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Spherical complexities and closed geodesics

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L-S category and critical points

Lusternik-Schnirelmann category and critical points

Theorem (Lusternik-Schnirelmann '34, Palais '65)

Let M be a Hilbert manifold and let $f \in C^{1,1}(M)$ be bounded from below and satisfy the Palais-Smale condition with respect to a complete Finsler metric on M. Then

$$\# \operatorname{Crit} f \geq \operatorname{cat}(M).$$

Remark

There are various generalisations, e.g. generalized Palais-Smale conditions (Clapp-Puppe '86), extensions to fixed points of self-maps (Rudyak-Schlenk '03).

Method of proof of the Lusternik-Schnirelmann theorem

$f \in C^{1,1}(M)$ bounded from below and satisfies PS condition w.r.t. Finsler metric on M. Put $f^a := f^{-1}((-\infty, a])$. Use properties of $\mathrm{cat}_M(\cdot)$ and minimax methods to show:

- If $[a, b]$ contains no critical value of f, then
$$\mathrm{cat}_M(f^b) = \mathrm{cat}_M(f^a).$$
- If c is a critical value of f, then
$$\mathrm{cat}_M(f^c) \leq \mathrm{cat}_M(f^{c-\varepsilon}) + \mathrm{cat}_M((\mathrm{Crit}\ f) \cap f^{-1}(\{c\})).$$

Combining these observations yields

$$\mathrm{cat}_M(f^a) \leq \#\left((\mathrm{Crit}\ f) \cap f^a\right) \quad \forall a \in \mathbb{R}$$

and finally the theorem.

Lusternik-Schnirelmann and closed geodesics

Let M be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric (e.g. $F(x, v) = \sqrt{g_x(v, v)}$ for g Riemannian metric),

$$E_F : \Lambda M \to \mathbb{R}, \qquad E_F(\gamma) = \int_0^1 F(\gamma(t), \dot{\gamma}(t))^2 \, dt.$$

Here, $\Lambda M := H^1(S^1, M) \overset{\simeq}{\subset} C^0(S^1, M)$ is a Hilbert manifold locally modelled on $H^1(S^1, \mathbb{R}^{\dim M}) = W^{1,2}(S^1, \mathbb{R}^{\dim M})$.

Then E_F is $C^{1,1}$ and satisfies the PS condition (Mercuri '77) with

$$\text{Crit } E_F = \{\text{closed geodesics of } F\} \cup \{\text{constant loops}\}.$$

Q: Can we use Lusternik-Schnirelmann theory to obtain lower bounds on
$\#\{$geometrically distinct non-constant closed geodesics of F$\}$?

Problems with the LS-approach and closed geodesics

There are problems:

- Since $\{\text{constant loops}\} \subset \text{Crit } E_F$ with $E_F(\text{const. loop}) = 0$, it holds for each $a \geq 0$ that $\#((\text{Crit } E_F) \cap E_F^a) = +\infty$.
- $\text{cat}_{\Lambda M}(\{\text{constant loops}\}) = ?$
- Critical points of E_F come in S^1-orbits, but $\text{cat}_{\Lambda M}(S^1 \cdot \gamma) \in \{1, 2\}$ for each $\gamma \in \Lambda M$.

Idea: Replace $\text{cat}_{\Lambda M} : \mathcal{P}(\Lambda M) \to \mathbb{N} \cup \{+\infty\}$ by a different function with similar properties.

There are several similar approaches to G-invariant functions, e.g. by Clapp-Puppe, Bartsch et al.

Spherical complexities

Definition of spherical complexities (M., 2019)

Definition (A. Schwarz, ’62)

Let $p : E \to B$ be a fibration. The *sectional category* or *Schwarz genus* of p is given by

$$\mathrm{secat}(p) = \inf\Big\{ r \in \mathbb{N} \ \Big|\ \exists \bigcup_{j=1}^{r} U_j = B \text{ open cover}, s_j : U_j \xrightarrow{C^0} E,\ \ p \circ s_j = \mathrm{incl}_{U_j}\ \forall j \Big\}.$$

Let X be a top. space, $n \in \mathbb{N}_0$, put $B_{n+1}X := C^0(B^{n+1}, X)$,
$S_nX := \{f \in C^0(S^n, X) \mid f \text{ is nullhomotopic}\}$.

Definition

The *n-spherical complexity of X* is given by

$$\mathrm{SC}_n(X) := \mathrm{secat}(r_n : B_{n+1}X \to S_nX, \quad \gamma \mapsto \gamma|_{S^n}).$$

For $A \subset S_nX$ define $\mathrm{SC}_{n,X}(A) := \mathrm{secat}(r_n|_{r_n^{-1}(A)} : r_n^{-1}(A) \to A)$.

Properties of spherical complexities

Remark $\mathrm{SC}_0(X) = \mathrm{TC}(X)$, the topological complexity of X.

In the following, let X be a metrizable ANR (e.g. a locally finite CW complex).

Proposition

Let $c_n : X \to S_nX$, $(c_n(x))(p) = x$ *for all* $p \in S^n$, $x \in X$. *Then* $\mathrm{SC}_{n,X}(c_n(X)) = 1$.

Consider the left $O(n+1)$-actions on S_nX and $B_{n+1}X$ by reparametrization, i.e.

$$(A \cdot \gamma)(p) = \gamma(A^{-1}p) \qquad \forall \gamma \in S_nX,\ A \in O(n+1),\ p \in S^n.$$

Proposition Let $G \subset O(n+1)$ be a closed subgroup and $\gamma \in S_nX$ and let G_γ denote its isotropy group. If G_γ is trivial or $n = 1$, then $\mathrm{SC}_{n,X}(G \cdot \gamma) = 1$.

A Lusternik-Schnirelmann-type theorem for SC_n

Theorem (M., 2019)

Let $G \subset O(n+1)$ be a closed subgroup, $\mathcal{M} \subset S_n X$ be a G-invariant Hilbert manifold, $f \in C^{1,1}(\mathcal{M})$ be G-invariant. Let

$$\nu(f,a) := \#\{\textit{non-constant } G\textit{-orbits in } \operatorname{Crit} f \cap f^a\}.$$

If

- *f satisfies the Palais-Smale condition w.r.t. a complete Finsler metric on $\mathcal{M}$,*
- *f is constant on $c_n(X)$,*
- *G acts freely on $(\operatorname{Crit} f) \cap f^a$ or $n = 1$,*

then

$$\mathrm{SC}_{n,X}(f^a) \leq \nu(f,a) + 1.$$

Consequences for closed geodesics

Corollary

Let M be a closed manifold, $F : TM \to [0, +\infty)$ be a Finsler metric and $a \in \mathbb{R}$. Let $E_F : \Lambda M \cap S_1 M \to \mathbb{R}$ be the restriction of the energy functional of F.

Let $\nu(F, a)$ be the number of $SO(2)$-orbits of non-constant contractible closed geodesics of F of energy $\leq a$. Then

$$\nu(F, a) \geq \mathrm{SC}_{1,M}(E_F^a) - 1.$$

If F is reversible, e.g. induced by a Riemannian metric, the same holds for the number of $O(2)$-orbits of contractible closed geodesics.

Remark The counting does not distinguish iterates of the same prime closed geodesic.

Lower bounds for spherical complexities

Sectional categories and cup length

Aim Find "computable" lower bounds on $\mathrm{SC}_{n,X}(A)$ using cohomology.

For X top. space, R commutative ring, $I \subset H^*(X;R)$ an ideal, let

$$\mathrm{cl}(I) := \sup\{r \in \mathbb{N} \mid \exists u_1, \ldots, u_r \in I \cap \tilde{H}^*(X;R) \text{ s.t. } u_1 \cup \cdots \cup u_r \neq 0\}.$$

Theorem (A. Schwarz, '62)

Let $p : E \to B$ be a fibration. Then

$$\mathrm{secat}(p) \geq \mathrm{cl}\Big(\ker\ [p^* : H^*(B;R) \to H^*(E;R)]\Big) + 1.$$

Consequences for spherical complexities

The previous theorem, some work and the long exact cohomology sequence of $(S_nX, c_n(X))$ yield:

Theorem

Let $A \subset S_nX$ and let $\iota : (A, \varnothing) \hookrightarrow (S_nX, c_n(X))$ be the inclusion of pairs. Then

$$\mathrm{SC}_{n,X}(A) \geq \mathrm{cl}\Big(\mathrm{im}\,[\iota^* : H^*(S_nX, c_n(X); R) \to H^*(A; R)]\,\Big) + 1.$$

Problem The cup product on $H^*(S_nX, c_n(X); R)$ might be either hard to compute or not that interesting. (E.g. the cup product on $H^*(LS^2, c_1(S^2); \mathbb{Q})$ vanishes.)

Idea Improve cup length bounds by associating $\mathbb{N}$-valued *weights* to cohomology classes.

Sectional category and fiberwise joins

Given fibrations $p : E \to B$, $p' : E' \to B$, let

$$p * p' : E *_f E' \to B$$

denote the *fiberwise join of p and* p'. The fiber over each $b \in B$ is $(E *_f E')_b = E_b * E'_b$. ($*$ = topological join)

Let $p : E \to B$ be a fibration. Define fibrations $p_r : E_r \to B$, $r \in \mathbb{N}$, recursively by

$$p_1 = p,\ E_1 = E, \qquad p_r = p * p_{r-1},\ E_r = E *_f E_{r-1}.$$

Theorem (A. Schwarz, '62)

$\mathrm{secat}(p) = \inf\{r \in \mathbb{N} \mid \exists s : B \overset{C^0}{\to} E_r \text{ with } p_r \circ s = \mathrm{id}_B\}$.

Sectional category weights

Let $p : E \to B$ be a fibration, R be a commutative ring.

Definition (Farber-Grant 2007; Fadell-Husseini '92, Rudyak '99)

Let $u \in \widetilde{H}^*(B;R)$, $u \neq 0$. The *weight of u with respect to p* is given by $\mathrm{wgt}_p(u) := \sup\{r \in \mathbb{N}_0 \mid p_r^* u = 0\}$.

Properties: Let $u, v \in \widetilde{H}^*(B;R)$ with $u \neq 0$, $v \neq 0$.

- If $\mathrm{wgt}_p(u) \geq k$, then $\mathrm{secat}(p) \geq k+1$.
- $\mathrm{wgt}_p(u \cup v) \geq \mathrm{wgt}_p(u) + \mathrm{wgt}_p(v)$.

Thus, if $k := \mathrm{cl}(\ker p^*)$ and $u_1, \dots, u_k \in \ker p^*$ with $u_1 \cup \dots \cup u_k \neq 0$, then

$$\mathrm{secat}(p) \geq \sum_{j=1}^{k} \mathrm{wgt}_p(u_j) + 1 \geq \mathrm{cl}(\ker p^*) + 1.$$

Construction of classes of weight ≥ 1 for $r_n(\gamma) = \gamma|_{S^n}$

Want to find classes in ker $[r_n^* : H^*(S_nX; R) \to H^*(B_{n+1}X; R)]$.

Lemma

Let X be a top. space, R be a commutative ring and

$$\mathrm{ev}_n : S_nX \times S^n \to X, \quad (\alpha, p) \mapsto \alpha(p).$$

For $k \geq n$ let

$$Z_n : H^k(X; R) \to H^{k-n}(S_nX; R), \qquad Z_n(\sigma) = \mathrm{ev}_n^* \sigma / [S^n],$$

where $\cdot/\cdot$ denotes the slant product. Then $Z_n(\sigma) \in \ker r_n^*$.

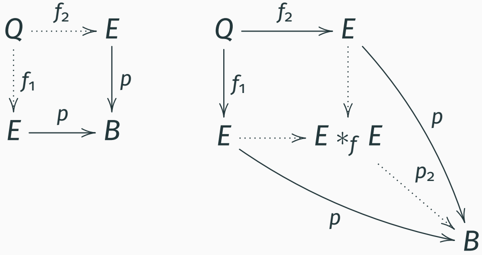
Remark If $n = 1$ and X is simply connected, then $Z_1 : \widetilde{H}^*(X; R) \to H^{*-1}(LX; R)$ is injective. (Jones '87)

Construction of classes of weight ≥ 2

(Generalization of methods from **Grant**-M. 2018)

If $p : E \to B$ is a fibration, then $p_2 : E *_f E \to B$ is constructed as a homotopy pushout of a pullback (double mapping cylinder):

pullback   homotopy pushout



Construction of classes of weight ≥ 2, cont.

As a homotopy pushout of a pullback, it has a Mayer-Vietoris sequence:

$$\begin{array}{ccccccccc} \dots & \longrightarrow & H^{k-1}(Q;R) & \xrightarrow{\delta} & H^k(E *_f E;R) & \longrightarrow & \oplus_{i=1}^2 H^k(E;R) & \longrightarrow & \dots \\ & & & & \uparrow p_2^* & \nearrow p^* \oplus p^* & & & \\ & & & & H^k(B;R) & & & & \end{array}$$

Want to find $u \in H^k(B;R)$ with $p_2^* u = 0$. If $u \in \ker p^*$, then $p_2^* u \in \mathrm{im}\delta$. Try to find $\alpha_u \in H^{k-1}(Q;R)$ with

$$\delta(\alpha_u) = p_2^* u,$$

find conditions that imply $\alpha_u = 0$.

Construction of classes of weight ≥ 2, cont.

Back to our setting: Here $p = r_n : B_{n+1}X \to S_nX$, $\gamma \mapsto \gamma|_{S^n}$, and the pullback is

$$Q = \{(\gamma_1, \gamma_2) \in (B_{n+1}X)^2 \mid \gamma_1|_{S^n} = \gamma_2|_{S^n}\} \approx C^0(S^{n+1}, X),$$

hence for $E_2 := B_{n+1}X *_f B_{n+1}X$ the Mayer-Vietoris sequence has the form

$$\cdots \to H^{k-1}(C^0(S^{n+1}, X); R) \xrightarrow{\delta} H^k(E_2; R) \to \oplus_{i=1}^2 H^k(B_{n+1}X; R) \to \ldots$$

Q: If $u \in H^k(X; R)$, $Z_n(u) \neq 0 \in H^{k-n}(S_nX; R)$, what is

$$\alpha_{Z_n(u)} \in H^{k-n-1}(C^0(S^{n+1}, X); R) \text{ ?}$$

Construction of classes of weight ≥ 2, cont.

Lemma

Let $u \in H^k(X;\mathbb{Q})$, $k \geq n+1$. Then

$$\alpha_{Z_n(u)} = e_{n+1}^* u/[S^{n+1}],$$

where $\quad e_{n+1} : C^0(S^{n+1}, X) \times S^{n+1} \to X, \quad e_{n+1}(\gamma, p) = \gamma(p).$

Use lemma and Mayer-Vietoris sequence to show:

Theorem

Let $u \in H^k(X;\mathbb{Q})$ with $Z_n(u) \neq 0$. If $f^*u = 0$ for all $f : S^{n+1} \times P \overset{C^0}{\to} X$, where P is any closed oriented manifold with $\dim P = k - n - 1$, then

$$\mathrm{wgt}(Z_n(u)) \geq 2.$$

Consequences for topological complexity (the case $n = 0$)

Theorem (Grant-M. '18, M. '19)

Let M be a closed or. manifold with $\dim M \geq 3$. If there exists $u \in H^2(M;\mathbb{Q})$ with $f^*u = 0$ for all $f \in C^0(T^2, M)$, then $\mathrm{TC}(M) \geq 6$.

Theorem (M. '19)

Let M be an even-dim. closed or. manifold. If $\mathrm{TC}(M) \leq 4$, then M is dominated by a manifold $P \times S^1$, i.e. there exists a degree-1 map $P \times S^1 \to M$. Here, P is a closed or. manifold with $\dim P = \dim M - 1$.

Proof.

Assume M is not dominated by ... Let $n := \dim M$, $u \neq 0 \in H^n(M;\mathbb{Q})$, $\bar{u} := 1 \times u - u \times 1 \in H^n(M \times M;\mathbb{Q})$. Then $\mathrm{wgt}(\bar{u}) \geq 2$ by the assumptions. Since n is even, $\bar{u}^2 = -2u \times u \neq 0$, hence $\mathrm{wgt}(\bar{u}^2) \geq 4$, so $\mathrm{TC}(M) \geq 5$. $\square$

Results on closed geodesics

Theorem (Lusternik-Fet, '51, for Riemannian manifolds)

Every Finsler metric on a closed manifold admits a non-constant closed geodesic.

Definition Two closed geodesics $\gamma_1, \gamma_2 : S^1 \to X$ are *positively distinct* if either $\gamma_1(S^1) \neq \gamma_2(S^1)$ or $\exists A \in O(2) \setminus SO(2)$ with $\gamma_1 = A \cdot \gamma_2$.

- Bangert-Long, 2007: every Finsler metric on S^2 has two positively distinct ones
- Rademacher, 2009: every *bumpy* Finsler metric on S^n has two positively distinct ones
- etc., Long-Duan 2009 for S^3, Wang 2019 for pinched metrics on S^n, ...

New results using spherical complexities

Theorem (M., 2020)

Let M be a closed oriented manifold, $F : TM \to [0, +\infty)$ be a Finsler metric of reversibility λ and flag curvature K. Let $\ell_F > 0$ be the length of the shortest non-const. closed geodesic of F.

a) *If $M = S^{2d}$, $d \geq 2$, $0 < K \leq 1$ and $F \leq \frac{1+\lambda}{\lambda}\sqrt{g_1}$, then F admits two pos. distinct closed geodesics of length $< 2\ell_F$. (g_1 = round metric of constant curvature 1)*

b) *If $M = \mathbb{C}P^n$ or $M = \mathbb{H}P^n$, $n \geq 3$, $0 < K \leq 1$ and $F \leq \frac{1+\lambda}{\lambda}\sqrt{g_1}$, then $\exists$ two pos. distinct closed geodesics of length $< 2\ell_F$.*

c) *If $M = S^{2d+1}$, $d \in \mathbb{N}$, $\frac{\lambda^2}{(1+\lambda)^2} < K \leq 1$ and $F \leq \frac{(k+1)(1+\lambda)}{m\lambda}\sqrt{g_1}$, then F admits $\lceil \frac{2m}{k} \rceil$ pos. distinct closed geodesics of length $< (k+1)\ell_F$.*

Method of proof of results for closed geodesics

Let $\iota_a : E_F^a \hookrightarrow \Lambda M$ be the inclusion. For parts a) and b):

- If γ is a closed geodesic of length ℓ_F, then its iterates satisfy $E_F(\gamma^k) \geq E_F(\gamma^2) = 4\ell_F^2 \ \forall k \geq 2$. Thus, if $a < 4\ell_F^2$ and $\nu(F, a) \geq 2$, then E_F^a contains two distinct closed geodesics.
- If $u \in H^*(\Lambda M; R)$ satisfies $\iota_a^* u \neq 0$, then $\nu(F, a) \geq \mathrm{wgt}(u)$. Thus, it suffices to find such u with $\mathrm{wgt}(u) \geq 2$ and $\iota_a^* u \neq 0$, where $a < 4\ell_F^2$.
- For classes of fixed degree, positive curvature bounds provide such energy bounds.

Perspectives and possible applications

- Equivariant versions, use richer ring structure in $H^*_{S^1}(LM, M; \mathbb{Q})$
- Applicable in greater generality to periodic Reeb orbits on contact manifolds? (generalizing closed geodesics on T^1M)
- Higher-dimensional applications, i.e. for $\mathrm{SC}_{n,M}$ if $n > 1$?
- Any ideas? Contact me!

Thank you for your attention!

talk based on:

S. Mescher, Spherical complexities, with applications to closed geodesics, arXiv:1911.03948, to appear in Algebr. Geom. Topol.

S. Mescher, Existence results for closed geodesics via spherical complexities, to appear in Calc. Var. 59, 2020.

(slides at `http://www.math.uni-leipzig.de/~mescher`)

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