Regularization result for an infinite-dimensional diffusion process

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Introduction

Define the following set of probability measures:

$$\mathcal{P}_2(\mathbb{R}) := \left\{ \mu \text{ probability measure on } \mathbb{R}\colon \int_{\mathbb{R}} x^2 \mathrm{d}\mu(x) < +\infty \right\},$$

and the following L^2-Wasserstein distance:

$$W_2(\mu, \nu) = \inf_{(X,Y) \text{ coupling of } (\mu,\nu)} \mathbb{E}\left[|X - Y|^2 \right]^{1/2}.$$

$(\mathcal{P}_2(\mathbb{R}), W_2)$ is called the Wasserstein *space*.

Fokker-Planck equation

We consider the following equation for the $\mathcal{P}_2(\mathbb{R})$-valued process $(\mu_t)_{t\in[0,T]}$: for all $u \in (0,1)$

$$\mathrm{d}y_t(u) = b(y_t(u), \mu_t)\mathrm{d}t,$$

where $u \in (0,1) \mapsto y_t(u)$ is the quantile function associated to the measure μ_t.

As for the transport equation, there is no uniqueness in general if we assume b to be only bounded and continuous, and not Lipschitz.

An interesting question: can we restore uniqueness by adding a small diffusive perturbation?

Regularization by Additive Noise

- Uniqueness in law for general parabolic equations in finite dimension (Stroock, Varadhan, 1970').
- Transport equation perturbed by a Brownian motion.

$$\mathrm{d}x_t = b(x_t)\mathrm{d}t + \mathrm{d}B_t$$

  Strong uniqueness for $d = 1$, b bounded and continuous (Zvonkin, 1974).
- Strong uniqueness in higher dimension (Veretennikov, 1980).
- Strong uniqueness for $b \in L^p$, $p > d$ (Krylov-Röckner, 2005).

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Outline

1. Construction of a diffusion process on the Wasserstein space
2. Variation of the initial condition
3. Regularization result and perspectives

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Construction of a diffusion process

We associate to a process $(\mu_t)_{t\in[0,T]}$ on the Wasserstein space $\mathcal{P}_2(\mathbb{R})$ the quantile process $(y_t)_{t\in[0,T]}$, where for each $t \in [0,T]$, $u \in [0,1] \mapsto y_t(u)$ is the quantile function associated to the measure μ_t.

$$y_t(u) := \inf\{x \in \mathbb{R} : \int_{-\infty}^{x} \mathrm{d}\mu_t > u\}.$$

For every $t \in [0,T]$, y_t belongs to $L^2([0,1])$.

SDE for the process $(y_t)_{t\in[0,T]}$

For each $u \in (0,1)$, the real process $(y_t(u))_{t\in[0,T]}$ represents the path of a single particle starting from $g(u)$.

$$y_t(u) = g(u) + \int_0^t \int_0^1 \frac{\varphi(y_s(u) - y_s(u'))}{\int_0^1 \varphi^2(y_s(u) - y_s(v))\mathrm{d}v} \mathrm{d}w(u', s), \qquad (\star)$$

where

- g is the initial condition. We assume g to belong to $L^2([0,1],\mathbb{R})$ and to have a non-decreasing and càdlàg representative.
  E.g. $g(u) = u$, that is $\mu_0 = \mathrm{Leb}\,|_{[0,1]}$.
- φ is a Gaussian interaction kernel. We take $\varphi(x) = e^{-x^2/2}$.
- w is a Brownian sheet.

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Brownian sheet

A random set function w on the Borel sets of $[0,1] \times [0,T]$ is said to be a $(\mathcal{F}_t)_{t \in [0,T]}$-*Brownian sheet* on $[0,1] \times [0,T]$ with respect to Lebesgue measure if

- for each $(\mathcal{F}_t)_{t \in [0,T]}$-progressively measurable function f defined on $[0,1] \times [0,T]$ such that $\int_0^T \int_0^1 f^2(u,s)\mathrm{d}u\mathrm{d}s < +\infty$ almost surely, the process $\left(\int_0^t \int_0^1 f(u,s)\mathrm{d}w(u,s)\right)_{t \in [0,T]}$ is a local martingale;
- for each f_1 and f_2 satisfying the same conditions as f,

$$\left\langle \int_0^\cdot \int_0^1 f_1(u,s)\mathrm{d}w(u,s), \int_0^\cdot \int_0^1 f_2(u,s)\mathrm{d}w(u,s) \right\rangle_t = \int_0^t \int_0^1 f_1(u,s) f_2(u,s)\mathrm{d}u\mathrm{d}s.$$

SDE for the process $(y_t)_{t\in[0,T]}$

$$y_t(u) = g(u) + \int_0^t \int_0^1 \frac{\varphi(y_s(u) - y_s(u'))}{\int_0^1 \varphi^2(y_s(u) - y_s(v))\,\mathrm{d}v} \mathrm{d}w(u', s). \qquad (\star)$$

The evolution of a particle depends on its mass $m_s(u) = \int_0^1 \varphi^2(y_s(u) - y_s(v))\,\mathrm{d}v$. Its quadratic variation is given by:

$$\langle y(u), y(u)\rangle_t = \int_0^t \frac{1}{m_s(u)} \mathrm{d}s.$$

The interaction between a particle $y_t(u)$ and a particle $y_t(v)$ is described by the covariation:

$$\langle y(u), y(v)\rangle_t = \int_0^t \int_0^1 \frac{\varphi(y_s(u) - y_s(u'))}{m_s(u)} \frac{\varphi(y_s(v) - y_s(u'))}{m_s(v)} \mathrm{d}u'\mathrm{d}s.$$

$$y_t(u) = g(u) + \int_0^t \int_0^1 \frac{\varphi(y_s(u) - y_s(u'))}{\int_0^1 \varphi^2(y_s(u) - y_s(v))\mathrm{d}v} \mathrm{d}w(u', s). \qquad (\star)$$

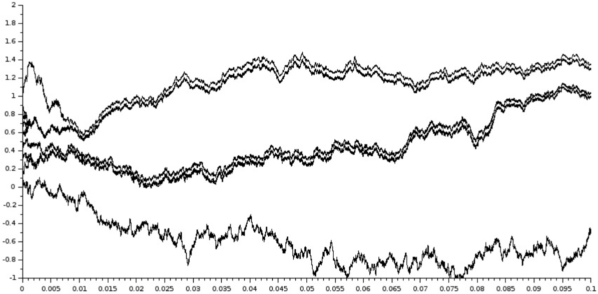

Figure: Example of 10 interacting particles, that is $\mu_0 = \frac{1}{10} \sum_{i=1}^{10} \delta_{i/10}$.

Solution of the SDE

$$y_t(u) = g(u) + \int_0^t \int_0^1 \frac{\varphi(y_s(u) - y_s(u'))}{\int_0^1 \varphi^2(y_s(u) - y_s(v))\mathrm{d}v} \mathrm{d}w(u', s). \qquad (\star)$$

- Existence of a solution of this equation in $L^2([0,1], \mathcal{C}[0,T])$.
- Continuity of the solution: it belongs to $\mathcal{C}([0,T] \times [0,1])$.
- Uniqueness of the solution in $\mathcal{C}([0,T] \times [0,1])$.

Outline

1 Construction of a diffusion process on the Wasserstein space

2 Variation of the initial condition

3 Regularization result and perspectives

Variation of the initial condition in a given direction

Let $g : [0,1] \to \mathbb{R}$ be $\mathcal{C}^\infty$ and (strictly) increasing, and denote by y^g the process starting from g and solution of $(\star)$.

Let $k \in \mathcal{C}^\infty([0,1],\mathbb{R})$ be a direction of perturbation. We want to compare y^g with $y^{g+\varepsilon k}$ when ε is small.

Define $\phi : L^2([0,1],\mathbb{R}) \to \mathbb{R}$ a bounded and continuous function. Let

$$\psi(g) := \mathbb{E}\left[\phi(y_T^g)\right].$$

We want to study the limit when ε goes to 0 of $\frac{\psi(g+\varepsilon k)-\psi(g)}{\varepsilon}$, and control the size of the perturbation with respect to T, to k and to its derivatives. We will follow a method inspired by the work of Thalmaier (1997) and Thalmaier-Wang (1998).

Construction of a map α

Fix $M \geqslant 1$ such that $g([0,1]) \subset [-M, M]$.
We will consider the following process $(y_t^{g+\varepsilon\alpha(t)k})_{t\in[0,T]}$, where $\alpha : [0,1] \to \mathbb{R}$ is a bounded, continuous and adapted process satisfying:

- $\alpha(0) = 1$.
- $\alpha(t) = 0$ if $t \geqslant \zeta_M := \inf\{s \geqslant 0 : \sup_{u\in[0,1]} |y_s(u)| \geqslant M\} \wedge T$.
- for every $p \geqslant 1$, $\mathbb{E}\left[\int_0^{\zeta_M} \dot\alpha(t)^p \mathrm{d}t\right] \leqslant C_p$.

Define $z_t^\varepsilon = y_t^{g+\varepsilon\alpha(t)k}$. We have:

$$z_0^\varepsilon(u) = y_0^{g+\varepsilon k}(u) = g(u) + \varepsilon k(u)$$
$$z_T^\varepsilon(u) = y_T^g(u).$$

Evolution of z_t^ε.

The process $(z_t^\varepsilon)_{t\in[0,T]} = (y_t^{g+\varepsilon\alpha(t)k})_{t\in[0,T]}$ satisfies:

$$z_{t\wedge\delta}^\varepsilon(u) = (g+\varepsilon k)(u) + \int_0^{t\wedge\delta}\int_0^1 \frac{\varphi(z_s^\varepsilon(u) - z_s^\varepsilon(u'))}{\int_0^1 \varphi^2(z_s^\varepsilon(u) - z_s^\varepsilon(v))\mathrm{d}v}\mathrm{d}w(u',s)$$
$$+ \int_0^{t\wedge\delta} D^{\varepsilon,s} y_s(u)(\varepsilon\dot{\alpha}(s)k)\mathrm{d}s.$$

For every $l \in \mathcal{C}[0,1]$, $D^{\varepsilon,t}y_s(u)(l)$ is the Gâteaux derivative, i.e. the limit when $h \to 0$ of

$$\frac{y_s^{g+\varepsilon\alpha(t)k+hl}(u) - y_s^{g+\varepsilon\alpha(t)k}(u)}{h}.$$

Girsanov Theorem

We want to compare

$$y_{t\wedge\delta}^{g+\varepsilon k}(u)=(g+\varepsilon k)(u)+\int_0^{t\wedge\delta}\int_0^1 \frac{\varphi(y_s^{g+\varepsilon k}(u)-y_s^{g+\varepsilon k}(u'))}{\int_0^1 \varphi^2(y_s^{g+\varepsilon k}(u)-y_s^{g+\varepsilon k}(v))\mathrm{d}v}\mathrm{d}w(u',s),$$

and

$$z_{t\wedge\delta}^{\varepsilon}(u)=(g+\varepsilon k)(u)+\int_0^{t\wedge\delta}\int_0^1 \frac{\varphi(z_s^{\varepsilon}(u)-z_s^{\varepsilon}(u'))}{\int_0^1 \varphi^2(z_s^{\varepsilon}(u)-z_s^{\varepsilon}(v))\mathrm{d}v}\mathrm{d}w(u',s)$$
$$+\int_0^{t\wedge\delta} D^{\varepsilon,s}y_s(u)(\varepsilon\dot{\alpha}(s)k)\mathrm{d}s.$$

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We want to compare

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$$+ \int_0^{t\wedge\delta} D^{\varepsilon,s} y_s(u)(\varepsilon\dot{\alpha}(s)k)\mathrm{d}s.$$

Girsanov Theorem

We want to compare

$$y^{g+\varepsilon k}_{t\wedge\delta}(u) = (g+\varepsilon k)(u) + \int_0^{t\wedge\delta}\int_0^1 \frac{\varphi(y^{g+\varepsilon k}_s(u) - y^{g+\varepsilon k}_s(u'))}{\int_0^1 \varphi^2(y^{g+\varepsilon k}_s(u) - y^{g+\varepsilon k}_s(v))\mathrm{d}v}\mathrm{d}w(u',s),$$

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$$+ \int_0^{t\wedge\delta}\int_0^1 \frac{\varphi(z^{\varepsilon}_s(u) - z^{\varepsilon}_s(u'))}{\int_0^1 \varphi^2(z^{\varepsilon}_s(u) - z^{\varepsilon}_s(v))\mathrm{d}v}\varepsilon\dot{\alpha}(s)h^{\varepsilon}_s(u')\mathrm{d}u'\mathrm{d}s.$$

(Fourier inversion)

Girsanov Theorem

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and

$$z^{\varepsilon}_{t\wedge\delta}(u) = (g+\varepsilon k)(u) + \int_0^{t\wedge\delta}\int_0^1 \frac{\varphi(z^{\varepsilon}_s(u) - z^{\varepsilon}_s(u'))}{\int_0^1 \varphi^2(z^{\varepsilon}_s(u) - z^{\varepsilon}_s(v))\mathrm{d}v}\mathrm{d}w(u',s)$$
$$+ \int_0^{t\wedge\delta}\int_0^1 \frac{\varphi(z^{\varepsilon}_s(u) - z^{\varepsilon}_s(u'))}{\int_0^1 \varphi^2(z^{\varepsilon}_s(u) - z^{\varepsilon}_s(v))\mathrm{d}v}\varepsilon\dot{\alpha}(s)h^{\varepsilon}_s(u')\mathrm{d}u'\mathrm{d}s.$$

(Fourier inversion)

We define

$$G^{\varepsilon}_t = \exp\left(-\varepsilon\int_0^{t\wedge\delta}\int_0^1 \dot{\alpha}(s)h^{\varepsilon}_s(u')\mathrm{d}w(u',s) - \frac{\varepsilon^2}{2}\int_0^{t\wedge\delta}\int_0^1 \dot{\alpha}(s)^2 h^{\varepsilon}_s(u')^2\mathrm{d}u'\mathrm{d}s\right).$$

Limit of $\frac{\psi(g+\varepsilon k)-\psi(g)}{\varepsilon}$

Finally, we obtain:

$$\frac{\psi(g+\varepsilon k)-\psi(g)}{\varepsilon} = \mathbb{E}\left[\psi(\zeta_M, y^g_{\zeta_M})\frac{G^\varepsilon_{\zeta_M}-1}{\varepsilon}\right] + \frac{R^\varepsilon}{\varepsilon},$$

where $|R^\varepsilon| \leqslant C\varepsilon^2$. We denote by $\psi(t,g) = \mathbb{E}\left[\phi(y_T^{(t,g)})\right]$, where $y^{(t,g)}$ is the solution of $(\star)$ with initial condition $y_t^{(t,g)} = g$.

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Recall that $\phi : L^2([0,1],\mathbb{R}) \to \mathbb{R}$ is a bounded and continuous function and that $\psi(g) := \mathbb{E}\left[\phi(y_T^g)\right]$.

Theorem

Let $g : [0,1] \to \mathbb{R}$ be a (strictly) increasing $\mathcal{C}^4$ function. Let M be such that $\|g\|_{L^\infty} \leqslant M$. For every $k : [0,1] \to \mathbb{R}$ with $\mathcal{C}^4$-regularity,

$$\lim_{\varepsilon \to 0} \frac{\psi(g+\varepsilon k) - \psi(g)}{\varepsilon} = -\mathbb{E}\left[\phi(y_T^g) \int_0^T \int_0^1 \dot{\alpha}(s) h_s(u) \mathrm{d}w(u,s)\right].$$

Furthermore,

$$\left|\mathbb{E}\left[\phi(y_T^g) \int_0^T \int_0^1 \dot{\alpha}(s) h_s(u) \mathrm{d}w(u,s)\right]\right| \leqslant \frac{C_g C_k}{\sqrt{T}} \|\phi\|_{L^\infty}.$$

where C_g depends on M, on $\|g^{(j)}\|_{L^\infty}$ for $j = 1, \ldots, 4$ and on $\|\frac{1}{g'}\|_{L^\infty}$, and where $C_k \leqslant C \sum_{j=0}^4 \|k^{(j)}\|_{L^\infty}$.

Perspective

Use the result obtained on the semi-group to get uniqueness for the following perturbed Fokker-Planck equation:

$$\mathrm{d}y_t(u) = b(y_t(u), \mu_t)\mathrm{d}t + \int_0^1 \frac{\varphi(y_t(u) - y_t(u'))}{\int_0^1 \varphi^2(y_t(u) - y_t(v))\mathrm{d}v}\mathrm{d}w(u', t).$$

Thank you for your attention!