Diffusive processes on the Wasserstein space: Coalescing models, Regularization properties and McKean-Vlasov equations

Soutenance de thèse de Victor Marx

Laboratoire J.A. Dieudonné, Nice (France)
sous la direction de François Delarue

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UNIVERSITÉ CÔTE D'AZUR

Brownian motion on $\mathbb{R}^d$

↓

Smoothing properties:

- Restoration of uniqueness

$$\mathrm{d}x_t = b(x_t)\mathrm{d}t$$

There might be several solutions to the Cauchy problem, if b is not smooth enough (e.g. $b(x) = \mathrm{sign}(x)\sqrt{|x|}$).

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$$\mathrm{d}x_t = b(x_t)\mathrm{d}t + \mathrm{d}B_t.$$

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$\hookrightarrow$ Stroock-Varadhan.
$\hookrightarrow$ Zvonkin, Veretennikov, Krylov-Röckner.

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- Associated semi-group: heat semi-group.

$$P_t\phi(x) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} \phi(y) e^{-\frac{(x-y)^2}{2t}} \mathrm{d}y.$$

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Our aim: obtain similar properties for a diffusion on a **space of probability measures**.

Interest of the space of probability measures:

- Optimal transportation:
  Let us define the L_2-Wasserstein space of $\mathbb{R}$:

$$\mathcal{P}_2(\mathbb{R}) := \left\{ \mu \text{ probability measure on } \mathbb{R}\colon \int_{\mathbb{R}} x^2 \mathrm{d}\mu(x) < +\infty \right\},$$

and the Wasserstein distance W_2 on $\mathcal{P}_2(\mathbb{R})$:

$$W_2(\mu,\nu) = \inf_{\gamma} \left(\iint_{\mathbb{R}^2} |x-y|^2 \, \gamma(\mathrm{d}x, \mathrm{d}y) \right)^{1/2}$$

where the infimum is taken over all probability measures γ on $\mathbb{R}^2$ having marginals μ and ν. $\hookrightarrow$ Monge-Kantorovich problem.

$\rightsquigarrow$ understand the geometric properties of the space of probability measures $\hookrightarrow$ Benamou-Brenier, McCann.
Gradient flows $\hookrightarrow$ Jordan-Kinderlehrer-Otto, Ambrosio-Gigli-Savaré.

Interest of the space of probability measures:

- Systems of interacting particles: McKean-Vlasov equation

  N (very large number) particles.
  Dynamics of particle i are influenced by

  * position of particle i;
  * distribution of all particles;
  * noises: common noise and proper (*idiosyncratic*) noise.

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  Convergence rate
  $\hookrightarrow$ Bossy-Talay, Bossy-Jabir-Talay, Jourdain-Reygner, Fournier-Guillin, ...

Finite dimension

Brownian motion on $\mathbb{R}^d$

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Smoothing properties

- Restoration of uniqueness
- Estimates on the gradient of the semi-group

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in this thesis $\longrightarrow$

Infinite dimension

Construction of a diffusion on the L_2-Wasserstein space
Part 1

↓

Smoothing properties

- Restoration of uniqueness: Part 2
- Estimates on the gradient of the semi-group: Part 3

Outline

1. Construction of a diffusion process on the Wasserstein space
2. Restoration of uniqueness of McKean-Vlasov equations
3. Gradient estimates for a Wasserstein diffusion on the torus

Wasserstein diffusion

The following diffusive properties of some stochastic processes $(\mu_t)_{t\in[0,T]}$ on $\mathcal{P}_2(\mathbb{R})$ have already been studied:

- large deviations in small time
  $\rightsquigarrow$ Varadhan formula: for any measurable subset A in $\mathcal{P}_2(\mathbb{R})$,

$$\varepsilon \ln \mathbb{P}\left[\mu_{t+\varepsilon} \in A \,|\, \mu_t\right] \underset{\varepsilon\to 0}{\longrightarrow} -\frac{W_2(\mu_t, A)^2}{2}.$$

- Itô formula: for any smooth map $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$

$$\mathrm{d}\phi(\mu_t) = \text{martingale term} \;+\; (\text{second order derivative of } \phi)(\mu_t)\mathrm{d}t,$$

  where the martingale term has the square of Wasserstein gradient of ϕ as local quadratic variation.

Examples:

- construction via Dirichlet forms and entropic measure on $\mathcal{P}_2[0,1]$
  $\hookrightarrow$ von Renesse-Sturm, Sturm, Andres-von Renesse
- construction via a system of coalescing particles
  $\hookrightarrow$ Konarovskyi, Konarovskyi-von Renesse, M.

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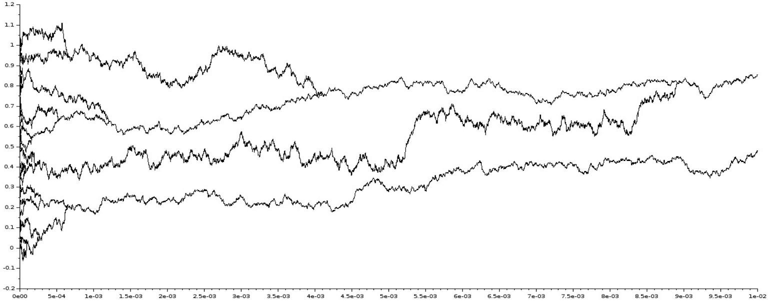
Konarovskyi's model

Let $N \geqslant 1$. For every $i \in \{1, \dots, N\}$, the particle $(x_t^i)_{t \in [0,T]}$ follows the equation:

$$x_t^i = g\left(\frac{i}{N}\right) + \int_0^t \frac{1}{N} \sum_{j=1}^N \frac{\mathbb{1}_{\{x_s^i = x_s^j\}}}{m_s^i} \sqrt{N} \mathrm{d}B_s^j,$$

where $B^1, \dots, B^N$ are N independent Brownian motions on $[0, T]$.

$m_s^i := \frac{1}{N} \sum_{k=1}^N \mathbb{1}_{\{x_s^i = x_s^k\}}$ represents the mass of particle i at time s.



We associate a process $(\mu_t^N)_{t\in[0,T]}$ on $\mathcal{P}_2(\mathbb{R})$ by taking the empirical measure

$$\mu_t^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{x_t^i}.$$

The sequence of the laws of the processes $(\mu_t^N)_{t\in[0,T]}$ is tight: there exists a limit process $(\mu_t)_{t\in[0,T]}$. $\hookrightarrow$ Konarovskyi

Two nice properties: for each time $t > 0$, μ_t supported by a finite set. Canonical representation as a quantile function $(y_t(u))_{t\in[0,T],u\in[0,1]}$ satisfying

(*i*) $y_0 = g$;

(*ii*) for each $t \in [0,T]$, $u \mapsto y_t(u)$ is non-decreasing;

(*iii*) for each $u \in [0,1]$, $(y_t(u))_{t\in[0,T]}$ is an L_2-martingale;

(*iv*) quadratic co-variation:

$$\langle y_\cdot(u), y_\cdot(u')\rangle_t = \int_0^t \frac{\mathbb{1}_{\tau_{u,u'}\leqslant s}}{m_s(u)} \mathrm{d}s,$$

where $m_s(u) = \mathrm{Leb}\{v \in [0,1] : \tau_{u,v} \leqslant s\}$.

Theorem 1

Let $g : [0,1] \to \mathbb{R}$ be an increasing L_p-function, $p > 2$.
Let y satisfy $(i) - (iv)$.
Then there exists a Brownian sheet w on $[0,1] \times [0,T]$ such that almost surely for all $u \in (0,1]$ and for all $t \in [0,T]$,

$$y_t(u) = g(u) + \int_0^t \int_0^1 \frac{\mathbb{1}_{\{y_s(u)=y_s(u')\}}}{m_s(u)} \mathrm{d}w(u',s).$$

Characterization of Brownian sheet:

- for each progressively measurable L_2-function f on $[0,1] \times [0,T]$, $(\int_0^t \int_0^1 f(u,s)\mathrm{d}w(u,s))_{t\in[0,T]}$ is a local martingale;
- for each f_1 and f_2,

$$\langle \int_0^\cdot \int_0^1 f_1(u,s)\mathrm{d}w(u,s), \int_0^\cdot \int_0^1 f_2(u,s)\mathrm{d}w(u,s)\rangle_t = \int_0^t \int_0^1 f_1(u,s) f_2(u,s)\mathrm{d}u\mathrm{d}s.$$

Open problem: Uniqueness of a process $(y_t(u))_{t\in[0,T],u\in[0,1]}$ satisfying $(i)-(iv)$.

$$y_t(u) = g(u) + \int_0^t \int_0^1 \frac{\mathbb{1}_0(y_s(u)-y_s(u'))}{m_s(u)} \mathrm{d}w(u',s).$$

Let us introduce a smooth approximation of the equation:

$$y_t^{\sigma,\varepsilon}(u) = g(u) + \int_0^t \int_0^1 \frac{\varphi_\sigma(y_s^{\sigma,\varepsilon}(u)-y_s^{\sigma,\varepsilon}(u'))}{\varepsilon + m_s^{\sigma,\varepsilon}(u)} \mathrm{d}w(u',s), \qquad \text{(SmoothDiff)}$$

where $m_s^{\sigma,\varepsilon} = \int_0^1 \varphi_\sigma^2(y_s^{\sigma,\varepsilon}(u)-y_s^{\sigma,\varepsilon}(v))\mathrm{d}v$.

- ▷ The indicator function $\mathbb{1}_0$ is replaced by a smooth function φ_σ.
- ▷ The denominator is bounded by below by ε.

$$y_t^{\sigma,\varepsilon}(u) = g(u) + \int_0^t \int_0^1 \frac{\varphi_\sigma(y_s^{\sigma,\varepsilon}(u) - y_s^{\sigma,\varepsilon}(u'))}{\varepsilon + m_s^{\sigma,\varepsilon}(u)} \mathrm{d}w(u', s), \qquad \text{(SmoothDiff)}$$

where $m_s^{\sigma,\varepsilon} = \int_0^1 \varphi_\sigma^2(y_s^{\sigma,\varepsilon}(u) - y_s^{\sigma,\varepsilon}(v))\mathrm{d}v$.

Theorem 2 - Tightness and characterization of the limit

Let $g : [0,1] \to \mathbb{R}$ be an increasing L_p-function, $p > 2$.

For each $\sigma > 0$ and for each $\varepsilon > 0$,

- there is a unique solution $y^{\sigma,\varepsilon}$ to equation (SmoothDiff) in $L_2([0,1], \mathcal{C}[0,T])$;
- almost surely for each time $t \in [0,T]$, $u \mapsto y_t^{\sigma,\varepsilon}(u)$ is non-decreasing and càdlàg.

Furthermore, there is a subsequence of $(y^{\sigma,\varepsilon})_{\sigma,\varepsilon \in \mathbb{Q}\cap(0,+\infty)}$ that converges in distribution to a limit y in $L_2([0,1], \mathcal{C}[0,T])$.
The limit process y

- satisfies properties $(i) - (iv)$;
- belongs to $\mathcal{D}((0,1), \mathcal{C}[0,T])$;
- for each time $t \in (0,T]$, $u \mapsto y_t(u)$ is a step function.

Key step in the proof

In order to prove tightness, we use the following uniform control on the inverse of mass:

Let $g \in L_p[0,1]$, $p > 2$. Let $\beta \in \left(1, \frac{3}{2} - \frac{1}{p}\right)$.
Then there is $C > 0$ independent of σ, ε such that for every $t \in [0,T]$,

$$\mathbb{E}\left[\int_0^t \int_0^1 \left(\frac{m_s^{\sigma,\varepsilon}(u)}{\left(\varepsilon + m_s^{\sigma,\varepsilon}(u)\right)^2}\right)^\beta \mathrm{d}u\mathrm{d}s\right] \leqslant C\sqrt{t}.$$

↳ The term $\frac{m_s^{\sigma,\varepsilon}(u)}{\left(\varepsilon + m_s^{\sigma,\varepsilon}(u)\right)^2}$ is arising when computing the quadratic variation of the martingale part of $y_t^{\sigma,\varepsilon}$.

Outline

McKean-Vlasov equations

Back to finite dimension.
Let us consider a system of N random particles $X^1, \dots, X^N$ interacting through:

$$\mathrm{d}X_t^i = b(t, X_t^i, \overline{\mu}_t^N)\mathrm{d}t + \sigma(t, X_t^i, \overline{\mu}_t^N)\mathrm{d}B_t^i$$

where $\overline{\mu}_t^N = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$.
$(B^i)_{1\leqslant i\leqslant N}$: idiosyncratic noise.

$\rightsquigarrow$ infinite-dimensional equation describes asymptotical behavior:

$$\mathrm{d}X_t = b(t, X_t, \mu_t)\mathrm{d}t + \sigma(t, X_t, \mu_t)\mathrm{d}B_t$$

where $\mu_t = \mathrm{Law}(X_t)$.

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where ~~$\mu_t = \mathrm{Law}(X_t)$~~ $\mu_t = \mathrm{Law}(X_t | \sigma((W_s)_{s\leqslant t}))$.
The process $(\mu_t)_{t\in[0,T]}$ is now random!

Assumptions on the drift function

Restoration of uniqueness by noise:

$$\mathrm{d}X_t = b(X_t, \mu_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t.$$

$\hookrightarrow$ Jourdain, Mishura-Veretennikov, Lacker, Chaudru de Raynal-Frikha, Röckner-Zhang

where the drift function $b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ satisfies

- μ fixed, $x \mapsto b(x, \mu)$ bounded measurable,
- x fixed, $\mu \mapsto b(x, \mu)$ Lipschitz-continuous with respect to the total variation distance d_{TV}, defined by

$$\mathrm{d}_{\mathsf{TV}}(\mu, \nu) := \sup \left\{ \left| \int f \mathrm{d}\mu - \int f \mathrm{d}\nu \right| ; f \text{ measurable}, \|f\|_{L_\infty} \leqslant 1 \right\}.$$

Idea: instead of a finite-dimensional Brownian motion, use our diffusion on $\mathcal{P}_2(\mathbb{R})$ to obtain a regularization result for a drift b with lower regularity than Lipschitz.

Infinite-dimensional common noise

Let $g : [0,1] \to \mathbb{R}$ be an initial quantile function.
Let $(y_t)_{t\in[0,T]}$ be a process of quantile functions satisfying for each $u \in [0,1]$:

$$\begin{cases} \mathrm{d}y_t(u) = b(y_t(u), \mu_t)\mathrm{d}t + \dfrac{1}{m_t(u)^{1/2}} \displaystyle\int_{\mathbb{R}} f(k)\Re(e^{-iky_t(u)}\mathrm{d}w(k,t)); \\ \mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}; \\ y_0(u) = g(u), \end{cases} \qquad (\star)$$

where

- $m_t(u) = \int_0^1 \varphi(y_t(u) - y_t(v))\mathrm{d}v$;
- $w = w^{\Re} + iw^{\Im}$ is a complex-valued Brownian sheet

$$\Re(e^{-iky_t(u)}\mathrm{d}w(k,t)) = \cos(ky_t(u))\mathrm{d}w^{\Re}(k,t) + \sin(ky_t(u))\mathrm{d}w^{\Im}(k,t);$$

- $f(k) = f_\alpha(k) := \frac{C_\alpha}{(1+k^2)^{\alpha/2}}$.
  $\alpha \nearrow \Rightarrow$ the map $u \mapsto y_t(u)$ is more regular.

The process $(\mu_t)_{t\in[0,T]}$ is here **random**.

$$\begin{cases} \mathrm{d}y_t(u) = b(y_t(u), \mu_t)\mathrm{d}t + \frac{1}{m_t(u)^{1/2}} \int_{\mathbb{R}} f(k)\Re(e^{-iky_t(u)}\mathrm{d}w(k,t)); \\ \mu_t = \mathrm{Leb}\,|_{[0,1]} \circ y_t^{-1}; \\ y_0(u) = g(u). \end{cases} \qquad (\star)$$

Theorem 3 - Restoration of uniqueness

Let g be a strictly increasing $\mathcal{C}^1$-function.
Let $f : k \mapsto \frac{1}{(1+k^2)^{\alpha/2}}$ with $\alpha > \frac{3}{2}$.
Let $b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a bounded measurable function such that

- for each $\mu \in \mathcal{P}_2(\mathbb{R})$, $x \mapsto b(x,\mu)$ is $\mathcal{C}^2$;
- $\partial_x b$ and $\partial_x^{(2)} b$ are uniformly bounded on $\mathbb{R} \times \mathcal{P}_2(\mathbb{R})$.

Then there is a unique weak solution to equation $(\star)$.

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Method of proof

Find an $L_2(\mathbb{R};\mathbb{C})$-valued process $(h_t)_{t\in[0,T]}$ such that

$$b(y_t(u),\mu_t) = \frac{1}{m_t(u)^{1/2}} \int_{\mathbb{R}} f(k) e^{-iky_t(u)} h_t(k) \mathrm{d}k.$$

and apply Girsanov's Theorem.
The solution h is of the form:

$$h_t(k) = \frac{1}{f(k)} \mathcal{F}^{-1}(b(\cdot,\mu_t)\cdot(\varphi * \mu_t))(k), \qquad \frac{1}{f(k)} = \frac{(1+k^2)^{\alpha/2}}{C_\alpha}.$$

$\alpha \nearrow \Rightarrow$ more difficult to obtain integrability of h.

Gap

$$\text{Velocity field} \quad b : \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$$
$$(x, \mu) \mapsto b(x, \mu)$$

Hypotheses on b	Jourdain, Mishura-Veretennikov, Lacker, ...		Theorem 3
Regularity in x	bounded and measurable		$\mathcal{C}^2$
Regularity in μ (d_{TV} distance)	Lipschitz		bounded and measurable
Example	$\mathbb{E}[b(X_t)]$, b bounded	$b(X_t, \mathbb{E}[X_t])$	$b(\mathbb{E}[X_t])$, b bounded or Hölder

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Hypotheses on b	Jourdain, Mishura-Veretennikov, Lacker, ...	Theorem 4 $\eta > \frac{3}{2}(1-\delta)$	Theorem 3
Regularity in x	bounded and measurable	H^η	$\mathcal{C}^2$
Regularity in μ (d_{TV} distance)	Lipschitz	$\mathcal{C}^\delta$, $\delta \in [0,1]$	bounded and measurable
Example	$\mathbb{E}[b(X_t)]$, b bounded	$b(X_t, \mathbb{E}[X_t])$	$b(\mathbb{E}[X_t])$, b bounded or Hölder

Diffusion model with idiosyncratic noise

- averaging over finite dimension with β;
- no quantile representation anymore;
- constant mass equal to one;
- notion of weak compatible solution.

$$\begin{cases} \mathrm{d}z_t = b(z_t, \mu_t)\mathrm{d}t + \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t} dw(k,t)) + \mathrm{d}\beta_t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t | \mathcal{G}_t^{\mu,w}), \quad (\mu, w) \perp\!\!\!\perp (\beta, \xi) \\ z_0 = \xi, \quad \mathcal{L}^{\mathbb{P}}(\xi) = \mu_0. \end{cases} \qquad (\blacktriangle)$$

For each t, (ξ, w, μ) and β conditionally independent given $\mathcal{G}_t$.

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For each t, (ξ, w, μ) and β conditionally independent given $\mathcal{G}_t$.

Theorem 4

Let $\eta > 0$ and $\delta \in [0,1]$ be such that $\eta > \frac{3}{2}(1-\delta)$ and let b be of class $(H^\eta, \mathcal{C}^\delta)$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(k) := \frac{1}{(1+k^2)^{\alpha/2}}$, with $\frac{3}{2} < \alpha \leqslant \frac{\eta}{1-\delta}$. Then existence and uniqueness of a weak compatible solution to equation $(\blacktriangle)$ hold.

Method of proof

Step 1: drift $b \equiv 0$.

$$\mathrm{d}z_t = \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t} dw(k,t)) + \mathrm{d}\beta_t.$$

$\exists!$ strong solution $z_t = \mathcal{Z}_t(\xi, w, \beta) \rightsquigarrow$ Yamada-Watanabe Theorem.
Moreover, if $(\Omega^i, \mathcal{G}^i, \mathbb{P}^i, z^i, w^i, \beta^i, \xi^i)$, $i = 1, 2$, are two solutions, then

$$\mathcal{L}^{\mathbb{P}^1}(z^1, w^1, \beta^1) = \mathcal{L}^{\mathbb{P}^2}(z^2, w^2, \beta^2). \qquad \textbf{(Uniq 1)}$$

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Step 2: drift $\widetilde{b}$ such that there is C satisfying

- $\widetilde{b}$ uniformly bounded: $\forall x, \mu$, $|\widetilde{b}(x,\mu)| \leqslant C$;
- $\widetilde{b}$ uniformly Lipschitz-continuous: $\forall x, \mu, \nu$, $|\widetilde{b}(x,\mu) - \widetilde{b}(x,\nu)| \leqslant C\mathrm{d}_{\mathrm{TV}}(\mu,\nu)$.

$$\begin{cases} \mathrm{d}z_t = \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t} dw(k,t)) + \mathrm{d}\beta_t + \widetilde{b}(z_t, \mu_t)\mathrm{d}t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t | \mathcal{G}_t^{\mu,w}). \end{cases}$$

Step 2 (foll.):

$$\begin{cases} \mathrm{d}z_t = \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t} dw(k,t)) + \mathrm{d}\beta_t + \widetilde{b}(z_t,\mu_t)\mathrm{d}t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t | \mathcal{G}_t^{\mu,w}). \end{cases}$$

$\exists!$ semi-strong solution, *i.e.* $(\mu_t)_{t\in[0,T]}$ is adapted with respect to $(\mathcal{G}_t^w)_{t\in[0,T]}$. Moreover, if $(\Omega^i, \mathcal{G}^i, \mathbb{P}^i, z^i, w^i, \beta^i, \xi^i)$, $i = 1, 2$, are two solutions, then

$$\mathcal{L}^{\mathbb{P}^1}(z^1, w^1) = \mathcal{L}^{\mathbb{P}^2}(z^2, w^2). \qquad \textbf{(Uniq 2)}$$

Idea: **Existence:** fixed-point argument $\hookrightarrow$ Jourdain, Lacker, ...
Girsanov argument with respect to β:

$$\frac{\mathrm{d}\mathbb{P}^\mu}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^t \widetilde{b}(z_s,\mu_s)\mathrm{d}\beta_s - \frac{1}{2}\int_0^t \widetilde{b}(z_s,\mu_s)^2 \mathrm{d}s \right).$$

Solve the fixed-point problem "$\mathcal{L}^{\mathbb{P}^\mu}(z_t) = \mu$".
Uniqueness: prove that each weak solution μ is solution to the fixed-point problem $\rightsquigarrow$ μ adapted with respect to $(\mathcal{G}_t^w)_{t\in[0,T]}$. Use (**Uniq 1**).

Step 3: drift b of class $(H^\eta, \mathcal{C}^\delta)$.

$$\begin{cases} \mathrm{d}z_t = \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t} dw(k,t)) + \mathrm{d}\beta_t + b(z_t,\mu_t)\mathrm{d}t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t | \mathcal{G}_t^{\mu,w}). \end{cases}$$

Let us write: $b = \underbrace{\widetilde{b}}_{\text{Lipschitz in } \mu \text{ w.r.t. } \mathrm{d}_{\mathrm{TV}}} + \underbrace{b - \widetilde{b}}_{\substack{\text{regular in } x \\ \text{low regularity in } \mu}}$.

$$\begin{cases} \mathrm{d}z_t = \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t} dw(k,t)) + \mathrm{d}\beta_t + \widetilde{b}(z_t,\mu_t)\mathrm{d}t + (b-\widetilde{b})(z_t,\mu_t)\mathrm{d}t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t | \mathcal{G}_t^{\mu,w}). \end{cases}$$

Step 3 (foll.):

$$\begin{cases} \mathrm{d}z_t = \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t} dw(k,t)) + \mathrm{d}\beta_t + \widetilde{b}(z_t,\mu_t)\mathrm{d}t + (b-\widetilde{b})(z_t,\mu_t)\mathrm{d}t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t | \mathcal{G}_t^{\mu,w}). \end{cases}$$

Idea: Fourier inversion (as previously):

$$h_t(k) = \frac{1}{f(k)} \mathcal{F}^{-1}((b-\widetilde{b})(\cdot,\mu_t))(k).$$

and Girsanov argument with respect to w

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^t \int_{\mathbb{R}} \Re(\overline{h_t(k)} \mathrm{d}w(k,t)) - \frac{1}{2}\int_0^t \int_{\mathbb{R}} |h_t(k)|^2 \mathrm{d}k\mathrm{d}t \right).$$

Prove that the solution $(\mu_t)_{t\in[0,T]}$ to

$$\begin{cases} \mathrm{d}z_t = \int_{\mathbb{R}} f(k)\Re(e^{-ikz_t} dw(k,t)) + \mathrm{d}\beta_t + \widetilde{b}(z_t,\mu_t)\mathrm{d}t, \\ \mu_t = \mathcal{L}^{\mathbb{P}}(z_t | \mathcal{G}_t^{w}) \end{cases}$$

satisfies $\mu_t = \mathcal{L}^{\mathbb{Q}}(z_t | \mathcal{G}_t^{\mu,\widetilde{w}})$, where

$$\widetilde{w}(k,t) = w(k,t) + \int_0^t \int_0^k h_s(l)\mathrm{d}l\mathrm{d}s.$$

Open problem

Influence of a **non-constant mass**:

- in Konarovskyi's model $\rightsquigarrow$ mass helps to obtain tightness.
- here, it could also bring some additional regularity to our system.

Outline

Finite-dimensional case

Let X_t^x be solution of

$$\begin{cases} \mathrm{d}X_t^x = b(t, X_t^x)\mathrm{d}t + \sigma(t, X_t^x)\mathrm{d}W_t; \\ X_0^x = x. \end{cases}$$

Associated semi-group

$$P_t\phi(x) = \mathbb{E}\left[\phi(X_t^x)\right].$$

Under some regularity assumptions on b and σ, Bismut-Elworthy-Li integration by parts formula yields

$$\nabla(P_t\phi)_{x_0}(v_0) = \frac{1}{t}\mathbb{E}\left[\phi(X_t^x)\int_0^t \langle V_s, \sigma(X_s^x)\mathrm{d}W_s\rangle\right],$$

where $(V_t)_{t\in[0,T]}$ is a certain adapted stochastic process starting at v_0.
$\hookrightarrow$ Bismut, Elworthy, Elworthy-Li
It follows that

$$\|\nabla P_t\phi\|_{L_\infty} \leqslant \frac{C}{\sqrt{t}}\|\phi\|_{L_\infty}.$$

Semi-groups and applications

Applications in finite dimension:

- Geometry: study of PDEs on manifolds. ↪ Thalmaier, Thalmaier-Wang
  ↳ Hypoelliptic diffusions: e.g. Bakry-Baudoin-Bonnefont-Chafaï.
- Finance: computing price sensitivies (Greeks).
- Numerical analysis: Euler schemes for SDEs.

McKean-Vlasov equations:
↪ Crisan-McMurray, Baños

Further works related to the study of semi-groups for infinite-dimensional equations arising from particle systems:
↪ Kolokoltsov, Buckdahn-Li-Peng-Rainer, Chassagneux-Crisan-Delarue

Study of a diffusion on the torus $\mathbb{T}$

Two main reasons for working on the torus $\mathbb{T}$:

- to avoid problems of density at infinity;
- to work only with measure-valued processes having a positive density everywhere on the state space.

Equation of the diffusion:

$$\mathrm{d}x_t^g(u) = \sum_{k\in\mathbb{Z}} f_k \Re(e^{-ikx_t^g(u)}\mathrm{d}W_t^k) + \mathrm{d}\beta_t; \quad u \in [0,1], t \in [0,T].$$

Assumptions:

- for each k, $W_\cdot^k = W_\cdot^{\Re,k} + iW_\cdot^{\Im,k}$;
- $((W_\cdot^{\Re,k})_{k\in\mathbb{Z}}, (W_\cdot^{\Im,k})_{k\in\mathbb{Z}}, \beta_\cdot)$ is a collection of independent real-valued Brownian motions;
- $f_k = f_{\alpha,k} := \frac{C_\alpha}{(1+k^2)^{\alpha/2}}$.

Role of α: controlling the regularity of the solution $u \mapsto x_t^g(u)$ for each t.

Quantile functions on the torus

The initial condition $g : \mathbb{R} \to \mathbb{R}$ is seen as a quantile function on the torus.
It satisfies:

- regularity: $g \in \mathcal{C}^1$;
- monotonicity: $g'(u) > 0$ for each $u \in \mathbb{R}$;
- pseudo-periodicity: $g(u+1) = g(u) + 2\pi$ for each $u \in \mathbb{R}$.

By the previous assumption, it can be seen as a function $g : [0,1] \to \mathbb{T}$.

If $g \in \mathcal{C}^{1+\theta}$, $\theta > 0$:
those properties are preserved almost surely for every t by the solution $u \mapsto x_t^g(u)$.

Role of β

$$\mathrm{d}x_t^g(u) = \sum_{k\in\mathbb{Z}} f_k \Re(e^{-ikx_t^g(u)}\mathrm{d}W_t^k) + \mathrm{d}\beta_t; \quad u \in [0,1], t \in [0,T].$$

As before, we study:

$$\begin{aligned} \nu_t^g &= \mathrm{Leb}_{[0,1]} \circ (x_t^g)^{-1} \\ \mu_t^g &= (\mathrm{Leb}_{[0,1]} \otimes \mathbb{P}^\beta) \circ (x_t^g)^{-1}. \end{aligned}$$

μ_t^g is also the conditional law of x_t^g given $\sigma((W_s^k)_{s\leqslant t})_{k\in\mathbb{Z}}$.

Equation on the density p_t^g of μ_t^g:

$$\mathrm{d}p_t^g(v) = -\partial_v \left(p_t(v) \sum_{k\in\mathbb{Z}} f_k \Re(e^{-ikv}\mathrm{d}W_t^k) \right) + \lambda p_t''(v)\mathrm{d}t,$$

$\hookrightarrow$ [Denis, Matoussi, Stoica]: threshold at $\lambda > \frac{\sum_{k\in\mathbb{Z}} f_k^2}{2}$.

without β: $\lambda = \frac{\sum_{k\in\mathbb{Z}} f_k^2}{2}$; with β: $\lambda = \frac{\sum_{k\in\mathbb{Z}} f_k^2 + 1}{2}$.

Semi-group

Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$.
We say that ϕ is $\mathbb{T}$-*stable* if $\phi(\mu) = \phi(\nu)$ if μ and ν have same trace on the torus.

Let us define the semi-group:

$$P_t\phi(\mu_0^g) = \mathbb{E}^W\left[\phi(\mu_t^g)\right]$$

Theorem 5 - Gradient estimate

Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be $\mathbb{T}$-stable and belong to $\mathcal{C}_b^{1,1}$.
Let $\theta \in (0,1)$ and $\alpha \geqslant \frac{7}{2} + \theta$.
Let g be an initial condition of class $\mathcal{C}^{3+\theta}$.
Let h be a 1-periodic perturbation of class $\mathcal{C}^1$.
Then $\rho \mapsto P_t\phi(\mu_0^{g+\rho h})$ is differentiable at $\rho = 0$ and there is $C > 0$ independent of h such that for every $t \in (0, T]$,

$$\left| \frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t\phi(\mu_0^{g+\rho h}) \right| \leqslant C \frac{\mathbb{V}^W \left[\phi(\mu_t^g)\right]^{1/2}}{t^{2+\theta}} \|h\|_{\mathcal{C}^1}.$$

Theorem 6

Let $\phi : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be a bounded, uniformly continuous and $\mathbb{T}$-stable function.
If h belongs to $\mathcal{C}^{3+\theta}$,
then there is $\rho_0 > 0$ and $C > 0$, depending on g and h such that for every $\rho \in (-\rho_0, \rho_0)$ and every $t \in (0, T]$,

$$\left| P_t\phi(\mu_0^{g+\rho h}) - P_t\phi(\mu_0^g) \right| \leqslant C|\rho| \frac{\|\phi\|_{L_\infty}}{t^{2+\theta}}.$$

Method of proof ↪ Thalmaier, Thalmaier-Wang

⇝ Apply Girsanov's Theorem:

$$\mathbb{E}^W\left[\phi(\mu_t^{g+\rho t h})\mathcal{E}_t^{g,\rho}\right] = \mathbb{E}^W\left[\phi(\mu_t^g)\right].$$

Method of proof $\hookrightarrow$ Thalmaier, Thalmaier-Wang

- Expansion of the process $(x_t^{g+\rho th})_{t\in[0,T]}$ $\hookrightarrow$ Kunita

$$x_t^{g+\rho th}(u) = x_0^g(u) + \sum_{k\in\mathbb{Z}} f_k \int_0^t \Re(e^{-ikx_s^g(u)} \mathrm{d}W_s^k) + \beta_t$$
$$+\rho \int_0^t \frac{\partial_u x_s^g(u)}{g'(u)} h(u)\mathrm{d}s + \mathcal{O}(\rho^2).$$

- Solve an inversion problem (as previously by Fourier transform): find a collection of adapted processes $((\lambda_t^k)_{t\in[0,T]})_{k\in\mathbb{Z}}$ such that for all $u\in[0,1]$, $s\in[0,T]$,

$$\frac{\partial_u x_s^g(u)}{g'(u)} h(u) = \sum_{k\in\mathbb{Z}} f_k \Re(e^{-ikx_s^g(u)} \lambda_s^k).$$

- Apply Girsanov's Theorem:

$$\mathbb{E}^W\left[\phi(\mu_t^{g+\rho th})\mathcal{E}_t^{g,\rho}\right] = \mathbb{E}^W\left[\phi(\mu_t^g)\right].$$

Method of proof $\hookrightarrow$ Thalmaier, Thalmaier-Wang

$\rightsquigarrow$ We differentiate with respect to ρ at point $\rho = 0$ the Girsanov equality:

$$\mathbb{E}^W \left[\phi(\mu_t^{g+\rho t h}) \mathcal{E}_t^{g,\rho} \right] = \mathbb{E}^W \left[\phi(\mu_t^g) \right].$$

$$\Big\downarrow \ \frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0}$$

$$t \frac{\mathrm{d}}{\mathrm{d}\rho}_{|\rho=0} P_t \phi(\mu_0^{g+\rho h}) - \mathbb{E}^W \mathbb{E}^\beta \left[\phi(\mu_t^g) \sum_{k \in \mathbb{Z}} \int_0^t \Re(\overline{\lambda_s^k} \mathrm{d} W_s^k) \right] = 0.$$

$\Lsh$ Bismut-Elworthy-Li integration by parts formula.

Problem:

$$\frac{\partial_u x_s^g(u)}{g'(u)} h(u) = \sum_{k \in \mathbb{Z}} f_k \Re(e^{-ikx_s^g(u)} \lambda_s^k), \qquad f_k = \frac{C_\alpha}{(1+k^2)^{\alpha/2}}.$$

Role of α:

$\alpha \nearrow \Rightarrow$ higher regularity on $u \mapsto \partial_u x_s^g(u)$ and more difficult to invert f_k.

$\rightsquigarrow$ not possible to find an appropriate exponent α.

- $\rightsquigarrow$ Split the left hand side into two terms:

$$\frac{\partial_u x_s^g(u)}{g'(u)} h(u) = \underbrace{A_s^{g,\varepsilon}}_{\text{smooth}}(u) + \underbrace{(A_s^g - A_s^{g,\varepsilon})}_{\text{remainder term}}(u).$$

- $\rightsquigarrow$ Apply Girsanov's Theorem with respect to the noise β to deal with the remainder term.

This leads to an integration by parts formula *with remainder term* and we get a gradient estimate with a deteriorated explosion rate $t^{-(2+\theta)}$.

Open problems

- improvement of the explosion rate $t^{-(2+\theta)}$:
  - Already done: improvement to $t^{-(1+\theta)}$ by an interpolation argument, assuming more regularity on g and the direction of perturbation h.
  - Already done: Application: gradient estimate with a non trivial source term.
- improvement of the explosion rate $\rightsquigarrow$ get closer to $t^{-1/2}$. Aim: obtain a gradient estimate for a diffusion **with a non-smooth drift function** b;
- **convergence rate** of the particle system associated to the diffusion on the torus.

Thank you for your attention.

Merci pour votre attention.