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Spherical complexities and closed geodesics

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Lusternik-Schnirelmann category and critical points

Definition

For a topological space X and $A \subset X$ put

$$\mathrm{cat}_X(A) := \inf\Big\{ r \in \mathbb{N} \;\Big|\; \exists U_1, \dots, U_r \subset X \text{ open, s.t. } U_j \hookrightarrow X \text{ nullhomotopic } \forall j \text{ and } A \subset \bigcup_{j=1}^{r} U_j \Big\}.$$

$\mathrm{cat}(X) := \mathrm{cat}_X(X)$ is the *Lusternik-Schnirelmann category of* X.

- $\mathrm{cat}(X)$ is a homotopy invariant of X.
- $\mathrm{cat}(X)$ is hard to compute explicitly.

Lusternik-Schnirelmann category and critical points

Theorem (Lusternik-Schnirelmann '34, Palais '65)

Let M be a Hilbert manifold and let $f \in C^{1,1}(M)$ be bounded from below and satisfy the Palais-Smale condition with respect to a complete Finsler metric on M. Then

$$\# \operatorname{Crit} f \geq \operatorname{cat}(M).$$

- There are various generalisations, e.g. generalized Palais-Smale conditions (Clapp-Puppe '86), extensions to fixed points of self-maps (Rudyak-Schlenk '03).
- Advantage over Morse inequalities: No nondegeneracy condition required.

Proposition

Let X be a normal ANR. Put $\nu(A) := \mathrm{cat}_X(A)$.

(1) *(Monotonicity)* $A \subset B \subset X \quad \Rightarrow \quad \nu(A) \leq \nu(B)$.

(2) *(Subadditivity)* $\nu(A \cup B) \leq \nu(A) + \nu(B) \quad \forall A, B \subset X$.

(3) *(Continuity)* *Every* $A \subset X$ *has an open neighborhood* U *with* $\nu(A) = \nu(U)$.

(4) *(Deformation monotonicity)* *If* $\Phi_t : A \to X$, $t \in [0, 1]$, *is a deformation, then* $\nu(\Phi_1(A)) \geq \nu(A)$.

A map $\nu : \mathcal{P}(X) \to \mathbb{N} \cup \{+\infty\}$ satisfying (1)-(4) is called an *index function*.

Method of proof of the Lusternik-Schnirelmann theorem

$f \in C^{1,1}(M)$ bounded from below and satisfies PS condition w.r.t. Finsler metric on M. Put $f^a := f^{-1}((-\infty, a])$. Use properties (1)-(4) and minimax methods to show:

- If $[a, b]$ contains no critical value of f, then

$$\mathrm{cat}_M(f^b) = \mathrm{cat}_M(f^a).$$

- If c is a critical value of f, then

$$\mathrm{cat}_M(f^c) \leq \mathrm{cat}_M(f^{c-\varepsilon}) + \mathrm{cat}_M(\mathrm{Crit}\ f \cap f^{-1}(\{c\})).$$

Combining these observations yields

$$\mathrm{cat}_M(f^a) \leq \#\,(\mathrm{Crit}\ f \cap f^a) \quad \forall a \in \mathbb{R}$$

and finally the theorem.

Lusternik-Schnirelmann and closed geodesics

Let M be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric (e.g. $F(x, v) = \sqrt{g_x(v, v)}$ for g Riemannian metric),

$$E_F : \Lambda M := H^1(S^1, M) \to \mathbb{R}, \qquad E_F(\gamma) = \int_0^1 F(\gamma(t), \dot\gamma(t))^2 \, dt.$$

Then E_F is $C^{1,1}$ and satisfies PS condition (Mercuri, '77) with

$$\mathrm{Crit}\ E_F = \{\text{closed geodesics of } F\} \cup \{\text{constant loops}\}.$$

Q: Can we use Lusternik-Schnirelmann theory to obtain lower bounds on $\#\{$non-constant closed geodesics of F$\}$?

Problems with the LS-approach and closed geodesics

There are problems:

- Since $\{\text{constant loops}\} \subset \mathrm{Crit}\ E_F$, it holds for each $a \geq 0$ that $\#(\mathrm{Crit}\ E_F \cap \Lambda M^a) = +\infty$.
- $\mathrm{cat}_{\Lambda M}(\{\text{constant loops}\}) = ?$
- Critical points of E_F come in S^1-orbits, but $\mathrm{cat}_{\Lambda M}(S^1 \cdot \gamma) \in \{1, 2\}$ for each $\gamma \in \Lambda M$.

Idea: Replace $\mathrm{cat}_{\Lambda M} : \mathcal{P}(\Lambda M) \to \mathbb{N} \cup \{+\infty\}$ by a different index function.

Spherical complexities

Definition of spherical complexities (M., 2019)

Let X top. space, $n \in \mathbb{N}_0$, $B_{n+1}X := C^0(B^{n+1}, X)$,
$S_nX := \{f \in C^0(S^n, X) \mid f \text{ is nullhomotopic}\}$.

Definition

- Let $A \subset S_nX$. A *sphere filling on A* is a continuous map $s : A \to B_{n+1}X$ with $s(\gamma)|_{S^n} = \gamma$ for all $\gamma \in A$.
- For $A \subset S_nX$ put

$$\mathrm{SC}_{n,X}(A) := \inf\Big\{ r \in \mathbb{N} \;\Big|\; \exists U_1, \dots, U_r \subset S_nX \text{ open and sphere fillings } s_j : U_j \to B_{n+1}X \;\forall j \text{ and } A \subset \bigcup_{j=1}^{r} U_j \Big\} \in \mathbb{N} \cup \{\infty\}.$$

Call $\mathrm{SC}_n(X) := \mathrm{SC}_{n,X}(S_nX)$ the *n-spherical complexity of X*.

Remark $\mathrm{SC}_0(X) = \mathrm{TC}(X)$, the topological complexity of X.

Properties of spherical complexities (1)

In the following, let X be a metrizable ANR (e.g. a locally finite CW complex).

Proposition

$\mathrm{SC}_{n,X} : \mathcal{P}(S_nX) \to \mathbb{N} \cup \{+\infty\}$ *is an index function on* S_nX.

Proposition

Let $c_n : X \to S_nX$, $(c_n(x))(p) = x$ *for all* $p \in S^n$, $x \in X$. *Then*

$\mathrm{SC}_{n,X}(c_n(X)) = 1$.

Proof.

Define a sphere filling $s : c_n(X) \to B_{n+1}X$ by

$$s(c_n(x)) = (B^{n+1} \to X, \ \ p \mapsto x) \quad \forall x \in X,$$

extend continuously to an open neighborhood. $\square$

Properties of spherical complexities (2)

Let X be a metrizable ANR. Consider the left $O(n+1)$-actions on S_nX and $B_{n+1}X$ by reparametrization, i.e.

$$(A \cdot \gamma)(p) = \gamma(A^{-1}p) \qquad \forall \gamma \in S_nX,\ A \in O(n+1),\ p \in S^n.$$

Proposition Let $G \subset O(n+1)$ be a closed subgroup and $\gamma \in S_nX$ and let G_γ denote its isotropy group. If G_γ is trivial or $n = 1$, then $\mathrm{SC}_{n,X}(G \cdot \gamma) = 1$.

Proof If G_γ trivial, take $\beta : B^{n+1} \overset{C^0}{\to} X$ with $\beta|_{S^n} = \gamma$, put

$s : G \cdot \gamma \to B_{n+1}X, \quad s(A \cdot \gamma) = A \cdot \beta\ \forall A \in G.$

If $n = 1$ and $G \cong \mathbb{Z}_k$ for $k \in \mathbb{N}$, s.t. $\gamma = \alpha^k$ for some $\alpha \in S_1X$, take $\beta \in B_2X$ with $\beta|_{S^1} = \alpha$ and define $s : G \cdot \gamma \to B_2X$ by

$$s(A \cdot \gamma) = A \cdot (\beta \circ p_k),$$

where $p_k : B^2 \to B^2$, $z \mapsto z^k$. Extend to open nbhd. of $G \cdot \gamma$.

A Lusternik-Schnirelmann-type theorem for SC_n

Theorem (M., 2019)

Let $G \subset O(n+1)$ be a closed subgroup, $\mathcal{M} \subset S_n X$ be a G-invariant Hilbert manifold, $f \in C^{1,1}(\mathcal{M})$ be G-invariant. Let

$$\nu(f, \lambda) := \#\{\text{non-constant } G\text{-orbits in } \operatorname{Crit} f \cap f^\lambda\}.$$

If

- *f satisfies the Palais-Smale condition w.r.t. a complete Finsler metric on $\mathcal{M}$,*
- *f is constant on $c_n(X)$,*
- *G acts freely on $\operatorname{Crit} f \cap f^\lambda$ or $n = 1$,*

then

$$\mathrm{SC}_{n,X}(f^\lambda) \leq \nu(f, \lambda) + 1.$$

Consequences for closed geodesics

Corollary

Let M be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric and $\lambda \in \mathbb{R}$. Let $E_F : H^1(S^1, M) \cap S_1M \to \mathbb{R}$ be the restriction of the energy functional of F.

Let $\nu(F, \lambda)$ be the number of $SO(2)$-orbits of non-constant contractible closed geodesics of F of energy $\leq \lambda$. Then

$$\nu(F, \lambda) \geq \mathrm{SC}_{1,M}(E_F^\lambda) - 1.$$

If F is reversible, e.g. induced by a Riemannian metric, the same holds for the number of $O(2)$-orbits of contractible closed geodesics.

Remark The counting does not distinguish iterates of the same prime closed geodesic.

Definition A *Finsler metric* on a manifold M is a map $F : TM \overset{C^0}{\to} [0, +\infty)$, such that $F|_{TM\setminus\{\text{zero-section}\}}$ is C^∞ and

- $F_x(\lambda v) = \lambda F_x(v)\ \forall (x, v) \in TM,\ \lambda \geq 0$,
- $F_x(v) = 0 \quad \Leftrightarrow \quad v = 0 \in T_x M$,
- $D^2 F_x$ is positive definite for all $x \in M$.

The *reversibility of* F is $\lambda := \sup\{F_x(-v) \mid F_x(v) = 1\}$.

F is *reversible* if $\lambda = 1$, i.e. if $F_x(-v) = F_x(v)$ for all $(x, v) \in TM$. A *closed geodesic of* F is a critical point of

$$E_F : \Lambda M := H^1(S^1, M) \to \mathbb{R}, \qquad E_F(\gamma) = \int_0^1 F(\gamma(t), \dot\gamma(t))^2\, dt.$$

Closed geodesics occur in $SO(2)$-orbits, if reversible, then in $O(2)$-orbits. Iterates are again closed geodesics.

Results on closed geodesics

Theorem (Lusternik-Fet, '51, for Riemannian manifolds)

Every Finsler metric on a closed manifold admits a non-constant closed geodesic.

Definition $\gamma_1, \gamma_2 : S^1 \to X$ are *geometrically distinct* if $\gamma_1(S^1) \neq \gamma_2(S^1)$. They are called *positively distinct* if they are either geom. distinct or $\exists A \in O(2) \setminus SO(2)$ with $\gamma_1 = A \cdot \gamma_2$.

Existence results for closed geodesics:

- Bangert-Long, 2007: every Finsler metric on S^2 has two positively distinct ones
- Rademacher, 2009: every *bumpy* Finsler metric on S^n has two positively distinct ones
- etc., Long-Duan 2009 for S^3, Wang 2019 for pinched metrics on S^n, ...

Theorem (M., 2019)

Let M be a closed oriented $2n$-dimensional manifold, $n \geq 3$. Assume that $\exists x \in H^k(M;\mathbb{Q})$, $2 \leq k < n$, with $x^2 \neq 0$ (e.g. $\mathbb{C}P^n$). Let $F : TM \to [0,+\infty)$ be a Finsler metric of reversibility λ whose flag curvature satisfies

$$\frac{1}{4}\left(\frac{\lambda}{1+\lambda}\right)^2 < K \leq 1, \qquad \left(\textit{i.e. if } F \textit{ reversible:} \quad \frac{1}{16} < K \leq 1,\right)$$

then F admits two positively distinct closed geodesics (geometrically distinct if F is reversible).

Idea of proof Find $a > 0$ such that $E_F^a = E_F^{-1}((-\infty, a])$ contains only prime closed geodesics and such that $SC_{1,M}(E_F^a) \geq 3$.

Lower bounds for spherical complexities

Lower bounds for spherical complexities

Aim Find "computable" lower bounds on $\mathrm{SC}_{n,X}(A)$.

Method Put spherical complexities in a bigger framework and use results by A. S. Schwarz from a more general context.

Spherical complexities and sectional category

Definition (A. Schwarz, ’62)

Let $p : E \to B$ be a fibration. The *sectional category* or *Schwarz genus* of p is given by

$$\mathrm{secat}(p) = \inf\Big\{ n \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{n} U_j = B \text{ open cover}, \\ s_j : U_j \overset{C^0}{\to} E,\;\; p \circ s_j = \mathrm{incl}_{U_j} \;\; \forall j \Big\}.$$

In our setting:

$$\mathrm{SC}_n(X) = \mathrm{secat}\left(r_n : B_{n+1}X \to S_nX,\;\; \gamma \mapsto \gamma|_{S^n} \right),$$

$$\mathrm{SC}_{n,X}(A) \geq \mathrm{secat}\left(r_n|_{r_n^{-1}(A)} : r_n^{-1}(A) \to A \right) \quad \forall A \subset S_nX.$$

Sectional categories and cup length

Given X top. space, R commutative ring, $I \subset H^*(X;R)$ an ideal, let

$\mathrm{cl}(I) := \sup\{r \in \mathbb{N} \mid \exists u_1, \ldots, u_r \in I \cap \tilde{H}^*(X;R) \text{ s.t. } u_1 \cup \cdots \cup u_r \neq 0\}.$

Theorem (Lusternik-Schnirelmann '34)

$$\mathrm{cat}(X) \geq \mathrm{cl}(H^*(X;R)) + 1.$$

Theorem (A. Schwarz, '62)

Let $p : E \to B$ be a fibration. Then

$$\mathrm{secat}(p) \geq \mathrm{cl}\Big(\ker\ [p^* : H^*(B;R) \to H^*(E;R)]\Big) + 1.$$

Consequences for spherical complexities

The previous theorem, some work and the long exact cohomology sequence of $(S_nX, c_n(X))$ yield:

Theorem

Let $A \subset S_nX$ and let $\iota : (A, \varnothing) \hookrightarrow (S_nX, c_n(X))$ be the inclusion of pairs. Then

$$\mathrm{SC}_{n,X}(A) \geq \mathrm{cl}\Big(\mathrm{im}\left[\iota^* : H^*(S_nX, c_n(X); R) \to H^*(A; R)\right]\Big) + 1.$$

Problem The cup product on $H^*(S_nX, X; R)$ might be either hard to compute or not that interesting. (E.g. the cup product on $H^*(LS^2, S^2; \mathbb{Q})$ vanishes.)

Idea Improve cup length bounds by associating $\mathbb{N}$-valued *weights* to cohomology classes.

Sectional category and fiberwise joins

Given fibrations $f_1 : E_1 \to B$, $f_2 : E_2 \to B$, let

$$f_1 * f_2 : E_1 *_f E_2 \to B$$

denote the *fiberwise join of* p_1 *and* p_2. The fiber over each $b \in B$ is $(E_1 *_f E_2)_b = (E_1)_b * (E_2)_b$.

Let $p : E \to B$ be a fibration. Define fibrations $p_n : E_n \to B$, $n \in \mathbb{N}$, recursively by

$$p_1 = p,\ E_1 = E, \qquad p_n = p * p_{n-1},\ E_n = E *_f E_{n-1}.$$

Theorem (A. Schwarz, '62)

secat$(p) = \inf\{n \in \mathbb{N} \mid \exists s : B \overset{C^0}{\to} E_n \textit{ with } p_n \circ s = \mathrm{id}_B.\}$

Sectional category weights

Let $p : E \to B$ be a fibration, R be a commutative ring.

Definition (Farber-Grant 2007; Fadell-Husseini '92, Rudyak '99)

Let $u \in H^*(B;R)$, $u \neq 0$. The *weight of u with respect to p* is given by $\mathrm{wgt}_p(u) := \sup\{n \in \mathbb{N}_0 \mid p_n^* u = 0\}$.

Properties: Let $u, v \in H^*(B;R)$ with $u \neq 0$, $v \neq 0$.

- If $\mathrm{wgt}_p(u) \geq k$, then $\mathrm{secat}(p) \geq k+1$.
- $\mathrm{wgt}_p(u \cup v) \geq \mathrm{wgt}_p(u) + \mathrm{wgt}_p(v)$.
- If $f : X \overset{C^0}{\to} B$ with $f^*u \neq 0$, then $\mathrm{wgt}_{f^*p}(f^*u) \geq \mathrm{wgt}_p(u)$.

Thus, can use weights to improve cup length bounds: if $k := \mathrm{cl}(\ker p^*)$ and $u_1, \dots, u_k \in \ker p^*$ with $u_1 \cup \dots \cup u_k \neq 0$, then

$$\mathrm{secat}(p) \geq \sum_{j=1}^{k} \mathrm{wgt}_p(u_j) + 1 \geq \mathrm{cl}(\ker p^*) + 1.$$

Let (M, F) be a Finsler manifold, $a > 0$ and let $\iota_a : E_F^a \hookrightarrow \Lambda M$ be the inclusion. From the above properties we obtain:

if $u \in H^*(\Lambda M, c_1(M); R)$ satisfies $\iota_a^* u \neq 0$, then

$$\nu(F, a) \geq \mathrm{wgt}(u) := \mathrm{wgt}_{r_1}(u).$$

Thus, it suffices to find such u with $\mathrm{wgt}(u) \geq 2$ for sufficiently small a.

Construction of classes of weight ≥ 1

Lemma

Let X be a simply connected top. space and R be a commutative ring. Let $LX = C^0(S^1, X)$ and

$$\mathrm{ev} : LX \times S^1 \to X, \quad (\alpha, t) \mapsto \alpha(t).$$

Then for $k \geq 2$,

$$Z : H^k(X; R) \to H^{k-1}(LX; R), \qquad Z(\sigma) = \mathrm{ev}^*\sigma/[S^1]$$

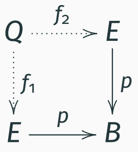
is injective and $\mathrm{wgt}(Z(\sigma)) \geq 1$ for all $\sigma \neq 0 \in \widetilde{H}^*(X; R)$.

(Here, $\cdot/\cdot$ denotes the slant product.)

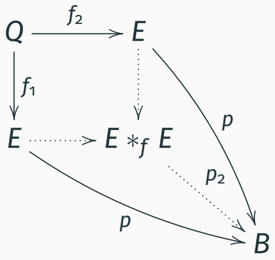
(Generalization of methods from Grant-M. 2018)

If $p : E \to B$ is a fibration, then $p_2 : E *_f E \to B$ is constructed as a homotopy pushout of a pullback (double mapping cylinder):

pullback



homotopy pushout



Construction of classes of weight ≥ 2, cont.

As a homotopy pushout of a pullback, it has a Mayer-Vietoris sequence:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & H^{k-1}(Q;R) & \xrightarrow{\delta} & H^k(E *_f E;R) & \longrightarrow & \oplus_{i=1}^2 H^k(E;R) & \longrightarrow & \dots \\ & & & & \uparrow{\scriptstyle p_2^*} & \nearrow{\scriptstyle p^*\oplus p^*} & & & \\ & & & & H^k(B;R) & & & & \end{array}$$

Want to find $u \in H^k(B;R)$ with $p_2^* u = 0$. If u lies in $\ker p^*$, then $p_2^* u \in \mathrm{im}\delta$. Try to find $\alpha_u \in H^{k-1}(Q;R)$ with $\delta(\alpha_u) = p_2^* u$, find conditions that imply $\alpha_u = 0$.

Back to our setting: Here $p = r_1 : B_2X \to S_1X$, $\gamma \mapsto \gamma|_{S^1}$, and the pullback is

$$\begin{aligned} Q &= \{(\gamma_1, \gamma_2) \in (B_2X)^2 \, | r_1(\gamma_1) = r_1(\gamma_2)\} \\ &= \{(\gamma_1, \gamma_2) \in (C^0(B^2, X))^2 \mid \gamma_1|_{S^1} = \gamma_2|_{S^1}\} \approx C^0(S^2, X), \end{aligned}$$

hence for $E_2 := B_2X *_f B_2X$ the Mayer-Vietoris sequence has the form

$$\cdots \to H^{k-1}(C^0(S^2, X); \mathbb{Q}) \xrightarrow{\delta} H^k(E_2; \mathbb{Q}) \to H^k(B_2X; \mathbb{Q}) \oplus H^k(B_2X; \mathbb{Q}) \to \dots$$

Lemma

Let $u \neq 0 \in H^k(X; \mathbb{Q})$, $k \geq 2$. Then $p_2^*(Z(u)) = \delta(a_u)$, where $a_u \in H^{k-2}(C^0(S^2, X); \mathbb{Q})$ is given by

$$a_u = e_2^* u / [S^2], \qquad e_2 : C^0(S^2, X) \times S^2 \to X, \;\; e_2(\gamma, p) := \gamma(p).$$

Theorem

Let $u \neq 0 \in H^k(X;\mathbb{Q})$, $k \geq 3$. If $f^*u = 0$ for all $f : S^2 \times P \xrightarrow{C^0} X$, where P is any closed oriented $(k-2)$-manifold, then

$$\mathrm{wgt}(Z(u)) \geq 2.$$

Proof.

By the previous lemma, $\mathrm{wgt}(Z(u)) \geq 2$ if $e_2^*u/[S^2] = 0$.

But since $H_*(C^0(S^2, X);\mathbb{Q})$ is representable, this will hold if for all closed or. $(k-2)$-manifolds P and $g : P \xrightarrow{C^0} C^0(S^2, X)$ it holds that

$$\left\langle e_2^*u/[S^2], g_*[P]\right\rangle = 0 \quad \Leftrightarrow \quad < f^*u, [S^2 \times P] >= 0,$$

where $f(p, x) = e_2(g(x), p)$. The claim immediately follows. □

Sketch of proof of the main result

Reminder By assumption, $x \in H^k(M;\mathbb{Q})$ with $2k < \dim M$ and $x^2 \neq 0$.

M even-dim., F positive curv. $\Rightarrow \pi_1(M) = 0 \Rightarrow Z(x^2) \neq 0$.

If $f^*(x^2) \stackrel{!}{=} 0$ for all $f : S^2 \times P \stackrel{C^0}{\to} M$, $\dim P = 2k-2$, then $\mathrm{wgt}(Z(x^2)) \geq 2$. But this is clear since

$$f^*(x^2) = (f^*x)^2 \in H^{2k}(S^2 \times P;\mathbb{Q}) \cong H^2(S^2;\mathbb{Q}) \otimes H^{2k-2}(P;\mathbb{Q}),$$

which can not contain nontrivial squares in degree $2k$.

Remains to show for $u := Z(x^2)$ that $\iota_a^* u \neq 0$ for some $a > 0$ for which E_F^a contains only prime geodesics.

Sketch of proof of the main result, cont.

Reminder By assumption, $\frac{1}{4}(\frac{\lambda}{1+\lambda})^2 < \delta \leq K \leq 1$.

Let γ_0 be a non-constant closed geodesic of F of shortest length ℓ_0.

- Since $K \leq 1$, $\ell_0 > \pi\frac{1+\lambda}{\lambda}$ by an injectivity radius estimate.
- Since $K \geq \delta$ and $\deg u < \dim M - 1$, it follows by comparison arguments (Ballmann-Thorbergsson-Ziller '82 for Riemannian metrics, Rademacher 2004 for Finsler) that $\iota_a^* u \neq 0$ for $a := \frac{\pi^2}{\delta}$, hence $\nu(F, a) \geq \mathrm{wgt}(u) = 2$.
- One derives from $\delta > \frac{1}{4}(\frac{\lambda}{1+\lambda})^2$ and the bound for ℓ_0 that $E_F(\gamma_0^2) > a$.

$\Rightarrow$ E_F^a contains two positively distinct closed geodesics.

(geometrically distinct, if F reversible).

Perspectives and possible applications

- Equivariant versions, use richer ring structure in $H^*_{S^1}(LM, M; \mathbb{Q})$ *($\longrightarrow$ work in progress)*
- Apply the methods to more general flows having Lyapunov functions.
- Applicable in greater generality to Reeb orbits on contact manifolds? (generalizing closed geodesics on T^1M)
- Higher-dimensional applications, i.e. for $\mathrm{SC}_{n,M}$ if $n > 1$?

Thank you for your attention!

talk based on:

S. Mescher, Spherical complexities, with applications to closed geodesics, arXiv:1911.03948

(slides at `http://www.math.uni-leipzig.de/~mescher`)