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Oriented robot motion planning in Riemannian manifolds

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Geometrietag 2018

Topology and robot motion planning

Topology and robot motion planning

Real-world situation

A robot is supposed to move autonomously from one location to another in its workspace (e.g. warehouse, grid network, ...).

Topological motion planning problem

Let X be a topological space. Given $x, y \in X$, find a path $\gamma \in PX = C^0([0,1], X)$ with $\gamma(0) = x$ and $\gamma(1) = y$.

Definition

Let X be a top. space, $A \subset X \times X$. A *motion planner over* A is a map $s : A \to PX$, such that $(s(x,y))(0) = x$, $s(x,y))(1) = y$, for all $(x,y) \in A$.

(i.e. a section over A of the fibration

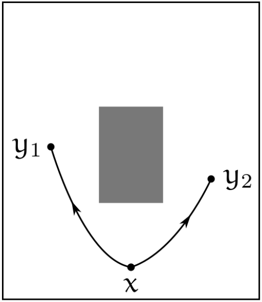
$$\pi : PX \to X \times X, \qquad \gamma \mapsto (\gamma(0), \gamma(1)).\ )$$

Existence of motion planners

Q: How "simple" can a motion planner over $X \times X$ be chosen?

Theorem (Farber, 2003)

Let X be a top. space. There exists a continuous motion planner over $X \times X$ if and only if X is contractible.



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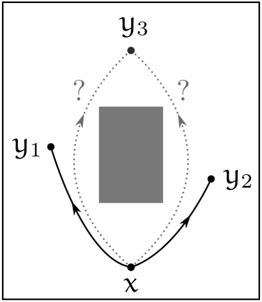


Topological complexity (Farber, 2003)

Definition

Let X be a top. space. The *topological complexity of X* is given by

$$\mathrm{TC}(X) = \inf \Big\{ n \in \mathbb{N} \;\Big|\; \exists \bigcup_{j=1}^{n} U_j = X \times X \text{ open cover, s.t.} \\ \forall j\, \exists s_j : U_j \to PX \text{ cont. motion planner} \Big\} \in \mathbb{N} \cup \{+\infty\}.$$

Theorem $\mathrm{TC}(X)$ is a homotopy invariant.

Theorem If X is an r-connected CW complex, then

$$\mathrm{TC}(X) \leq \frac{2 \dim X + 1}{r + 1}.$$

Examples of TC(X)

- $\mathrm{TC}(X) = 1 \quad \Leftrightarrow \quad X$ is contractible.
- S^n, n odd. Fix a nowhere vanishing vector field X on S^n.

  $U_1 = \{(x, y) \in S^n \times S^n \mid y \neq -x\}, \quad s_1 : U_1 \to PS^n,$

  $s_1(x, y) =$ param. minimal great circle segment from x to y,

  $U_2 =$ small neighborhood of $\{(x, -x) \mid x \in S^n\}, s_2 : U_2 \to PS^n,$

  $s_2(x, -x) =$ param. great circle segment tangent to $X(x)$,

  extend continuously to all of $U_2 \qquad \Rightarrow \quad \mathrm{TC}(S^n) = 2.$

- **Theorem (Farber, Tabachnikov, Yuzvinsky, 2003)**

$$\mathrm{TC}(\mathbb{R}P^n) = \begin{cases} I_n & n \in \{1, 3, 7\}, \\ I_n + 1 & \text{else}, \end{cases}$$

  where $I_n = \inf\{k \in \mathbb{N} \mid \exists \text{ immersion } \mathbb{R}P^n \to \mathbb{R}^k\}$.

A more general notion: sectional category

Definition Let $p : E \to B$ be a fibration. The *sectional category* or *Schwarz genus* of p is given by

$$\mathrm{secat}(p) = \inf \left\{ n \in \mathbb{N} \;\middle|\; \exists \bigcup_{j=1}^{n} U_j = B \text{ open cover}, \right.$$
$$\left. s_j : U_j \overset{C^0}{\to} E,\;\; p \circ s_j = \mathrm{incl}_{U_j} \;\; \forall j \right\}.$$

Two important cases:

$$\mathrm{TC}(X) = \mathrm{secat}\Big(\pi : PX \to X \times X,\; \gamma \mapsto (\gamma(0), \gamma(1))\Big),$$

$$\mathrm{cat}(X) = \mathrm{secat}\Big(p : P_{x_0}X = \{\gamma \in PX \mid \gamma(0) = x_0\} \to X,\; \gamma \mapsto \gamma(1)\Big)$$

(cat = Lusternik-Schnirelmann category)
Relations between cat and TC:

$\mathrm{cat}(X) \leq \mathrm{TC}(X) \leq \mathrm{cat}(X \times X)$, $\qquad \mathrm{TC}(G) = \mathrm{cat}(G)$ if G top. group.

Lower bounds by cohomology methods

Given a commutative ring R, a topological space Y and an ideal $I \subset H^*(Y;R)$, let

$$\mathrm{cl}_R(I) := \sup\{n \mid \exists u_1, \ldots, u_n \in I,\ \deg u_i > 0\ \forall i,\ \text{s.t. } u_1 \cup \cdots \cup u_n \neq 0\}.$$

Theorem (Schwarz, 1961)
Let $p : E \to B$ be a fibration, R be a commutative ring. Then

$$\mathrm{secat}(p) \geq \mathrm{cl}_R\left(\ker\left[p^* : H^*(B;R) \to H^*(E;R)\right]\right) + 1.$$

Put $Z_R(X) := \ker\left[\Delta^* : H^*(X \times X;R) \to H^*(X;R)\right]$,

where $\Delta(x) = (x,x)$. We call elements of $Z_R(X)$ *zero-divisors.*

Corollary (Farber, 2003) $\mathrm{TC}(X) \geq \mathrm{zcl}_R(X) + 1 := \mathrm{cl}_R(Z_R) + 1.$

Note If $\sigma \in H^*(X;R)$, then $\bar{\sigma} := 1 \times \sigma - \sigma \times 1 \in Z_R(X)$.

Recent topics

- Compute TC for relevant types of spaces (e.g. configuration spaces $F(X, k)$, collision-free motion planning).
- Construct "optimal" motion planners for useful spaces, i.e. having $\mathrm{TC}(X)$ domains of continuity.
- π discrete group, compute $\mathrm{TC}(K(\pi, 1))$ as invariant of π. (cf. Eilenberg-Ganea: $\mathrm{cat}(K(\pi, 1)) = \mathrm{cd}(\pi) + 1$ if $\mathrm{cd}(\pi) \neq 2$.) For example:

Theorem (Farber-M., 2017)

Let π be a group for which there is a finite CW complex of type $K(\pi, 1)$. If the centraliser of g is infinite cyclic for every $g \neq 1 \in \pi$, then

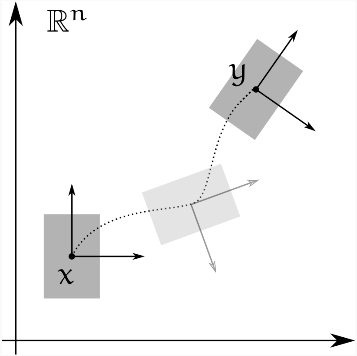
$$\mathrm{TC}(K(\pi, 1)) \in \{\mathrm{cd}(\pi \times \pi), \mathrm{cd}(\pi \times \pi) + 1\}.$$

Oriented robot motion planning

Motion planning with orientations in $\mathbb{R}^n$

Assume that the robot's workspace is a subset of $\mathbb{R}^n$. To represent its relative orientation, add an *orthonormal basis* to the representation.

(cf. J.-C. Latombe, *Robot motion planning*)



Linear oriented motion planning problem

Let $U \subset \mathbb{R}^n$. Given $x, y \in U$ and $B_1, B_2 \in SO(n)$, find curves $\gamma \in PU$ and $\alpha \in P(SO(n))$ with $(\gamma(0), \alpha(0)) = (x, B_1)$ and $(\gamma(1), \alpha(1)) = (y, B_2)$.

Oriented motion planning in Riemannian manifolds

Generalize to oriented Riemannian manifolds (M, g) as workspaces. Let $F(M, g)$ be its positive orthonormal frame bundle, i.e.

$$F(M,g)_x = \{\text{positive orthonormal bases of } (T_xM, g_x)\}.$$

$F(M, g) \to M$ is a principal $SO(n)$-bundle, where $n \in \mathbb{N}$.

Oriented motion planning problem

Let (M, g) be an oriented Riemannian manifold. Given $x, y \in M$, positive orthonormal bases B_1 of T_xM and B_2 of T_yM find $\gamma \in P(F(M, g))$ with $\gamma(0) = (x, B_1)$ and $\gamma(1) = (y, B_2)$.

One sees that: oriented motion planning in (M, g)
$\widehat{=}$ topological motion planning in $F(M, g)$

Aim: Study $\mathrm{TC}(F(M, g))$ and its properties.

Two motivating examples

1) $\mathsf{TC}(\mathbb{R}^n) = 1$ for all $n \in \mathbb{N}$, since $\mathbb{R}^n$ is contractible. But

$$\mathsf{TC}(F(\mathbb{R}^n, g)) = \mathsf{TC}(SO(n) \times \mathbb{R}^n) = \mathsf{TC}(SO(n)) = \mathrm{cat}(SO(n)) \geq n.$$

2) (S^n, g_n), where g_n is the round metric. Then

$$\mathsf{TC}(S^n) = \begin{cases} 2 & n \text{ odd}, \\ 3 & n \text{ even}. \end{cases}$$

But $F(S^n, g_n) \cong SO(n+1)$, hence

$$\mathsf{TC}(F(S^n, g_n)) = \mathsf{TC}(SO(n+1)) = \mathrm{cat}(SO(n+1)) \geq n+1.$$

Question Is it always true that $\mathsf{TC}(F(M, g)) \geq \dim M$?

The main results

Upper bounds for the TC of frame bundles (M., 2018)

Let (M, g) be an n-dimensional oriented Riemannian manifold.

Theorem

If M is parallelizable, then $\mathrm{TC}(F(M,g)) \leq \mathrm{cat}(SO(n)) + \mathrm{TC}(M) - 1$.

Theorem

Let G be a compact Lie group that acts freely, smoothly and orientation-preservingly on M. Then

$$\begin{aligned}\mathrm{TC}(F(M,g)) &\leq \frac{n(n+3)}{2} - \dim G + 1 \\ &\quad (= 2\dim M + \dim SO(n) - \dim G + 1)\end{aligned}$$

Idea of proof Apply an inequality for the TC of G-spaces shown by M. Grant (2012), use that $F(M,g)$ has a free $SO(n)$-action.

Lower bounds for the TC of frame bundles (M., 2018)

Theorem

Let (M, g) be an n-dimensional oriented Riemannian manifold. If one of the following holds:

a) *the map $i^* : H^*(F(M,g);\mathbb{K}) \to H^*(SO(n);\mathbb{K})$ induced by fiber inclusion is surjective for some field $\mathbb{K}$,*

b) *M has a spin structure,*

then $\mathrm{TC}(F(M,g)) \geq n$.

Idea of proof for part a) Use the *Leray-Hirsch theorem* and the ring structure of $H^*(SO(n);\mathbb{K})$ to show that

$$\mathrm{zcl}_{\mathbb{K}}(F(M,g)) \geq n.$$

Proof of the lower bound for spin manifolds (1)

Lemma (Farber-Tabachnikov-Yuzvinsky, 2003)

X top. space, $p : \widetilde{X} \to X$ a regular G-covering. Then

$$\mathrm{TC}(X) \geq \mathrm{secat}\left[q := p \times_G p : \widetilde{X} \times_G \widetilde{X} \to X \times X\right],$$

where $\widetilde{X} \times_G \widetilde{X}$ is the orbit space of the diagonal G-action.

Put $F(M) := F(M, g)$. By definition, a spin structure on M is a $\mathbb{Z}_2$-covering $\mathrm{Spin}(M) \to F(M)$, where $\mathrm{Spin}(M) \to M$ is a principal $\mathrm{Spin}(n)$-bundle.

Consider the bundle $\mathcal{S}(M) := \mathrm{Spin}(M) \times_{\mathbb{Z}_2} \mathrm{Spin}(M)$. Then

$$\mathrm{TC}(F(M)) \geq \mathrm{secat}\left[q : \mathcal{S}(M) \to F(M) \times F(M)\right].$$

Proof of the lower bound for spin manifolds (2)

The cohomology lower bound of secat yields

$$\mathrm{TC}(F(M)) \geq \mathrm{cl}_{\mathbb{Z}_2}\left(\ker\left[q^* : H^*(F(M)^2;\mathbb{Z}_2) \to H^*(\mathcal{S}(M);\mathbb{Z}_2)\right]\right) + 1.$$

$q : \mathcal{S}(M) \to F(M)^2$ is a $\mathbb{Z}_2$-covering. Let $w(q) \in H^1(F(M)^2;\mathbb{Z}_2)$ be its characteristic class / Stiefel-Whitney class. Have *transfer exact sequence*

$$\cdots \to H^{i-1}(F(M)^2;\mathbb{Z}_2) \xrightarrow{w(q)\cup\cdot} H^i(F(M)^2;\mathbb{Z}_2) \xrightarrow{q^*} H^i(\mathcal{S}(M);\mathbb{Z}_2) \xrightarrow{\mathrm{tr}^*} \dots$$

which yields

$$\begin{aligned}\mathrm{TC}(F(M)) &\geq \mathrm{cl}_{\mathbb{Z}_2}\left(\mathrm{im}\left[w(q)\cup\cdot : H^*(F(M)^2;\mathbb{Z}_2) \to H^*(F(M)^2;\mathbb{Z}_2)\right]\right) + 1 \\ &\geq \mathrm{height}(w(q)) + 1 := \sup\left\{n \in \mathbb{N}\,|w(q)^n \neq 0\right\} + 1.\end{aligned}$$

Proof of the lower bound for spin manifolds (3)

Proposition

Let $p : \widetilde{X} \to X$ be a $\mathbb{Z}_2$-covering. Then

$$\textit{height}(w(p)) \geq \sup\{n \in \mathbb{N} \mid \exists\, \mathbb{Z}_2\text{-}\textit{equivariant } f : S^n \xrightarrow{C^0} \widetilde{X}\}.$$

Thus, to show that $\mathrm{TC}(F(M)) \geq n$ it suffices to show:

Claim There exists a $\mathbb{Z}_2$-equivariant continuous map

$$f : S^{n-1} \to \mathcal{S}(M) = \mathrm{Spin}(M) \times_{\mathbb{Z}_2} \mathrm{Spin}(M).$$

$\mathcal{S}(M) \to M$ is a bundle with fiber $\mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{Spin}(n)$. The fiber inclusion $i : \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{Spin}(n) \to \mathcal{S}(M)$ is $\mathbb{Z}_2$-equivariant. Thus, it suffices to show:

Proof of the lower bound for spin manifolds (4)

Proposition

There exists a $\mathbb{Z}_2$-equivariant continuous map

$$f : S^{n-1} \to \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{Spin}(n).$$

Proof.

We identify

$$\mathrm{Spin}(n) = \{v_1 v_2 \dots v_{2k} \in \mathrm{Cl}(n) \mid v_1, \dots, v_{2k} \in S^{n-1}\}.$$

If $p : \mathrm{Spin}(n) \times \mathrm{Spin}(n) \to \mathrm{Spin}(n) \times_{\mathbb{Z}_2} \mathrm{Spin}(n)$ denotes the projection, then

$$f(v) = p(ve, s_0)$$

does the job, where $e \in S^{n-1}$, $s_0 \in \mathrm{Spin}(n)$ arbitrary. □

Some computations using these estimates (M., 2018)

- **Closed oriented surfaces** It is known that

$$\mathrm{TC}(\Sigma_g) = \begin{cases} 3 & \text{if } g \in \{0,1\}, \\ 5 & \text{if } g \geq 2. \end{cases}$$

  Can show that

$$\mathrm{TC}(F(S^2)) = \mathrm{TC}(F(T^2)) = 4, \quad \mathrm{TC}(F(\Sigma_g)) \in \{5,6\} \text{ for } g \geq 2.$$

- **Complex projective spaces** Shown by Farber that $\mathrm{TC}(\mathbb{C}P^n) = 2n+1$ for all $n \in \mathbb{N}$. We can show that

$$\mathrm{TC}(F(\mathbb{C}P^n)) \geq 4n \qquad \forall n \in \mathbb{N}.$$

Bonus: Topological complexity of symplectic manifolds

Estimates for cat and TC of symplectic manifolds

Question How are cat and TC of closed symplectic manifolds related to their symplectic topology?

Basic observation

Let (M,ω), $\omega \in \Omega^2(M)$, be a closed symplectic manifold, $\dim M = 2n$. Then $\omega^n \in \Omega^{2n}(M)$ is a volume form. Hence,

$$[\omega]^n \neq 0 \in H^{2n}(M;\mathbb{R}) \quad \Rightarrow \quad \mathrm{cl}_{\mathbb{R}}(M) \geq n.$$

Put $\bar{\omega} := 1 \times \omega - \omega \times 1 \in \Omega^2(M \times M)$. Then

$$[\bar{\omega}]^{2n} = (-1)^n \binom{2n}{n} [\omega]^n \times [\omega]^n \neq 0 \quad \Rightarrow \quad \mathrm{zcl}_{\mathbb{R}}(M) \geq 2n.$$

We derive

$$n+1 \leq \mathrm{cat}(M) \leq 2n+1, \qquad 2n+1 \leq \mathrm{TC}(M) \leq 4n+1.$$

Maximality of cat and TC for symplectic manifolds

Let (M, ω) be a closed symplectic manifold.

Theorem (Rudyak-Oprea, 1999)

If (M, ω) *is symplectically aspherical, i.e.* $\int_{S^2} f^*\omega = 0$ *for all* $f : S^2 \stackrel{C^0}{\to} M$*, then*

$$\mathrm{cat}(M) = \dim M + 1.$$

Theorem (Grant-M., 2018)

If (M, ω) *is symplectically atoroidal, i.e.* $\int_{T^2} f^*\omega = 0$ *for all* $f : T^2 \stackrel{C^0}{\to} M$*, then*

$$\mathrm{TC}(M) = 2\dim M + 1.$$

Remark sympl. atoroidal $\Rightarrow$ sympl. aspherical

cat, TC **and symplectic torus actions**

- If (M, ω) is symplectically aspherical, then it admits no nontrivial Hamiltonian S^1-actions.
- If (M, ω) is symplectically atoroidal, then it admits no nontrivial symplectic S^1-actions.

Open question Do cat and TC detect the maximal rank of Hamiltonian/symplectic torus actions on symplectic manifolds?

Thank you for your attention!

(slides at `http://www.math.uni-leipzig.de/~mescher`)