} \). Set
\[ W = Ar^\kappa v_1 \equiv w(x) \]
where \( A \) and \( \kappa \) are positive constants. Then \( w > 0 \) in \( \mathbb{R}^n \setminus \mathcal{C} \) and
\[ \triangle w = Ar^{\kappa - 2} v_1 \left( \kappa (\kappa - 1) + (n - 1) \kappa - \alpha_1 \right), \]
where
\[ \alpha_1 = -\frac{n - 2}{2} + \sqrt{\left(\frac{n - 2}{2}\right)^2 + \nu_1^2}. \]

Consequently, we have \( \triangle w \leq 0 \) in \( \mathbb{R}^n \setminus \mathcal{C} \), provided that \( \kappa > 0 \) is sufficiently small.

### 2.4 Generalizations

The Perron method can be applied to the Dirichlet problem for a more general class of linear elliptic equations of second order, see for instance [10], pp. 102. The previous discussion in the case of the Laplace equation shows that we need a strong maximum principle, the existence of solutions of the Dirichlet problem on a ball with continuous boundary data and some estimates for the derivatives. Then we are able to prove the existence of a solution of the equation in the given domain. The problem of the boundary behavior requires an additional discussion.
2.5 Exercises

1. Let $\Omega \subset \mathbb{R}^n$ be a connected domain. Consider the eigenvalue problem

$$
-\Delta u = \lambda u \quad \text{in } \Omega,
$$

$$
u = 0 \quad \text{on } \partial \Omega.
$$

Suppose that $\lambda_0 \geq 0$ is an eigenvalue and $u_0 \in C^2(\Omega) \cap C(\overline{\Omega})$ an associated eigenfunction satisfying $u_0(x) \geq 0$ in $\Omega$. Show that $u_0(x) > 0$ in $\Omega$.

2. Prove the second corollary to Theorem 2.1.1.

3. Prove Lemma 2.2.4.
Chapter 3

Maximum principles

Maximum principles provide powerful tools for linear and nonlinear elliptic equations of second order. In these notes we consider linear equations.

3.1 Basic maximum principles

Set

\[
Mu = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u + \sum_{i=1}^{n} b_{i}(x) D_{i} u,
\]

\[
Lu = Mu + c(x)u,
\]

where \(a_{i,j}, b_{i}\) and \(c\) are real and defined on a simply connected domain \(\Omega \subset \mathbb{R}^n\). We assume \(a_{ij} = a_{ji}\). Let \(\lambda(x)\) be the minimum of the eigenvalues of the symmetric matrix defined by the coefficients \(a_{ij}\) and let \(\Lambda(x)\) be the maximum of these eigenvalues.

**Definition.** \(L\) is called *elliptic* in \(\Omega\) if \(\lambda(x) > 0\) in \(\Omega\). \(L\) is said to be *strictly elliptic* in \(\Omega\) if \(\lambda(x) \geq \lambda_0 > 0\) in \(\Omega\), where \(\lambda_0\) is a constant. An elliptic \(L\) is called *uniformly elliptic* if \(\Lambda/\lambda\) is bounded in \(\Omega\).

In the following we suppose that \(L\) is at least elliptic, and for each \(i\)

\[
\sup_{x \in \Omega} \frac{|b_{i}(x)|}{\lambda(x)} < \infty. \quad (3.1)
\]

**Theorem 3.1.1** (Weak maximum principle). *Let \(L\) be elliptic in the bounded domain \(\Omega\). Assume that a function \(u \in C^2(\Omega) \cap C(\bar{\Omega})\) satisfies \(Mu \geq 0\) and \(u \leq M\) on \(\partial \Omega\) for some constant \(M > 0\). Then \(u \equiv M\).*
(Mu ≤ 0) in Ω. Then the supremum (infimum) of u on \(\overline{\Omega}\) is achieved on \(\partial\Omega\).

Proof. Assume initially that \(Mu > 0\) in \(\Omega\). Then \(u\) cannot achieve an interior maximum since \(\nabla u(x_0) = 0\) at this point where \(u\) achieves its maximum, and since the matrix \(D^2u(x_0) = [D_{ij}u(x_0)]\) is non-positive (necessary condition of second order). It follows that, see an exercise in [17], for instance,

\[
Mu(x_0) = \sum_{i,j=1}^{n} a_{ij}(x_0)D_{ij}u(x_0) \leq 0
\]

since the matrix \([a_{ij}(x_0)]\) is non-negative (even positive) by assumption. This inequality is a contradiction to our assumption.

For a positive sufficiently large constant \(\gamma\) we calculate

\[
Me^{\gamma x_1} = (\gamma^2 a_{11} + \gamma b_1)e^{\gamma x_1} \\
\geq \lambda(\gamma^2 - \gamma c_1)e^{\gamma x_1} > 0.
\]

We recall that \(a_{11} \geq \lambda\) and \(|b_1|/\lambda \leq c_1\), where \(c_1\) is a positive constant, see assumption (3.1). Consequently, for any \(\epsilon > 0\) we have in \(\Omega\)

\[
M(a + \epsilon e^{\gamma x_1}) > 0.
\]

Using the above result, we conclude that

\[
\sup_{\Omega} (a + \epsilon e^{\gamma x_1}) = \sup_{\partial\Omega} (a + \epsilon e^{\gamma x_1}).
\]

Letting \(\epsilon \to 0\), we obtain

\[
\sup_{\Omega} a = \sup_{\partial\Omega} a.
\]

The next theorem is the strong maximum principle. It follows from the boundary point lemma due to E. Hopf [11]. The proof of this lemma needs the previous weak maximum principle. The strong maximum principle is the essential tool to show existence of a solution of the Dirichlet problem via the Perron method.

**Lemma 3.1.1** (E. Hopf, 1952). Let \(L\) be uniformly elliptic. Assume that \(u \in C^2(\Omega)\) satisfies \(Mu \geq 0\) in \(\Omega\). Let \(x_0 \in \partial\Omega\) and suppose that

(i) \(u\) is continuous at \(x_0\),
(ii) \( u(x_0) > u(x) \) for all \( x \in \Omega \cap B_a(x_0) \) for an \( a > 0 \),

(iii) \( \partial \Omega \) satisfies the interior sphere condition at \( x_0 \).

Then the outer normal derivative of \( u \) at \( x_0 \), if it exists, satisfies

\[
\frac{\partial u}{\partial \nu}(x_0) > 0.
\]

Proof. Let \( B = B_R(y) \) be the ball related to the interior sphere condition, see Fig. 3.1.

Consider the function

\[
v(x) = e^{-\alpha r^2} - e^{-\alpha R^2},
\]

where \( r = |x - y| > \rho \) and \( \alpha \) is a positive constant which we will determine later. A calculation leads to

\[
Mv = e^{-\alpha r^2} \left( 4\alpha^2 \sum_{i,j=1}^{n} a_{ij}(x_i - y_i)(x_j - y_j) 
- 2\alpha \left( \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_i(x_i - y_i) \right) \right) 
\geq e^{-\alpha r^2} \left( 4\alpha^2 \lambda(x) r^2 - 2\alpha \left( \sum_{i=1}^{n} a_{ii} + |b|r \right) \right),
\]

Figure 3.1: Proof of the Hopf boundary point lemma
where \( b = (b_1, \ldots, b_n) \). Since by assumption \( a_{ii}/\lambda \) and \(|b|/\lambda\) are bounded, we may choose \( \alpha \) large enough such that \( Mv \geq 0 \) in the annular domain \( \mathcal{A} := B_R(y) \setminus B_\rho(y) \). Since \( u(x) - u(x_0) < 0 \) on \( \partial B_\rho(y) \) there is a constant \( \epsilon > 0 \) such that \( u(x) - u(x_0) + \epsilon v(x) \leq 0 \) on \( \partial B_\rho(y) \). This inequality is also satisfied on \( \partial B_R(y) \) by assumption on \( u \) and since \( v = 0 \) on \( \partial B_R(y) \).

We have \( M(u(x) - u(x_0) + \epsilon v(x)) = Mu + \epsilon Mv \). Then the weak maximum principle implies that \( u - u(x_0) + \epsilon v \leq 0 \) on \( \partial B_\rho(y) \). This inequality is also satisfied on \( \partial B_R(y) \) by assumption on \( u \) and since \( v = 0 \) on \( \partial B_R(y) \).

We have \( M(u(x) - u(x_0) + \epsilon v(x)) = Mu + \epsilon Mv \). Then the weak maximum principle implies that \( u - u(x_0) + \epsilon v \leq 0 \) on \( \partial B_\rho(y) \). Thus

\[
\frac{\partial u}{\partial \nu}(x_0) \geq -\epsilon V'(R),
\]

provided that the normal derivative exists. \( \square \)

**Remark.** If the normal derivative does not exist, then

\[
\liminf_{x \to x_0} \frac{u(x_0) - u(x)}{|x_0 - x|} > 0,
\]

where the angle between \( x_0 - x \) and the exterior normal \( \nu \) is less then \((\pi/2) - \delta\) for a fixed \( \delta > 0 \).

**Corollary.** Suppose that \( \partial \Omega \) satisfies the interior sphere condition at \( x_0 \in \partial \Omega \), \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) satisfies \( Lu \geq 0 \) and \( u(x_0) > u(x) \) in \( \Omega \cap \mathcal{U} \), where \( \mathcal{U} \) is a neighborhood of \( x_0 \). If additionally \( c \leq 0 \) in \( \Omega \cap \mathcal{U} \) and \( u(x_0) > 0 \) then \( \partial u/\partial \nu(x_0) > 0 \), provided the normal derivative exists.

**Proof.** \( Mu = Lu - cu \geq 0 \) in \( \Omega \cap \mathcal{V} \), where \( \mathcal{V} \) is a neighborhood of \( x_0 \). \( \square \)

In generalization to the strong maximum principle for \( \triangle \) we have

**Theorem 3.1.2** (Strong maximum principle). Let \( L \) be uniformly elliptic. Assume that \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \) satisfies \( Mu \geq 0 \) (\( Mu \leq 0 \)) in a connected domain \( \Omega \), not necessarily bounded. Then if \( u \) achieves its supremum (infimum) in the interior of \( \Omega \), \( u \) is a constant.

**Proof.** Consider the case of a maximum. Assume that \( u \) is not constant and achieves its maximum \( m \) in the interior of \( \Omega \). Set \( \Omega_1 = \{ x \in \Omega : u(x) = m \} \)
and $\Omega_2 = \{ x \in \Omega : u(x) < m \}$. By assumption $\Omega_1$ is not empty. We will show that $\Omega_1$ is open. Then $\Omega_2 = \emptyset$ since we suppose that $\Omega$ is connected. Let $x^1 \in \Omega_1$. Consider a ball $B = B_{2\rho_0}(x^1) \subset \subset \Omega$. If $\Omega_1$ is not open, then there is an $x^2 \in B_{\rho_0}(x^1)$ such that $u(x^2) < m$. Consequently, there is a ball $B_{\rho}(x^2)$, where $0 < \rho \leq |x^2 - x^1|$, and $u(x) < m$ in $B_{\rho}(x^2)$ and there is an $x^3 \in \partial B_{\rho}(x^2)$ such that $u(x^3) = m$. See Fig. 3.2 for notations.

![Figure 3.2: Proof of the strong maximum principle](image)

The Hopf lemma says that $\partial u/\partial \nu > 0$ at $x^3$, where $\nu$ is the exterior normal derivative on $x^3 \in \partial B_{\rho}(x^2)$ at $u(x^3)$ which is a contradiction to the fact that $u$ attains an interior maximum at $x^3$. $\square$

In many cases the assumption $c \equiv 0$ in $\Omega$ is not satisfied. If $c(x) \leq 0$ in $\Omega$, then we have the following corollary to the previous theorem. If $c(x)$ is positive on a subset of $\Omega$, then the situation is more complicated. In this case one has to study an associated eigenvalue problem.

**Corollary.** Let $\Omega$ be a connected domain, not necessarily bounded. Suppose that $L$ is uniformly elliptic and $c(x) \leq 0$ in $\Omega$. Assume that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$ ($Lu \leq 0$). If $u$ achieves its **positive** supremum (**negative** infimum) in the interior of $\Omega$ then $u$ is a constant.

**Proof.** Consider the case of a maximum. Set $m = \sup_\Omega u(x)$ and $\Omega_1 = \{ x \in \Omega : u(x) = m \}$, $\Omega_2 = \{ x \in \Omega : u(x) < m \}$. By assumption $\Omega_1$ is not empty. We show that $\Omega_1$ is an open set. Let $x^1 \in \Omega_1$. Then there is ball $B_{\rho}(x^1) \subset \Omega$
where \( u \) is non-negative. Thus
\[
Mu \equiv Lu - c(x)u \geq 0
\]
in \( B_\rho(x^1) \). The above strong maximum principle (Theorem 3.1.2) says that
\( u(x) = m \) for all \( x \in B_\rho(x^1) \).

From this corollary follows a result which is important for many applications:

**Theorem 3.1.3 (Comparison principle).** Let \( \Omega \) be a bounded and connected domain. Suppose that \( L \) is uniformly elliptic and \( c(x) \leq 0 \) in \( \Omega \). Assume that \( u, v \) are in \( C^2(\Omega) \cap C(\overline{\Omega}) \) and satisfy \( Lu \geq Lv \) in \( \Omega \) and \( u \leq v \) on \( \partial \Omega \). Then \( u \leq v \) in \( \Omega \).

**Proof.** Set \( w = u - v \). Then \( Lw \geq 0 \) in \( \Omega \) and \( w \leq 0 \) on \( \partial \Omega \). From the above corollary we see that \( w \) can not achieve a positive maximum in \( \Omega \).

### 3.1.1 Directional derivative boundary value problem

As an application of the previous corollary we consider a generalization of the Neumann problem. Let \( \Omega \subset \mathbb{R}^n \) be a bounded and connected domain, and assume that \( \partial \Omega \) is sufficiently smooth. Consider
\[
Lu = f, \quad \text{in } \Omega, \quad \tag{3.2}
\]
\[
\frac{\partial u}{\partial \alpha} = \phi \quad \text{on } \partial \Omega, \quad \tag{3.3}
\]
where \( f, \phi \) are given and sufficiently regular, and the direction \( \alpha \) is not tangential on \( \partial \Omega \) at each point of \( \partial \Omega \), see Fig. 3.3.

![Figure 3.3: Directional derivative boundary value problem](image-url)
3.1. BASIC MAXIMUM PRINCIPLES

Proposition 3.1.1. Suppose that $c \leq 0$ in $\Omega$ and let $u_1, u_2 \in C^2(\Omega) \cap C^1(\overline{\Omega})$ are solutions of (3.2), (3.3). Then $u_1 - u_2 \equiv \text{const. in } \Omega$.

Proof. Set $u = u_1 - u_2$, then $Lu = 0$ in $\Omega$ and $\partial u/\partial \alpha = 0$ on $\partial \Omega$. Suppose that $u$ is not constant, then we can assume that

$$\max_{\Omega} u > 0.$$ 

This maximum is achieved at $x_0 \in \partial \Omega$ and $u(x_0) > u(x)$ for all $x \in \Omega$, see the corollary to the strong maximum principle. For the tangential derivative we have $(\partial u/\partial t)(x_0) = 0$, and by assumption implies that $(\partial u/\partial \alpha)(x_0) = 0$. Let $\nu = a \, t + b \alpha$, then

$$(\partial u/\partial \nu)(x_0) = a(x_0)(\partial u/\partial t)(x_0) + b(x_0)(\partial u/\partial \alpha)(x_0) = 0,$$

which is a contradiction to the corollary to the Hopf boundary point lemma which says that $\partial u/\partial \nu > 0$ at $x_0$.

3.1.2 Behavior near a corner

Set $Lu = \sum_{i,j=1}^{2} a_{ij}(x)u_{x_i,x_j}$ and consider the Dirichlet problem

$$Lu = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial \Omega,$$

where $f$ is given. Suppose that the boundary of $\Omega \subset \mathbb{R}^2$ has a corner. Without loss of generality we suppose that the corner is the origin. Set $\Omega_\rho = \Omega \cap B_\rho(0)$. Then we assume that there is a $\rho > 0$ such that in $\Omega_\rho$ we have $a_{ij} = a_{ji}$, $L$ is uniformly elliptic, $a_{ij} \in C^\alpha(\overline{\Omega_\rho})$. An appropriate rotation with center at the origin and a stretching of the axis transforms (3.4), (3.5) into a Dirichlet problem where $a_{ij}(0,0) = \delta_{ij}$. Here we denote the transformed coefficients $a_{ij}(C^{-1}y)$ and the transformed right hand side $f(C^{-1}y)$ by $a_{ij}$ and $f$, resp., and $y$ by $x$. The new domain is denoted by $\Omega_\rho$ again. The new interior angle $\omega$ can be calculated from the original interior angle $\gamma$ and the original coefficients $[a_{ij}(0,0)]$, see an exercise. After this mapping we arrived at the problem

$$Lu = \Delta u + \sum_{i,j=1}^{2} (a_{ij}(x) - \delta_{ij})u_{x_i,x_j} = f \text{ in } \Omega_\rho,$$
$$u = 0 \text{ on } \partial \Omega \cap \overline{\Omega_\rho},$$
where

\[ |a_{ij}(x) - \delta_{ij}| \leq c|x|^\alpha. \]

Suppose that \( \Omega_\rho \) is contained in a domain, see Fig. 3.5, defined by \( 0 < r < \rho \) and \( \theta_1(\rho) \leq \theta \leq \theta_2(\rho) \), where \( r = (x_1^2 + x_2^2)^{1/2} \). Set

\[ \omega = \omega(\rho) = \theta_2(\rho) - \theta_1(\rho) \]

and consider the function

\[ v(\theta) = \sin \left( \left( \frac{\pi}{\omega} - h(\epsilon) \right) (\theta - \theta_1(\rho) + \epsilon) \right), \]

where \( 0 < \epsilon \leq \epsilon_0, \epsilon_0 \) sufficiently small, and \( h(\epsilon) = 2\pi\epsilon/\omega^2 \). There is an \( \epsilon_0 > 0 \) and a positive constant \( c \), independent on \( \epsilon \) such that \( v(\theta) \geq c\epsilon \) for all \( 0 < \epsilon \leq \epsilon_0 \) and \( \theta_1(\rho) \leq \theta \leq \theta_2(\rho) \) (exercise). Set

\[ \kappa = \frac{\pi}{\omega} - h(\epsilon) \]

and consider the function

\[ W(r, \theta) = Ar^{\kappa-\eta}v(\theta) \]

\[ = w(x_1, x_2), \]

where \( A \) and \( \eta \) are constants, \( 0 < \eta < \kappa \). In polar coordinates we have

\[ \Delta w = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial W}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 W}{\partial \theta^2}. \]
Then
\[ \Delta w = A r^{\kappa - \eta - 2} ((\kappa - \eta)^2 - \kappa^2) v \]
and
\[ |w_{x, x_j}| \leq c |A| r^{\kappa - \eta - 2}. \]
The constant \( c \), here and in the following formulas, are independent of \( r \) and \( \theta \). Then
\[ Lw = A r^{\kappa - \eta - 2} ((\kappa - \eta)^2 - \kappa^2) v + O \left( A r^{\kappa - \eta - 2 + \alpha} \right). \]
Suppose that the constant \( A \) is positive, then from the above considerations it follows that for given \( \delta > 0 \), \( 0 < \delta \leq \delta_0 \), \( \delta_0 \) sufficiently small, there are positive constants \( c(\delta) \) and \( \rho = \rho(\delta) \) such that
\[ Lw \leq -A c(\delta) r^{(\pi/\omega) - \delta - 2} \]
in \( \Omega_\rho \).

**Proposition 3.1.2.** Assume that the right hand side \( f \) of (3.4) satisfies
\[ |f| \leq c r^{(\pi/\omega) - 2 - \delta + \tau} \]
in \( \Omega_{\rho_0}, \rho_0 > 0 \), for a \( \tau > 0 \). Assume \( u \in C^2(\Omega_{\rho_0}) \) and \( \sup_{\Omega_{\rho_0}} |u(x)| < \infty \). Then for given (small) \( \epsilon > 0 \) there exists positive constants \( c(\epsilon) \) and \( \rho(\epsilon) > 0 \) such that
\[ |u| \leq c(\epsilon) |x|^{(\pi/\omega) - \epsilon} \]
in \( \Omega_{\rho(\epsilon)} \).

**Proof.** We have \( Lw \leq Lu \) in \( \Omega_\rho \) and \( w \geq u \) on \( \partial \Omega_\rho \), provided \( \rho > 0 \) is sufficiently small. \( \square \)

**Remark.** The additional assumption that \( u \) remains bounded up to the corner is essential for the previous proposition since there exists also solutions which are unbounded near the corner. An example is the boundary value problem \( \Delta u = 0 \) in \( \Omega_\alpha \), \( u = 0 \) on \( \partial \Omega_\alpha \), where \( \Omega_\alpha \) is the sector defined by \( r > 0 \) and \( 0 < \theta < \alpha \), where \( 0 < \alpha < 2\pi \). Solutions are given by
\[ u(x) = r^{\pi k/\alpha} \sin ((\pi/\alpha)k\theta), \]
where \( k \in \{ \pm 1, \pm 2, \ldots \} \). For a class of quasilinear non-uniformly boundary value problems the behavior of the solution near the corner does not require such an additional assumption near the boundary, see [8, 16, 3, 18].
The behavior follows from the problem itself. The reason for this striking difference is that essential information is lost through the linearization.

Remark. Asymptotic expansions near a corner yield more precise behavior, see [15] for a class of uniformly quasilinear elliptic Dirichlet problems. In general, the expansion, depends on the solution considered, in contrast to some nonuniform problems, see [16], [19]

3.1.3 An a priori estimate

Consider for given bounded functions $f$, $\Phi$ defined on $\Omega$ and on $\partial\Omega$, respectively, the Dirichlet problem

$$
Lu = \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u + \sum_{i=1}^{n} b_{i}(x) u_{x_{i}} + c(x) = f(x) \quad \text{in } \Omega, \quad (3.8)
$$

$$
u = \Phi \quad \text{on } \partial\Omega, \quad (3.9)
$$

where $a_{i,j}$, $b_{i}$ and $c$ are real and defined on a simply connected and bounded domain $\Omega \subset \mathbb{R}^{n}$. We assume that $a_{ij} = a_{ji}$, $L$ is strictly elliptic and

$$
\sup_{\Omega} |b_{i}(x)| < \infty
$$

for every $i = 1, \ldots, n$. Let $K$ be an upper bound of $-b_{1}$ and set

$$
\alpha = \frac{1}{\lambda_{0}} \left( K + (K^{2} + 4\lambda_{0})^{1/2} \right), \quad (3.10)
$$

where the positive constant $\lambda_{0}$ is a lower bound of the minimum of the eigenvalues of the matrix $[a_{ij}(x)]$. Set $d = \text{diam } \Omega$.

**Proposition 3.1.3.** Suppose that $c(x) \leq 0$ in $\Omega$ and let $u \in C^{2}(\Omega) \cap C(\Omega)$ be a solution of (3.8), (3.9). Then

$$
\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |\Phi| + \left( e^{\alpha d} - 1 \right) \sup_{\Omega} |f|.
$$

**Proof.** Suppose that $\Omega$ is contained in the strip defined by $0 \leq x_{1} \leq d$. Consider the function

$$
g(x) = e^{\alpha d} - e^{\alpha x_{1}},
$$
where $\alpha$ is positive constant which will be determined later. We have $g(x) \geq 0$ if $x \in \Omega$. We get
\[
Lg = -(a_{11}\alpha^2 + b_1\alpha)e^{\alpha x_1} + cg \\
\leq (\lambda_0\alpha^2 + K\alpha)e^{\alpha x_1} \\
\leq -e^{\alpha x_1} \\
\leq -1,
\]
provided that $\alpha$ is large enough. We can choose $\alpha$ given by (3.10). Set
\[
h = \sup_{\partial\Omega} |\Phi| + g(x) \sup_{\Omega} |f|.
\]
Then
\[
Lh = Lg \sup_{\Omega} |f| + c \sup_{\partial\Omega} |\Phi| \\
\leq - \sup_{\Omega} |f|.
\]
We recall that $c \leq 0$ on $\Omega$. Summarizing, we have
\[
Lh \leq - \sup_{\Omega} |f| \quad \text{in} \quad \Omega, \\
h \geq \sup_{\Omega} |\Phi| \quad \text{on} \quad \partial\Omega.
\]
Set $v = u - h$, then
\[
Lv = f - Lh \geq f + \sup_{\Omega} |f| \geq 0 \quad \text{in} \quad \Omega, \\
v = \Phi - \sup_{\partial\Omega} |\Phi| \leq 0 \quad \text{on} \quad \partial\Omega.
\]
The comparison principle says that $v \leq 0$ in $\Omega$. The same argument leads to the inequality $u \geq -h$ in $\Omega$ if we set $v = -u - h$. Then
\[
Lv = -f - Lh \geq -f + \sup_{\Omega} |f| \geq 0 \quad \text{in} \quad \Omega, \\
v = -\Phi - h \leq -\Phi - \sup_{\partial\Omega} |\Phi| \leq 0 \quad \text{on} \quad \partial\Omega.
\]
3.2 A discrete maximum principle

To simplify the presentation we consider here a subclass of elliptic boundary value problems in a domain \( \Omega \subseteq \mathbb{R}^2 \), see [12], pp. 458, for instance. Set

\[
Mu = \Delta u + b_1(x,y)u_x + b_2(x,y)u_y.
\]

Suppose that there is a constant \( K \) such that

\[
\sup_{\Omega} (|b_1(x,y)| + |b_2(x,y)|) \leq K. \tag{3.11}
\]

Let

\[
Lu = Mu + c(x,y)u,
\]

where \( c(x,y) \) is defined on \( \Omega \). Then we consider for given \( f \) defined on \( \Omega \), and \( \Phi \) defined on \( \partial \Omega \), the Dirichlet problem

\[
Lu = f \quad \text{in } \Omega, \tag{3.12}
\]

\[
u = \Phi \quad \text{on } \partial \Omega. \tag{3.13}
\]

Let \( h > 0 \) be a (small) constant. We define an associated difference operator to \( Mu \) by

\[
M_h u = \frac{u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)}{h^2} + b_1(x,y) \frac{u(x+h,y) - u(x-h,y)}{2h} + b_2(x,y) \frac{u(x,y+h) - u(x,y-h)}{2h}.
\]

We have \( \lim_{h \to 0} M_h u = Mu \) on every compact sub-domain of \( \Omega \), provided that \( u \in C^2(\Omega) \).

Set \( P_0 = (x,y) \) and assume \( P_0 \in \Omega \). The four points \( P_{01} = (x+h,y), P_{02} = (x, y+h), P_{03} = (x-h,y), P_{04} = (x, y-h) \) are called \( h \)-neighborhood of \( P_0 \) or a star around \( P_0 \). Consider the intersection of \( \Omega \) with an \( h \)-net \( N_h \) of \( \mathbb{R}^2 \) defined by

\[
N_h(x,y) = \{(x + lh, y + kh) \in \mathbb{R}^2 : k, l = 0, \pm 1, \pm 2, \ldots \},
\]

where \( (x,y) \in \mathbb{R}^2 \) is given. Let \( \Omega_h \) be the set of the points \( P_i \) of \( \Omega \cap N_h \) such that each point of the star associated to \( P_i \) are contained in \( \Omega \). The set of all star points not in \( \Omega_h \) is denoted by \( \partial \Omega_h \), see Fig. 3.5.

Then we define \( \overline{\Omega_h} = \Omega_h \cup \partial \Omega_h \). Suppose that \( \Omega_h \) is connected in the sense that for given \( P, R \in \Omega_h \) there exists points \( Q_1, Q_2, \ldots, Q_s \in \Omega_h \), where \( Q_1 = P \), and \( Q_s = R \), such that \( Q_j \) is a point of the \( h \)-neighborhood
of $Q_{j-1}$, $j = 1, \ldots, s + 1$. We assume that $h > 0$ is sufficiently small such that
\[ hK < 1. \quad (3.14) \]

**Theorem 3.2.1.** Let $u$ be defined on $\overline{\Omega_h}$. Suppose that $M_h u \geq 0$ in $\Omega_h$ and $u$ attains its maximum in $\Omega_h$. Then $u$ is constant on $\overline{\Omega_h}$.

**Proof.** Let $P_i \in \Omega_h$. From the definition of $M_h u$ and the assumption $M_h u \geq 0$ we obtain that
\[ u(P_i) \leq \sum_{j=1}^{4} \lambda_{ij} u(P_{ij}), \]
where $P_{ij}$ are the points of the $h$-neighborhood of $P_i$ and
\[
\lambda_{i1} = \frac{1}{4} \left( 1 + \frac{h b_1(P_i)}{2} \right), \quad \lambda_{i2} = \frac{1}{4} \left( 1 + \frac{h b_2(P_i)}{2} \right), \\
\lambda_{i3} = \frac{1}{4} \left( 1 - \frac{h b_1(P_i)}{2} \right), \quad \lambda_{i4} = \frac{1}{4} \left( 1 - \frac{h b_2(P_i)}{2} \right). 
\]
We have $\sum_{j=1}^{4} \lambda_{ij} = 1$ and, see inequality (3.14), $\lambda_{ij} > 0$. Assume that
\[ m := \max_{\Omega_h} u(x) = u(P_i). \]
Then $u = m$ in all points of the $h$-neighborhood of $P_i$. Since $\Omega_h$ is connected, the theorem is shown. \qed

Set $L_h u = M_h u + c(x) u$. The following corollary is the discrete version of the corollary to the above Theorem 3.2.
Corollary. Suppose that \( c(x) \leq 0 \) on \( \Omega \), and \( u \) defined on \( \overline{\Omega}_h \) satisfies \( L_h u \geq 0 \) on \( \Omega_h \). If \( u \) achieves its non-negative supremum \( m \) in \( \Omega_h \) then \( u \) is constant on \( \overline{\Omega}_h \).

Proof. Set \( \Omega_{h,1} = \{ x \in \Omega : u(x) = m \} \) and \( \Omega_{h,2} = \{ x \in \Omega : u(x) < m \} \).

By assumption \( \Omega_{h,1} \) is not empty. The set \( \Omega_{h,1} \) is open in the sense that if \( P_i \in \Omega_{h,1} \) then \( u(P_{il}) = m \) for every point \( P_{il}, l = 1, 2, 3, 4 \), of the associated star to \( P_i \). This follows from the inequality

\[
    u(P_i) \leq \sum_{j=1}^{3} \lambda_{ij} u(P_{ij}) + \frac{h^2}{4} c(P_i) u(P_i).
\]

Thus \( u = m \) on \( \overline{\Omega}_h \) since \( \Omega_h \) is connected by assumption. \( \square \)

From this corollary it follows

Theorem 3.2.2 (Comparison principle). Let \( \Omega \) be a bounded and connected domain. Suppose that \( c(x) \leq 0 \). Assume that \( u, v \) are defined on \( \overline{\Omega}_h \) and satisfy \( L_h u \geq L_h v \) in \( \Omega_h \) and \( u \leq v \) on \( \partial \Omega_h \). Then \( u \leq v \) on \( \overline{\Omega} \).

Proof. Set \( w = u - v \). Then \( L_h w \geq 0 \) in \( \Omega_h \) and \( w \leq 0 \) on \( \partial \Omega_h \). From the above corollary we see that \( w \) can not achieve a nonnegative maximum in \( \Omega \). \( \square \)

Suppose that \( \Omega_h \) is bounded and connected. We consider the discrete Dirichlet problem

\[
    L_h u = f \quad \text{in} \ \Omega_h, \quad \text{(3.15)}
\]
\[
    u = \Phi \quad \text{on} \ \partial \Omega_h, \quad \text{(3.16)}
\]

where \( f \) is defined in \( \Omega_h \) and \( \Phi \) on \( \partial \Omega_h \). Assume that \( c \leq 0 \) and \( h \) is sufficiently small such that the inequality (3.14) is satisfied.

Corollary. There exists a unique solution of the discrete Dirichlet problem (3.15), (3.16).

Proof. The Dirichlet problem defines a linear system of \( N \) equations in \( N \) unknowns. From the comparison principle it follow that there is at most one solution. From the linear algebra it is known that uniqueness implies existence. \( \square \)
3.2. A DISCRETE MAXIMUM PRINCIPLE

Proposition 3.2.1 (A priori estimate). Assume that \( u \) is a solution of the discrete Dirichlet problem (3.15), (3.16), where \( c(x) \leq 0 \). Then

\[
\max_{\Omega_h} |u| \leq \max_{\partial \Omega_h} |\Phi| + c \max_{\Omega_h} |f|,
\]

where the constant \( c \) is independent of \( u \) and \( h \), and \( 0 < h < h_0, \ h_0 \) sufficiently small, see the following proof for an explicit \( h_0 \).

Proof. The proof is the same as the proof of the apriori estimate of Theorem 3.4. Concerning notations in the following formula see the proof of this theorem. For a sufficiently large \( \alpha \) we have

\[
L_h \left( e^{\alpha d} - e^{\alpha x} \right) = -\alpha^2 e^{\alpha x} \left( \frac{\sinh(\alpha h/2)}{\alpha h/2} \right)^2 - b_1(x) \alpha e^{\alpha x} \frac{\sinh(\alpha h)}{\alpha h} + c(x) \left( e^{\alpha d} - e^{\alpha x} \right) \leq -e^{\alpha x} \left( \alpha^2 \left( \frac{\sinh(\alpha h/2)}{\alpha h/2} \right)^2 + b_1 \alpha \frac{\sinh(\alpha h)}{\alpha h} \right).
\]

Suppose that \( \alpha h \leq 1 \), then

\[
L_h \left( e^{\alpha d} - e^{\alpha x} \right) \leq -e^{\alpha x} \left( \alpha^2 - K \alpha \cosh(1) \right) \leq -1,
\]

if we take an \( \alpha \) which satisfies

\[
\alpha \geq (K/2) \cosh(1) + \sqrt{((K/2) \cosh(1))^2 + 1}.
\]

We recall that we assume that \( \Omega \) is bounded and is contained in the strip \( 0 < x_1 < d \), where \( d \) is the diameter of \( \Omega \).

The following result says that the solution of the discrete Dirichlet problem is an approximation of the solution of the original problem, provided that this solution is sufficiently smooth,

Corollary. Suppose that \( u \in C^3(\overline{\Omega}) \) is a solution of the continuous Dirichlet problem (3.12), (3.13), where \( c(x) \leq 0 \) and \( \Phi \) is defined on a boundary strip and is in \( C^1 \) in the closed strip. Let \( u_h \) be a solution of the discrete problem (3.15), (3.16). Then

\[
\max_{\Omega_h} |u(x) - u_h(x)| \leq ch,
\]
where the constant $c$ is independent on $h < h_0$, $h_0$ sufficiently small.

Proof. Here we make the additional assumption that $\partial \Omega_h \subset \partial \Omega$, see Fig. 3.5 for an example. The proof of the general case is left as an exercise. The assumption on $u$ implies that

$$|L_h u - Lu| \leq ch. \quad (3.17)$$

Let $U$ be defined on $\overline{\Omega_h}$. Then we have from the above the a priori estimate

$$\max_{\Omega_h} |U| \leq \max_{\partial \Omega_h} |U| + c \max_{\Omega_h} |L_h U|.$$ 

Set $U = u - u_h$, then

$$\max_{\Omega_h} |u - u_h| \leq c \max_{\Omega_h} |L_h u_h - L_h u|$$

since $u - u_h = 0$ on $\partial \Omega_h$. Finally, we have on $\Omega_h$ that

$$L_h u_h - L_h u = L_h u_h - Lu + Lu - L_h u = Lu - L_h u$$

since $L_h u_h = f$ and $Lu = f$ on $\Omega_h$. Then the estimate of the corollary follows from the estimate (3.17). \qed
3.3. Exercises

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and assume that $u \in C(\Omega)$ and

$$\sup_{\Omega} (u + \epsilon e^{\gamma x_1}) = \sup_{\partial\Omega} (u + \epsilon e^{\gamma x_1})$$

for each $\epsilon$. Show that

$$\sup_{\Omega} u = \sup_{\partial\Omega} u.$$

2. Let $[a_{ij}]$ be a real regular symmetric matrix in $\mathbb{R}^n$. Find a regular matrix $C$ in $\mathbb{R}^n$ such that

$$\sum_{i,j=1}^{n} a_{ij}u_{x_i x_j} = \sum_{i=1}^{n} U_{y_i y_i},$$

where $U(y) := u(C^{-1}y)$.

*Hint:* let $Z_1, Z_2, \ldots, Z_n$ be an orthonormal system of eigenvectors to $[a_{ij}]$ to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, resp. Set $B = (Z_1, Z_2, \ldots, Z_n)$, then

$$C = \begin{pmatrix}
\frac{1}{\sqrt{\lambda_1}} & 0 & \ldots & 0 \\
0 & \frac{1}{\sqrt{\lambda_2}} & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
0 & \ldots & 0 & \frac{1}{\sqrt{\lambda_n}}
\end{pmatrix} B^T.$$

3. Let $\gamma$ be the interior angle of a sector with its corner at the origin in $\mathbb{R}^2$. Calculate the interior angle $\omega$ of the sector transformed by the above mapping.

4. Let $u \in C^1(\Omega) \cap C^2(\Omega)$ be a solution of

$$\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = f \text{ in } \Omega,$$

$$u = \Phi \text{ on } \partial\Omega,$$

where $\Omega \in \mathbb{R}^2$ and $f, \Phi$ are given. Suppose that the origin is a corner of $\Omega$ with interior angle $\gamma$, $0 < \gamma < \pi$. Show that $\omega$, see the previous exercise, is the opening angle of the surface $S$ defined by $z = u(x_1, x_2)$, over the origin, see Fig. 3.6.

*Hint:*

$$\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = \sum_{i,j=1}^{2} a_{ij}(x)u_{x_i x_j},$$
where
\[ a_{ij} = \frac{1}{(1 + |\nabla u|^2)^{3/2}} \left( \delta_{ij} - \frac{u_{x_i} u_{x_j}}{1 + |\nabla u|^2} \right). \]

5. Let \( \Omega_\alpha \) be the sector in \( \mathbb{R}^2 \) defined by \( r > 0 \) and \( 0 < \theta < \alpha \), where \( 0 < \alpha < 2\pi \). Set \( \Omega_{\alpha,\rho} = \Omega_\alpha \cap B_\rho(0) \). Suppose that \( u \in C^2(\Omega_{\alpha,\rho} \setminus \{0\}) \) and \( \sup_{\Omega_{\alpha,\rho}} |u| < \infty \) is a solution of \( \Delta u = 0 \) in \( \Omega_{\alpha,\rho} \), \( u = 0 \) on \( (\partial \Omega_\alpha \cap B_\rho(0)) \setminus \{0\} \).

Show that there is a constant \( c \) such that
\[ |u| \leq c|x|^{\pi/\alpha} \]
in \( \Omega_{\alpha,\rho} \).

**Hint:** choose the comparison function \( W = A r^{\pi/\alpha} \sin(\pi \theta / \alpha) \) and show that \( -AW \leq u \leq AW \) on \( \Omega_\alpha \cap \partial B_\rho(0) \) provided the positive constant \( A \) is sufficiently large.

6. Let \( m, \lambda_j, c_j \) be real numbers satisfying \( c_j \leq m \), \( \lambda_j > 0 \) and \( \sum_{j=1}^{4} \lambda_j = 1 \). Show that \( m \leq \sum_{j=1}^{4} \lambda_j c_j \) implies that \( c_j = m \).

7. Prove the corollary to Proposition 3.3.

**Hint:** the inclusion \( \partial \Omega_h \subset \partial \Omega \) is not assumed. The result follows since for given \( x \in \partial \Omega_h \) there exists an \( x^1 \in \partial \Omega \) such that \( |x - x^1| < h \).
Bibliography


Index

a priori estimate, 74
a priory estimate
discrete, 79
barrier, 56
local, 56, 59
boundary behavior, 56
boundary point
regular, 57
Boundary value problem
directional derivative, 70
comparison principle, 51, 70
discrete, 78
cone condition
exterior, 61
corner, 71
derivative
interior normal, 21
Dirichlet problem, 8
Fredholm theorems, 31
harmonic lifting, 51
Hopf boundary point lemma, 67
integral equations, 26
jump relations, 19, 24
Laplace equation, 7
maximum principle, 49
discrete, 76
strong, 66
weak, 65
mean value theorem, 47
Neumann problem, 8
Newton
attraction rule, 7
opening angle, 81
partition of unity, 9
Perron method, 47
potential
dipole, 8, 11
double layer, 19
single layer, 8, 20, 25
volume, 37
regularity, 30
sphere condition
exterior, 60
subfunction, 52
subharmonic function, 50
superfunction, 52
superharmonic function, 50
surface
parallel, 32