19. Adjoint operators

1. Some examples of orthonormal bases.

a) Legendre polynomials

We consider the inner product space $L^2[-1,1]$. We are interested in finding of an orthonormal basis of functions that are easy to handle. For that we consider linearly independent set of polynomials

$$M = \{ x_n, \ n \geq 1 \},$$

where $x_n(t) = t^n$, $t \in [1,1]$.

The set M is not orthonormal, in particular

$$\langle x_k, x_l \rangle = \int_{-1}^{1} t^k t^l \, dt \neq 0$$

if $k+l$ - even. We remark that

$$\overline{\operatorname{span} M} = L_2[-1,1].$$

(every function from $L_2[-1,1]$ can be approximated by a continuous function on $[-1,1]$ and every continuous function can be approximated by polynomials)

To build an orthonormal set from $\mathcal{M}$ we need to apply the Gram-Schmidt procedure:

$$e_0(t) = \frac{x_0(t)}{\|x_0\|} = \frac{1}{\sqrt{2}}$$

$$\|x_0\|^2 = \int_{-1}^{1} x_0^2(t)\,dt = \int_{-1}^{1} 1\,dt = 2$$

$$v_1(t) = x_1(t) - \langle x_1, e_0 \rangle e_0(t) = t - 0 = t$$

$$e_1(t) = \frac{v_1(t)}{\|v_1\|^2} = t$$

$$e_2(t) = \sqrt{\frac{5}{2}}\,\frac{1}{2}(3t^2-1)$$

$$e_3(t) = \sqrt{\frac{7}{2}}\,\frac{1}{2}(5t^3-3t)$$

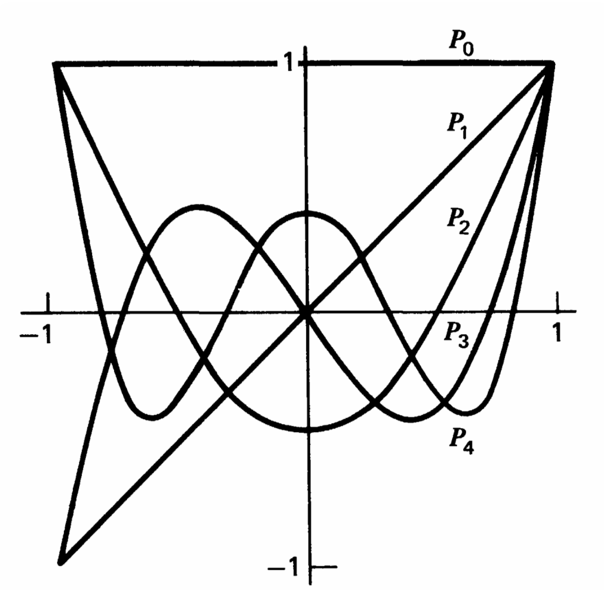
. . . . .

One can show that

$$e_n(t) = \sqrt{\frac{2n+1}{2}}\,P_n(t),$$

where

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n}\left[(t^2-1)^n\right].$$



So, functions $\{e_n, n \geq 0\}$ form a basis in $L^2[-1,1]$, that is

$$x = \sum_{n=0}^{\infty} \langle x, e_n \rangle e_n \qquad \forall\, x \in L^2[-1,1].$$

b) Hermite polynomials

It is clear that polynomials do not belong to $L^2(\mathbb{R})$. To construct an orthonormal basis we take

$$x_n(t) = t^n e^{-\frac{t^2}{2}}, \; t \in \mathbb{R}, \quad n \geq 0$$

Similarly,

$$\overline{\operatorname{span}\{x_n,\ n\geq 0\}} = L^2(\mathbb{R}).$$

So, $\mathcal{M}$ is total. One can show that

$$e_n(t) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-\frac{t^2}{2}} H_n(t),$$

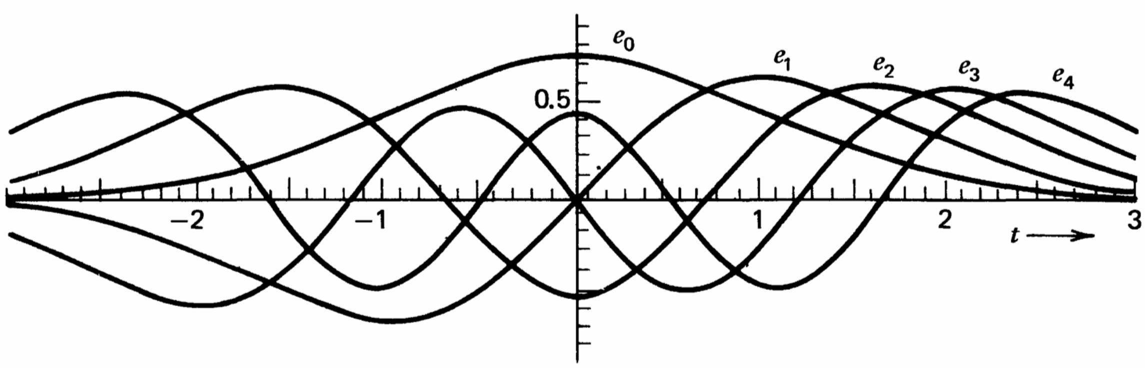
where $H_0(t) = 1,$

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n}\left(e^{-t^2}\right), \quad n = 1, 2, \dots$$

— Hermite polynomials

Hence, $\{e_n,\ n\geq 0\}$ is an orthonormal basis in $L^2(\mathbb{R})$, i.e.

$$x = \sum_{n=0}^{\infty} \langle x, e_n\rangle e_n, \qquad \forall x \in L^2(\mathbb{R}).$$



c) Laguerre polynomials.

Similarly as before, using the Gram-Schmidt procedure to

$$t^n e^{-\frac{t}{2}}, \quad t \geq 0, \quad n \geq 1,$$

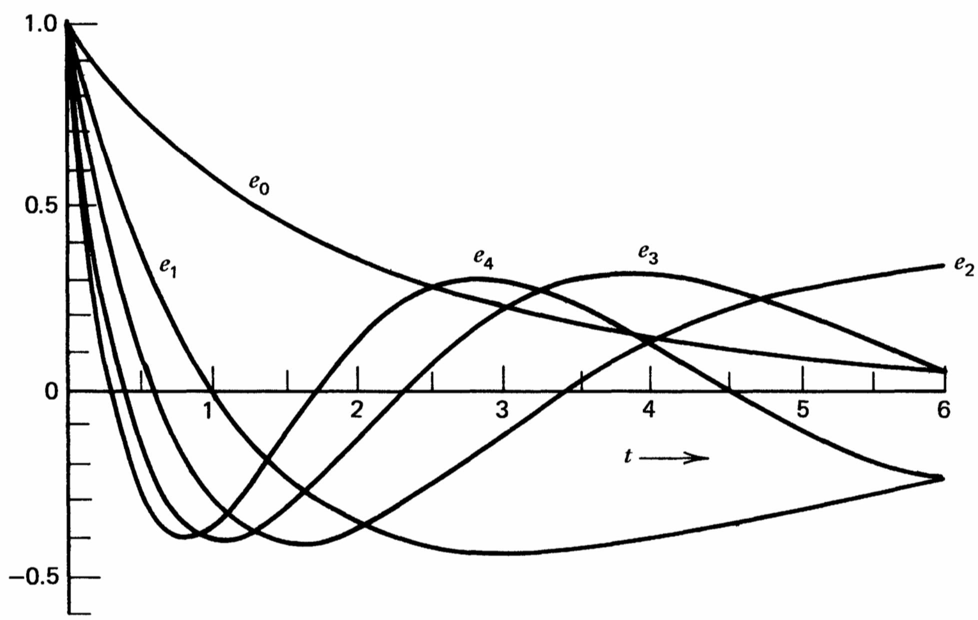
in $L^2([0, +\infty))$, we obtain

$$e_n(t) = e^{-\frac{t}{2}} L_n(t),$$

where

$$L_0(t) = 1, \quad L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} \left(t^n e^{-t} \right)$$

– Laguerre polynomials



Again $\{e_n, n \geq 1\}$ is an orthonormal basis in $L^2([0,\infty))$. So,

$$x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n, \quad x \in L^2([0,\infty)).$$

2. Adjoint operators.

Let H be a Hilbert space.

Th 19.1 (Riesz's representation theorem)
Every bounded linear functional f on H can be represented in terms of inner product:

$$f(x) = \langle x, z \rangle,$$

where z is uniquely determined (dependent on f) element of H, and

$$\|f\| = \|z\|.$$

Def 19.2 Let H_1, H_2 be Hilbert spaces. Let $T: H_1 \to H_2$ be a bounded linear operator. Then the *adjoint operator* T^* of T is the operator

$$T^*: H_2 \to H_1$$

such that for all $x \in H_1$, $y \in H_2$

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

Th 19.3 The adjoint operator T^* of T exists, is unique and is a bounded linear operator with norm

$$\|T^*\| = \|T\|.$$

The existence of operator T^* follows from Riesz's representation theorem. Namely, for fixed $y \in H_2$, the map

$$x \mapsto \langle Tx, y \rangle$$

is a bounded linear functional on H_1

$$\left(|\langle Tx, y \rangle| \le \|Tx\| \|y\| \le \underbrace{\|T\| \|y\|}_{C} \|x\| \right)$$

Hence, $\exists!\ z \in H_1$ s.t.

$$\langle Tx, y \rangle = \langle x, z \rangle.$$

Then define $T^* y := z$.

Th 19.4. Let H_1, H_2 be Hilbert spaces and $T, S : H_1 \to H_2$ be bounded linear operators, $\alpha \in K$. Then

a) $\langle T^* y, x \rangle = \langle y, Tx \rangle$

b) $(S+T)^* = S^* + T^*$

c) $(\alpha T)^* = \bar{\alpha} T^*$

d) $(T^*)^* = T$

e) $\|T^* T\| = \|T T^*\| = \|T\|^2$

f) $T^* T = 0 \iff T = 0$

g) $(ST)^* = T^* S^*$ (if $H_1 = H_2$)

3. Self-adjoint, unitary and normal operators

In this section, we assume that H is a Hilbert space.

Def 19.5 A bounded linear operator $T: H \to H$ on a Hilbert space H is said to be

- **self-adjoint** or **Hermitian** if
$$T^* = T$$
- **unitary** if T is bijective and
$$T^* = T^{-1}$$
- **normal** if
$$TT^* = T^*T.$$

We remark that if T is self-adjoint or unitary, then T is normal. The inverse is not trye

Example 19.6 Take $T = 2iI$, where I is the identity operator, i.e. $Ix = x$. Then $T^* = -2iI$. Indeed,

$$\langle Tx, y\rangle = \langle 2ix, y\rangle = 2i\langle x, y\rangle =$$
$$= \langle x, \overline{2i}y\rangle = \langle x, -2iy\rangle$$

So, $T^*y = -2iy \Rightarrow T^* = -2iI.$
So, $TT^* = T^*T$. But $T^* \neq T^{-1} = -\frac{1}{2}iI$, and $T \neq T^*$.

Example 19.7 We consider $\mathbb{C}^n$ with the inner product

$$\langle x, y \rangle = \sum_{k=1}^{n} \xi_k \overline{\eta_k},$$

$x = (\xi_k)_{k=1}^n$, $y = (\eta_k)_{k=1}^n$.

Any (bounded) linear operator

$$T : \mathbb{C}^n \to \mathbb{C}^n$$

can be given by matrix M_T.

Namely,

$$y = Tx$$

can be expressed as

$$\begin{pmatrix} \eta_1 \\ \vdots \\ \eta_n \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ - & - & - \\ a_{n1} & \dots & a_{nn} \end{pmatrix}}_{M_T} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}$$

transposition

Then $$M_{T^*} = \overline{M}_T^{\,T} = \begin{pmatrix} \overline{a_{11}} & \dots & \overline{a_{n1}} \\ - & - & - \\ \overline{a_{1n}} & \dots & \overline{a_{nn}} \end{pmatrix}.$$

Th 19.8 Let $T: H \to H$ be a bounded linear operator on a Hilbert space H. Then

a) If T is self-adjoint, $\langle Tx, x\rangle$ is real $\forall\, x \in H$

b) If H is complex ($K = \mathbb{C}$) and $\langle Tx, x\rangle$ is real $\forall\, x \in H$, then T is self-adjoint.

Proof a) If T is self-adjoint, then

$$\overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle Tx, x\rangle .$$

b) If $\langle Tx, x\rangle$ is real, then

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \overline{\langle x, T^{*}x\rangle} = \langle T^{*}x, x\rangle$$

Hence

$$0 = \langle Tx, x\rangle - \langle T^{*}x, x\rangle =$$

$$= \langle (T - T^{*})x, x\rangle$$

This implies that $T - T^{*} = 0$.

(To see that, in complex Hilbert space, $\langle Tx, x\rangle = 0$ implies $T = 0$,

one needs to consider
$$\langle T(\alpha x+y), \alpha x+y\rangle, \quad \alpha\in K, x,y\in H)$$

$\blacksquare$

Th 13.9. a) The product of two bounded self-adjoint operators S and T is self-adjoint iff
$$ST = TS$$

b) Let T_n, $n\geq 1$ be self-adjoint operators on H, and $T_n \to T$ in $B(H,H)$, i.e.
$$\|T_n - T\| \to 0, \quad n\to\infty.$$
Then T is self-adjoint.

Proof b) We neet to show that $T=T^*$.

Consider
$$\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\| \to 0$$

So, $T_n^* \to T^*$. Since $T_n = T^*_n$, then $T_n \to T^*$. This implies that
$$T = T^*.$$

$\blacksquare$