Regularization result for an infinite-dimensional diffusion process

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Introduction

Define the following set of probability measures:

$$\mathcal{P}_2(\mathbb{R}) := \left\{ \mu \text{ probability measure on } \mathbb{R} : \int_{\mathbb{R}} x^2 \, d\mu(x) < +\infty \right\},$$

and the following $L^2$-Wasserstein distance:

$$W_2(\mu, \nu) = \inf_{(X, Y) \text{ coupling of } (\mu, \nu)} \mathbb{E} \left[ |X - Y|^2 \right]^{1/2}.$$ 

$(\mathcal{P}_2(\mathbb{R}), W_2)$ is called the Wasserstein space.
We consider the following equation for the $\mathcal{P}_2(\mathbb{R})$-valued process $(\mu_t)_{t \in [0,T]}$: for all $u \in (0,1)$

$$dy_t(u) = b(y_t(u), \mu_t)dt,$$

where $u \in (0,1) \mapsto y_t(u)$ is the quantile function associated to the measure $\mu_t$.

As for the transport equation, there is no uniqueness in general if we assume $b$ to be only bounded and continuous, and not Lipschitz.

An interesting question: can we restore uniqueness by adding a small diffusive perturbation?
Regularization by Additive Noise

- Uniqueness in law for general parabolic equations in finite dimension (Stroock, Varadhan, 1970').

- Transport equation perturbed by a Brownian motion.

\[ dx_t = b(x_t)dt + dB_t \]

Strong uniqueness for \( d = 1, b \) bounded and continuous (Zvonkin, 1974).

- Strong uniqueness in higher dimension (Veretennikov, 1980).

- Strong uniqueness for \( b \in L^p, p > d \) (Krylov-Röckner, 2005).
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Outline

1. Construction of a diffusion process on the Wasserstein space

2. Variation of the initial condition

3. Regularization result and perspectives
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Construction of a diffusion process

We associate to a process \((\mu_t)_{t \in [0,T]}\) on the Wasserstein space \(\mathcal{P}_2(\mathbb{R})\) the quantile process \((y_t)_{t \in [0,T]}\), where for each \(t \in [0, T]\), \(u \in [0,1] \mapsto y_t(u)\) is the quantile function associated to the measure \(\mu_t\).

\[
y_t(u) := \inf\{x \in \mathbb{R} : \int_{-\infty}^{x} d\mu_t > u\}.
\]

For every \(t \in [0, T]\), \(y_t\) belongs to \(L^2([0,1])\).
SDE for the process \((y_t)_{t\in[0,T]}\)

For each \(u \in (0,1)\), the real process \((y_t(u))_{t\in[0,T]}\) represents the path of a single particle starting from \(g(u)\).

\[
y_t(u) = g(u) + \int_0^t \int_0^1 \frac{\varphi(y_s(u) - y_s(u'))}{\int_0^1 \varphi^2(y_s(u) - y_s(v))dv} dw(u', s),
\]

where

- \(g\) is the initial condition. We assume \(g\) to belong to \(L^2([0,1], \mathbb{R})\) and to have a non-decreasing and càdlàg representative. E.g. \(g(u) = u\), that is \(\mu_0 = \text{Leb}_{|[0,1]}\).
- \(\varphi\) is a Gaussian interaction kernel. We take \(\varphi(x) = e^{-x^2/2}\).
- \(w\) is a Brownian sheet.
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Brownian sheet

A random set function $w$ on the Borel sets of $[0, 1] \times [0, T]$ is said to be a $(\mathcal{F}_t)_{t \in [0, T]}$-Brownian sheet on $[0, 1] \times [0, T]$ with respect to Lebesgue measure if

- for each $(\mathcal{F}_t)_{t \in [0, T]}$-progressively measurable function $f$ defined on $[0, 1] \times [0, T]$ such that $\int_0^T \int_0^1 f^2(u, s)\,du\,ds < +\infty$ almost surely, the process $\left(\int_0^t \int_0^1 f(u, s)\,dw(u, s)\right)_{t \in [0, T]}$ is a local martingale;
- for each $f_1$ and $f_2$ satisfying the same conditions as $f$,

\[
\langle \int_0^t \int_0^1 f_1(u, s)\,dw(u, s), \int_0^t \int_0^1 f_2(u, s)\,dw(u, s) \rangle_t = \int_0^t \int_0^1 f_1(u, s)f_2(u, s)\,du\,ds.
\]
SDE for the process \((y_t)_{t \in [0,T]}\)

\[
y_t(u) = g(u) + \int_0^t \int_0^1 \frac{\varphi(y_s(u) - y_s(u'))}{\int_0^1 \varphi^2(y_s(u) - y_s(v)) \, dv} \, dw(u', s).
\]  \(\star\)

The evolution of a particle depends on its mass \(m_s(u) = \int_0^1 \varphi^2(y_s(u) - y_s(v)) \, dv\).

Its quadratic variation is given by:

\[
\langle y(u), y(u) \rangle_t = \int_0^t \frac{1}{m_s(u)} \, ds.
\]

The interaction between a particle \(y_t(u)\) and a particle \(y_t(v)\) is described by the covariation:

\[
\langle y(u), y(v) \rangle_t = \int_0^t \int_0^1 \frac{\varphi(y_s(u) - y_s(u')) \varphi(y_s(v) - y_s(u'))}{m_s(u) m_s(v)} \, du' ds.
\]
Construction of a diffusion process on the Wasserstein space

\[ y_t(u) = g(u) + \int_0^t \int_0^1 \frac{\varphi(y_s(u) - y_s(u'))}{\int_0^1 \varphi^2(y_s(u) - y_s(v)) dv} dw(u', s). \]  

(\star)

Figure: Example of 10 interacting particles, that is \( \mu_0 = \frac{1}{10} \sum_{i=1}^{10} \delta_{i/10} \).
Solution of the SDE

\[ y_t(u) = g(u) + \int_0^t \int_0^1 \frac{\varphi(y_s(u) - y_s(u'))}{\int_0^1 \varphi^2(y_s(u) - y_s(v)) dv} dw(u', s). \] (⋆)

- Existence of a solution of this equation in \( L^2([0, 1], C[0, T]) \).
- Continuity of the solution: it belongs to \( C([0, T] \times [0, 1]) \).
- Uniqueness of the solution in \( C([0, T] \times [0, 1]) \).
Outline

1 Construction of a diffusion process on the Wasserstein space
2 Variation of the initial condition
3 Regularization result and perspectives
Variation of the initial condition in a given direction

Let \( g : [0, 1] \to \mathbb{R} \) be \( C^\infty \) and (strictly) increasing, and denote by \( y^g \) the process starting from \( g \) and solution of (\( \star \)).

Let \( k \in C^\infty ([0, 1], \mathbb{R}) \) be a direction of perturbation. We want to compare \( y^g \) with \( y^{g+\varepsilon k} \) when \( \varepsilon \) is small.

Define \( \phi : L^2 ([0, 1], \mathbb{R}) \to \mathbb{R} \) a bounded and continuous function. Let

\[
\psi(g) := \mathbb{E} [\phi(y^g_T)].
\]

We want to study the limit when \( \varepsilon \) goes to 0 of \( \frac{\psi(g + \varepsilon k) - \psi(g)}{\varepsilon} \), and control the size of the perturbation with respect to \( T \), to \( k \) and to its derivatives. We will follow a method inspired by the work of Thalmaier (1997) and Thalmaier-Wang (1998).
Construction of a map $\alpha$

Fix $M \geq 1$ such that $g([0, 1]) \subset [-M, M]$.
We will consider the following process $(y_t^g + \varepsilon \alpha(t)k)_{t \in [0, T]}$, where $\alpha : [0, 1] \to \mathbb{R}$ is a bounded, continuous and adapted process satisfying:

- $\alpha(0) = 1$.
- $\alpha(t) = 0$ if $t \geq \zeta_M := \inf\{s \geq 0 : \sup_{u \in [0, 1]} |y_s(u)| \geq M\} \wedge T$.
- For every $p \geq 1$, $\mathbb{E} \left[ \int_0^{\zeta_M} \dot{\alpha}(t)^p \, dt \right] \leq C_p$.

Define $z^\varepsilon_t = y_t^g + \varepsilon \alpha(t)k$. We have:

$$z_0^\varepsilon(u) = y_0^g + \varepsilon k(u) = g(u) + \varepsilon k(u)$$
$$z_T^\varepsilon(u) = y_T^g(u).$$
Evolution of $z_t^\varepsilon$.

The process $(z_t^\varepsilon)_{t\in[0,T]} = (y_t^{g+\varepsilon\alpha(t)k})_{t\in[0,T]}$ satisfies:

$$z_{t\wedge\delta}(u) = (g + \varepsilon k)(u) + \int_0^{t\wedge\delta} \int_0^1 \frac{\varphi(z_s^\varepsilon(u) - z_s^\varepsilon(u'))}{\int_0^1 \varphi^2(z_s^\varepsilon(u) - z_s^\varepsilon(v))dv}dw(u', s)$$

$$+ \int_0^{t\wedge\delta} D^{\varepsilon,s} y_s(u)(\varepsilon\dot{\alpha}(s)k)ds.$$

For every $l \in C[0,1]$, $D^{\varepsilon,t} y_s(u)(l)$ is the Gâteaux derivative, i.e. the limit when $h \to 0$ of

$$\frac{y_s^{g+\varepsilon\alpha(t)k+h}l(u) - y_s^{g+\varepsilon\alpha(t)k}(u)}{h}.$$
Girsanov Theorem

We want to compare

\[ y_{t^\land \delta}^{g+\varepsilon k}(u) = (g + \varepsilon k)(u) + \int_0^{t^\land \delta} \int_0^1 \frac{\varphi(y_s^{g+\varepsilon k}(u) - y_s^{g+\varepsilon k}(u'))}{\int_0^1 \varphi^2(y_s^{g+\varepsilon k}(u) - y_s^{g+\varepsilon k}(v))} \, dv \, dw(u', s), \]

and

\[ z_{t^\land \delta}^{\varepsilon}(u) = (g + \varepsilon k)(u) + \int_0^{t^\land \delta} \int_0^1 \frac{\varphi(z_s^{\varepsilon}(u) - z_s^{\varepsilon}(u'))}{\int_0^1 \varphi^2(z_s^{\varepsilon}(u) - z_s^{\varepsilon}(v))} \, dv \, dw(u', s) \]

\[ + \int_0^{t^\land \delta} D^{\varepsilon,s} y_s(u)(\varepsilon \dot{\alpha}(s) k) \, ds. \]
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\[ + \int_0^{t^{\wedge}\delta} \int_0^1 \frac{\varphi(z_s^\varepsilon(u) - z_s^\varepsilon(u'))}{\int_0^1 \varphi^2(z_s^\varepsilon(u) - z_s^\varepsilon(v))} \varepsilon \dot{\alpha}(s) h_s^\varepsilon(u') du' ds. \]

(Fourier inversion)
Girsanov Theorem

We want to compare

\[ y_{t\wedge\delta}^{g+\varepsilon k}(u) = (g + \varepsilon k)(u) + \int_0^{t\wedge\delta} \int_0^1 \frac{\varphi(y_s^{g+\varepsilon k}(u) - y_s^{g+\varepsilon k}(u'))}{\int_0^1 \varphi^2(y_s^{g+\varepsilon k}(u) - y_s^{g+\varepsilon k}(v))} \, dw(u', s), \]

and

\[ z_{t\wedge\delta}^{\varepsilon}(u) = (g + \varepsilon k)(u) + \int_0^{t\wedge\delta} \int_0^1 \frac{\varphi(z_s^{\varepsilon}(u) - z_s^{\varepsilon}(u'))}{\int_0^1 \varphi^2(z_s^{\varepsilon}(u) - z_s^{\varepsilon}(v))} \, dw(u', s) \]

\[ + \int_0^{t\wedge\delta} \int_0^1 \frac{\varphi(z_s^{\varepsilon}(u) - z_s^{\varepsilon}(u'))}{\int_0^1 \varphi^2(z_s^{\varepsilon}(u) - z_s^{\varepsilon}(v))} \, \varepsilon \dot{\alpha}(s) h_s^{\varepsilon}(u') \, du'ds. \]

(Fourier inversion)

We define

\[ G_t^{\varepsilon} = \exp \left( -\varepsilon \int_0^{t\wedge\delta} \int_0^1 \dot{\alpha}(s) h_s^{\varepsilon}(u') \, dw(u', s) - \frac{\varepsilon^2}{2} \int_0^{t\wedge\delta} \int_0^1 \dot{\alpha}(s)^2 h_s^{\varepsilon}(u')^2 \, du'ds \right). \]
Limit of \[ \frac{\psi(g + \varepsilon k) - \psi(g)}{\varepsilon} \]

Finally, we obtain:

\[ \frac{\psi(g + \varepsilon k) - \psi(g)}{\varepsilon} = \mathbb{E} \left[ \psi(\zeta_M, y_{\zeta_M}^g) \frac{G_{\zeta_M}^\varepsilon - 1}{\varepsilon} \right] + \frac{R^\varepsilon}{\varepsilon}, \]

where \(|R^\varepsilon| \leq C\varepsilon^2\). We denote by \(\psi(t, g) = \mathbb{E} \left[ \phi(y_{T}^{(t,g)}) \right]\), where \(y^{(t,g)}\) is the solution of (\(*\)) with initial condition \(y_{t}^{(t,g)} = g\).
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Recall that $\phi : L^2([0, 1], \mathbb{R}) \to \mathbb{R}$ is a bounded and continuous function and that $\psi(g) := \mathbb{E}[\phi(y_T^g)]$.

**Theorem**

Let $g : [0, 1] \to \mathbb{R}$ be a (strictly) increasing $C^4$ function. Let $M$ be such that $\|g\|_{L^\infty} \leq M$. For every $k : [0, 1] \to \mathbb{R}$ with $C^4$-regularity,

$$\lim_{\varepsilon \to 0} \frac{\psi(g + \varepsilon k) - \psi(g)}{\varepsilon} = -\mathbb{E} \left[ \phi(y_T^g) \int_0^T \int_0^1 \dot{\alpha}(s) h_s(u) dw(u, s) \right].$$

Furthermore,

$$\left| \mathbb{E} \left[ \phi(y_T^g) \int_0^T \int_0^1 \dot{\alpha}(s) h_s(u) dw(u, s) \right] \right| \leq \frac{C_g C_k}{\sqrt{T}} \|\phi\|_{L^\infty}.$$

where $C_g$ depends on $M$, on $\|g^{(j)}\|_{L^\infty}$ for $j = 1, \ldots, 4$ and on $\|\frac{1}{g'}\|_{L^\infty}$, and where $C_k \leq C \sum_{j=0}^4 \|k^{(j)}\|_{L^\infty}$. 


Use the result obtained on the semi-group to get uniqueness for the following perturbed Fokker-Planck equation:

\[ \mathrm{d}y_t(u) = b(y_t(u), \mu_t) \mathrm{d}t + \int_0^1 \frac{\varphi(y_t(u) - y_t(u'))}{\int_0^1 \varphi^2(y_t(u) - y_t(v)) \mathrm{d}v} \mathrm{d}w(u', t). \]
Thank you for your attention!