17. Hilbert spaces

1. Definitions of inner product and Hilbert spaces

Let X be a vector space over field K, where $K = \mathbb{R}$ or $\mathbb{C}$.

Def. 17.1 • An inner product on X is a map

$$\langle \cdot, \cdot \rangle : X \times X \to K$$

which satisfy is the following properties: $\forall x, y, z \in X$, $\forall \alpha$

(IP1) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, x \rangle$

(IP2) $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$

(IP3) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ ← complex conjugate

(IP4) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \iff x = 0$.

• A vector space X with an inner product on it is called an inner product space

Example 17.2 a) Euclidean space $\mathbb{R}^n$

$$\langle x, y \rangle = \xi_1 \eta_1 + \dots + \xi_n \eta_n$$

b) Unitary space $\mathbb{C}^n$

$$\langle x, y \rangle = \xi_1 \overline{\eta_1} + \dots + \xi_n \overline{\eta_n}$$

c) Space $\ell^2 = \{ x = (\xi_k)_{k=1}^{\infty} : \xi_k \in K, k \geq 1, \sum_{k=1}^{\infty} |\xi_k|^2 < \infty \}$

$$\langle x, y \rangle = \sum_{k=1}^{\infty} \xi_k \overline{\eta_k}$$

d) Space $L^2[a,b] = \{ x : [0,1] \to K : \int_0^1 |x(t)|^2 dt < +\infty \}$

$$\langle x, y \rangle = \int_a^b x(t) \overline{y(t)} \, dt$$

e) Space $L^2(\mathbb{R}) = \{ x : \mathbb{R} \to K : \int_{-\infty}^{+\infty} |x(t)|^2 dt < +\infty \}$

$$\langle x, y \rangle = \int_{-\infty}^{+\infty} x(t) \overline{y(t)} \, dt.$$

2. Properties of inner product spaces

Define $\|x\| := \sqrt{\langle x, x \rangle}$, $x \in X$

it is easily to see that $\|\cdot\|$ satisfies properties (N1)-(N3) of a norm. Property (N4) will follow from the next lemma. Hence, the space X with norm $\|\cdot\|$ induced by the inner product is a normed space.

Lemma 17.3 (Cauchy-Schwarz inequality, triangle inequality)

a) $\forall\, x, y \in X$

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|,$$

and the equality holds iff x and y are linearly dependent

b) $\forall\, x, y \in X$

$$\|x+y\| \leq \|x\| + \|y\|$$

Exercise 17.4 check that a norm on an inner product space satisfies the parallelogram inequality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \quad (17.1)$$

Remark 17.5 Exercise 17.4 implies that ℓ^p, $L^p[a,b]$, $p \neq 2$ and $C[a,b]$ are not an inner product spaces.
Let us exploit this for ℓ^p. Take

$$x = (1, 1, 0, 0, \ldots), \; y = (1, -1, 0, 0, \ldots) \in \ell^p$$

Then $\|x\| = \|y\| = 2^{\frac{1}{p}}$ and $\|x+y\| = \|x-y\| = 2$.

So, (17.1) is not satisfied for such x and y for $p \neq 2$:

$$2^3 = 2^2 + 2^2 \neq 2\left(2^{\frac{2}{p}} + 2^{\frac{2}{p}}\right) = 2^{\frac{2}{p}+2}$$

Lemma 17.6. Let $x_n \to x$ and $y_n \to y$ in X. Then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.

Proof

$$|\langle x_n, y_n \rangle - \langle x, y \rangle| = |\langle x_n, y_n \rangle - \langle x_n, y \rangle + \langle x_n, y \rangle - \langle x, y \rangle| \leq$$

$$|\langle x_n, y_n - y \rangle| + |\langle x_n - x, y \rangle| \leq$$

$$\leq \underbrace{\|x_n\|}_{\substack{\text{bounded} \\ \text{as a convergent} \\ \text{sequence}}} \underset{\searrow 0}{\|y_n - y\|} + \underset{\searrow 0}{\|x_n - x\|} \underset{< +\infty}{\|y\|} \to 0$$

$\blacksquare$

Def 17.7 A complete (in norm generated by inner product) inner product space X is said to be a **Hilbert space**.

So, a Hilbert space is a Banach space.

A **subspace** Y of an inner product space X is defined to be a vector subspace of X taken with the inner product on X restricted to $Y \times Y$.

Th 17.8 Let Y be a subspace of a Hilbert space H. Then

(a) Y is complete iff Y is closed in H.

(b) if Y is finite dimensional, then Y is complete

(c) if H is separable, so is Y.

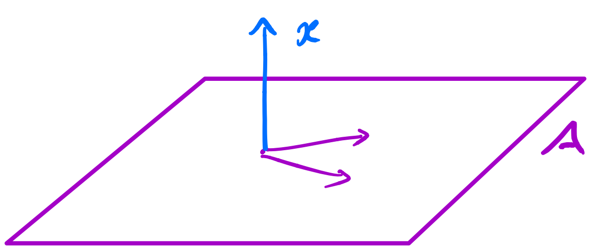
Proof (a) is a direct consequence of Th 13.7. (b) follows from the fact that every finite dimensional space is closed. ∎

3. Orthogonality

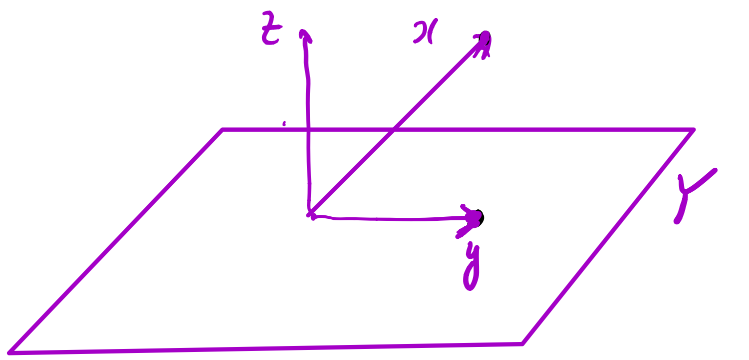
Def 17.9. An element x of an inner product space X is said to be **orthogonal** to an element $y \in X$ if

$$\langle x, y \rangle = 0.$$

We also say that x and y are orthogonal and write $x \perp y$. Similarly, for subsets $A, B \subset X$ $x \perp A$ if $x \perp a$ $\forall a \in A$; and $A \perp B$ if $a \perp b$ $\forall a \in A, \forall b \in B$.



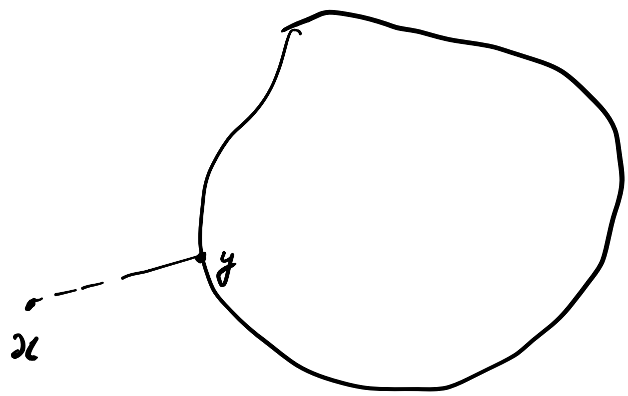
Now we are interesting in a "finding of perpendicular from x to a subspace Y.



Let M be a non-empty subset of X. Let us define the distance from x to M

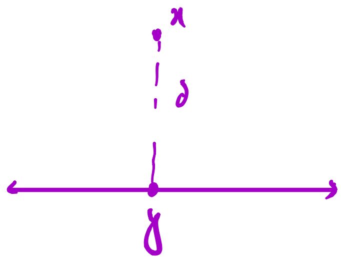
$$\delta = \inf_{\tilde{y} \in M} \|x - \tilde{y}\|.$$

We are interesting if there exists a unique $y \in M$ such that $\delta = \|x - y\|$.

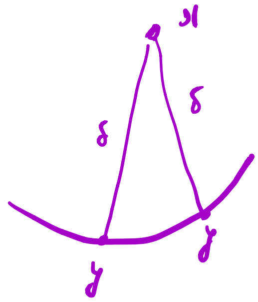


Example 17.10 $X = \mathbb{R}^2$

(y does not exists in M)



(exists unique y)



(exists infinitely many y)

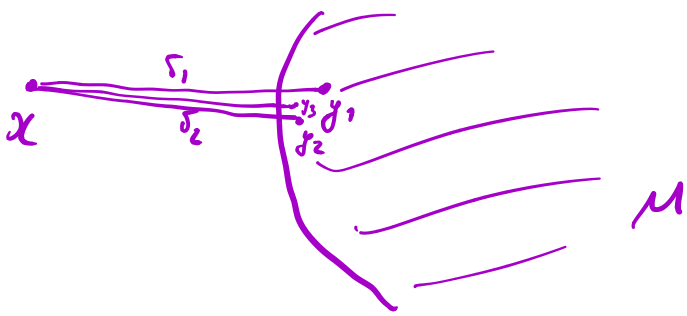
A subset M of X is **convex** if $\forall x, y \in M$

$$\alpha x + (1-\alpha) y \in M \quad \forall \alpha \in [0,1].$$

Th 17.11 Let X be an inner product space and $M \neq \emptyset$ be a convex subset which is complete. Then for every given $x \in X$ there exists a unique $y \in M$ such that

$$\delta = \inf_{\tilde{y} \in M} \| x - \tilde{y} \| = \| x - y \|.$$

Proof (Idea) One need to take a sequence $y_n \in M$ such that $\delta_n = \| x - y_n \| \to \delta$, $n \to \infty$, and show that it is a Cauchy sequence in M.



Then $\exists y \in M$ s.t. $y_n \to y$. ∎

Lemma 17.12 If in Th 17.11 $M = Y$, where Y is a complete subspace of X and $x \in X$ is fixed. Then $z = x - y$ is orthogonal to Y.

Next, let H be a Hilbert space and Y be a closed subspace Y. Define orthogonal complement

$$Y^{\perp} = \{ z \in H : z \perp Y \}.$$

It is a vector subspace