UNIVERSITÄT
LEIPZIG

# Spherical complexities and closed geodesics

Stephan Mescher (Universität Leipzig)

DMV-Jahrestagung 2020
Minisymposium Nonlinear Analysis
17. September 2020

# L-S category and critical points

# The Lusternik-Schnirelmann category of a space

**Definition**

For a topological space $X$ and $A \subset X$ put

$$\mathrm{cat}_X(A) := \inf\Big\{ r \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{r} U_j \supset A \text{ open cover}, \\ \text{s.t. } U_j \hookrightarrow X \text{ nullhomotopic } \forall j \in \{1, 2, \ldots, r\} \Big\}.$$

$\mathrm{cat}(X) := \mathrm{cat}_X(X)$ is the *Lusternik-Schnirelmann category of* $X$.

- $\mathrm{cat}(X)$ is a homotopy invariant of $X$.
- $\mathrm{cat}(X)$ is hard to compute explicitly.

# Lusternik-Schnirelmann category and critical points

**Theorem (Lusternik-Schnirelmann ’34, Palais ’65)**

*Let $M$ be a Hilbert manifold and let $f \in C^{1,1}(M)$ be bounded from below and satisfy the Palais-Smale condition with respect to a complete Finsler metric on $M$. Then*

$$\# \operatorname{Crit} f \geq \operatorname{cat}(M).$$

# Method of proof of the Lusternik-Schnirelmann theorem

$f \in C^{1,1}(M)$ bounded from below and satisfies PS condition w.r.t. Finsler metric on $M$. Put $f^a := f^{-1}((-\infty, a])$. Use properties of $\mathrm{cat}_X$ and minimax methods to show:

- If $[a, b]$ contains no critical value of $f$, then

$$\mathrm{cat}_M(f^b) = \mathrm{cat}_M(f^a).$$

- If $c$ is a critical value of $f$, then

$$\mathrm{cat}_M(f^c) \leq \mathrm{cat}_M(f^{c-\varepsilon}) + \mathrm{cat}_M(\mathrm{Crit}\ f \cap f^{-1}(\{c\})).$$

Combining these observations yields

$$\mathrm{cat}_M(f^a) \leq \#(\mathrm{Crit}\ f \cap f^a) \quad \forall a \in \mathbb{R}$$

and finally the theorem.

## Lusternik-Schnirelmann and closed geodesics

Let $M$ be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric (e.g. $F(x, v) = \sqrt{g_x(v, v)}$ for $g$ Riemannian metric), $\Lambda M := H^1(S^1, M) = W^{1,2}(S^1, M)(\simeq C^0(S^1, M))$,

$$E_F : \Lambda M \to \mathbb{R}, \quad E_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))^2 \, dt.$$

Then $\Lambda M$ is a Hilbert manifold, $E_F$ is $C^{1,1}$ and satisfies the PS condition (Mercuri, '77).

$$\text{Crit } E_F = \{\text{closed geodesics of } F\} \cup \{\text{constant loops}\}.$$

**Q:** Can we use Lusternik-Schnirelmann theory to obtain lower bounds on
$\#\{$geometr. distinct non-constant closed geodesics of F$\}$?

## Problems with the LS-approach and closed geodesics

There are problems:

- Since $\{\text{constant loops}\} \subset \mathrm{Crit}\ E_F$, it holds for each $a \geq 0$ that $\#(\mathrm{Crit}\ E_F \cap \Lambda M^a) = +\infty$.
- $\mathrm{cat}_{\Lambda M}(\{\text{constant loops}\}) =?$
- Critical points of $E_F$ come in $S^1$-orbits, but $\mathrm{cat}_{\Lambda M}(S^1 \cdot \gamma) \in \{1, 2\}$ for each $\gamma \in \Lambda M$.

**Idea:** Replace $\mathrm{cat}_{\Lambda M} : \mathfrak{P}(\Lambda M) \to \mathbb{N} \cup \{+\infty\}$ by a different function with similar properties.

There are several similar approaches to $G$-invariant functions, e.g. by Clapp-Puppe, Bartsch et al.

# Spherical complexities

# Definition of spherical complexities (M., 2019)

Let $X$ top. space, $n \in \mathbb{N}_0$, $B_{n+1}X := C^0(B^{n+1}, X)$,
$S_nX := \{f \in C^0(S^n, X) \mid f \text{ is nullhomotopic}\}$.

**Definition**

Let $A \subset S_nX$. A *sphere filling on $A$* is a continuous map $s : A \to B_{n+1}X$ with $s(\gamma)|_{S^n} = \gamma$ for all $\gamma \in A$.

$$\mathrm{SC}_{n,X}(A) := \inf\Big\{r \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{r} U_j \supset A \text{ open cover, sphere fillings } s_j : U_j \to B_{n+1}X \ \forall j \in \{1, 2, \dots, r\}\Big\} \in \mathbb{N} \cup \{\infty\}.$$

Call $\mathrm{SC}_n(X) := \mathrm{SC}_{n,X}(S_nX)$ the *$n$-spherical complexity of $X$*.

**Remark** $\mathrm{SC}_0(X) = \mathrm{TC}(X)$, the topological complexity of $X$.

(Farber, '03)

# Properties of spherical complexities

Let $X$ be a metrizable ANR (e.g. a locally finite CW complex).

**Proposition** Let $c_n : X \to S_nX$, $(c_n(x))(p) = x$ for all $p \in S^n$, $x \in X$. Then $\mathrm{SC}_{n,X}(c_n(X)) = 1$.

Consider the left $O(n+1)$-actions on $S_nX$ and $B_{n+1}X$ by reparametrization, i.e.

$$(A \cdot \gamma)(p) = \gamma(A^{-1}p) \qquad \forall \gamma \in S_nX,\ A \in O(n+1),\ p \in S^n.$$

**Proposition** Let $G \subset O(n+1)$ be a closed subgroup and $\gamma \in S_nX$ and let $G_\gamma$ denote its isotropy group. If $G_\gamma$ is trivial or $n = 1$, then $\mathrm{SC}_{n,X}(G \cdot \gamma) = 1$.

**Proof for $G_\gamma$ trivial:** Take $\beta \in B_{n+1}X$ with $\beta|_{S^n} = \gamma$, put

$$s : G \cdot \gamma \to B_{n+1}X, \quad s(A \cdot \gamma) = A \cdot \beta \ \forall A \in G.$$

# A Lusternik-Schnirelmann-type theorem for $\mathrm{SC}_n$

**Theorem (M., 2019)**

*Let $G \subset O(n+1)$ be a closed subgroup, $\mathcal{M} \subset S_n X$ be a $G$-invariant Hilbert manifold, $f \in C^{1,1}(\mathcal{M})$ be $G$-invariant. Let*

$$\nu(f,a) := \#\{\textit{non-constant } G\textit{-orbits in } \operatorname{Crit} f \cap f^a\}.$$

*If*

- *$f$ satisfies the Palais-Smale condition w.r.t. a complete Finsler metric on $\mathcal{M}$,*
- *$f$ is constant on $c_n(X)$,*
- *$G$ acts freely on $\operatorname{Crit} f \cap f^a$ or $n = 1$,*

*then*

$$\mathrm{SC}_{n,X}(f^a) \leq \nu(f,a) + 1.$$

# Lower bounds and cohomology

# Spherical complexities and sectional category

**Aim** Find "computable" lower bounds on $\mathrm{SC}_{n,X}(A)$.

**Method** Put spherical complexities in a bigger framework and use results by A. S. Schwarz from a more general context.

**Definition (A. Schwarz, '62)**

Let $p : E \to B$ be a fibration. The *sectional category* of $p$ is given by

$$\mathrm{secat}(p) = \inf\Big\{ r \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{r} U_j = B \text{ open cover, } s_j : U_j \xrightarrow{C^0} E, \\ p \circ s_j = \mathrm{incl}_{U_j} \;\; \forall j \in \{1, 2, \dots, r\} \Big\}.$$

Here: $\mathrm{SC}_n(X) = \mathrm{secat}\left( r_n : B_{n+1}X \to S_nX, \;\; \gamma \mapsto \gamma|_{S^n} \right)$,

$$\mathrm{SC}_{n,X}(A) \geq \mathrm{secat}\left( r_n|_{r_n^{-1}(A)} : r_n^{-1}(A) \to A \right) \quad \forall A \subset S_nX.$$

# Sectional categories and cup length

**Theorem (Lusternik-Schnirelmann '34)**

*X top. space, R commutative ring, Then*

$$\mathrm{cat}(X) \geq \textit{cup-length}(H^*(X;R)) + 1.$$

**Theorem (A. Schwarz, '62)**

*Let $p : E \to B$ be a fibration. Then*

$$\mathrm{secat}(p) \geq \textit{cup-length}\Big(\ker\ [p^* : H^*(B;R) \to H^*(E;R)]\Big) + 1.$$

Improve bounds using *weights* $\mathrm{wgt}_p : \widetilde{H}^*(B;R) \to \mathbb{N}$. If $u_1, \ldots, u_k \in \ker p^*$ with $u_1 \cup \cdots \cup u_k \neq 0$, then

$$\mathrm{secat}(p) \geq \sum_{i=1}^{k} \mathrm{wgt}_p(u_i) + 1.$$

(Fadell-Husseini '92, Rudyak '99, Farber-Grant '07).

## Consequences for spherical complexities

The previous theorem, some work and the long exact cohomology sequence of $(S_nX, c_n(X))$ yield:

**Theorem**

*Let $A \subset S_nX$ and let $\iota : (A, \varnothing) \hookrightarrow (S_nX, c_n(X))$ be the inclusion of pairs. Then*

$$\mathrm{SC}_{n,X}(A) \geq \textit{cup-length}\Big(\mathrm{im}\,[\iota^* : H^*(S_nX, c_n(X); R) \to H^*(A; R)]\,\Big) + 1.$$

*(plus improvements using weights)*

# Results on closed geodesics

# Main estimate for the number of closed geodesics

Let $M$ be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric and let $E_F : \Lambda M \cap S_1 M \to \mathbb{R}$ be its energy functional.

**Theorem**

*Let $\nu(F, a)$ be the number of $SO(2)$-orbits of non-constant contractible closed geodesics of F of energy $\leq a$. Then*

$$\nu(F, a) \geq SC_{1,M}(E_F^a) - 1.$$

*If F is reversible, i.e. if $F_x(v) = F_x(-v)\ \forall (x, v) \in TM$, the same holds for the number of $O(2)$-orbits of contractible closed geodesics.*

**Remark** The counting does not distinguish iterates of the same prime closed geodesic.

# Results on closed geodesics

**Theorem (Lusternik-Fet, '51, for Riemannian manifolds)**

*Every Finsler metric on a closed manifold admits a non-constant closed geodesic.*

**Definition** Two closed geodesics $\gamma_1, \gamma_2 : S^1 \to X$ are *positively distinct* if either $\gamma_1(S^1) \neq \gamma_2(S^1)$ or $\exists A \in O(2) \setminus SO(2)$ with $\gamma_1 = A \cdot \gamma_2$.

- Bangert-Long, 2007: every Finsler metric on $S^2$ has two positively distinct ones
- Rademacher, 2009: every *bumpy* Finsler metric on $S^n$ has two positively distinct ones *(generic condition)*
- etc., Long-Duan 2009 for $S^3$, Wang 2019 for pinched metrics on $S^n$, ...

# New results using spherical complexities

**Theorem (M., 2020)**

*Let $M$ be a closed oriented manifold, $F : TM \to [0, +\infty)$ be a Finsler metric of reversibility $\lambda$ and flag curvature $K$. Let $\ell_F > 0$ be the length of the shortest non-const. closed geodesic of $F$.*

a) *If $M = S^{2d}$, $d \geq 2$, $0 < K \leq 1$ and $F \leq \frac{1+\lambda}{\lambda}\sqrt{g_1}$, then $F$ admits two pos. distinct closed geodesics of length $< 2\ell_F$. ($g_1$ = round metric of constant curvature 1)*

b) *If $M = S^{2d+1}$, $d \in \mathbb{N}$, $\frac{\lambda^2}{(1+\lambda)^2} < K \leq 1$ and $F \leq \frac{(k+1)(1+\lambda)}{m\lambda}\sqrt{g_1}$, then $F$ admits $\lceil \frac{2m}{k} \rceil$ pos. distinct closed geodesics of length $< (k+1)\ell_F$.*

c) *If $M = \mathbb{C}P^n$ or $M = \mathbb{H}P^n$, $n \geq 3$, $0 < K \leq 1$ and $F \leq \frac{1+\lambda}{\lambda}\sqrt{g_1}$, then $\exists$ two pos. distinct closed geodesics of length $< 2\ell_F$.*

# Method of proof of results for closed geodesics

For parts a) and c):

- Use lower bounds by cup length and cohomology weights, ring structure on $H^*(\Lambda M; \mathbb{Q})$ is well-known for these spaces (Vigué-Poirrier/Sullivan '76).
- Use algebraic topology to establish criteria on cohomology classes to have weight two.
- Use energy bounds provided by conditions on $F$ to show that a class of weight two is supported on $E_F^{<4\ell_F^2}$.
- Conclude that $E_F^{<4\ell_F^2}$ contains two closed geodesics, distinctness follows from energy bound.

# Perspectives and possible applications

- Equivariant versions, use richer ring structure in $H^*_{S^1}(LM, M; \mathbb{Q})$
- Applicable in greater generality to periodic orbits of Reeb flows on contact manifolds? (generalizing geodesic flows on $T^1M$)
- Higher-dimensional applications, i.e. for $\mathrm{SC}_{n,M}$ if $n > 1$?
- Any other ideas? Contact me!

# Thank you for your attention!

talk based on:

S. Mescher, Spherical complexities, with applications to closed geodesics, arXiv:1911.03948, to appear in Algebr. Geom. Topol.

S. Mescher, Existence results for closed geodesics via spherical complexities, to appear in Calc. Var. 59, 2020.

(slides at `http://www.math.uni-leipzig.de/~mescher`)

mescher@math.uni-leipzig.de