Problem sheet 1

Solutions has to be uploaded into Moodle:
https://lernen.min.uni-hamburg.de/mod/assign/view.php?id=28452
until 20:00, November 17.

1. Show that
$$\int_x^{+\infty} e^{-\frac{y^2}{2}} dy \sim \frac{1}{x} e^{-\frac{x^2}{2}}, \quad x \to +\infty.$$

2. Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of positive real numbers. We say that they are **logarithmically equivalent** and write $a_n \simeq b_n$ if
$$\lim_{n\to\infty} \frac{1}{n} (\ln a_n - \ln b_n) = 0.$$
   (a) Show that $a_n \simeq b_n$ iff $b_n = a_n e^{o(n)}$.
   (b) Show that $a_n \sim b_n$ implies $a_n \simeq b_n$ and that the inverse implication is not correct.

   HW1 [3 points] Show that $a_n + b_n \simeq \max\{a_n, b_n\}$.

3. Let $\xi_1, \xi_2, \ldots$ be independent normal distributed random variables with mean μ and variance σ^2. Let also $S_n = \xi_1 + \cdots + \xi_n$. Compute $\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left\{\frac{1}{n} S_n \geq x\right\}$ for $x > \mu$.

4. Let $\xi_1, \xi_2, \ldots$ be independent Bernoulli distributed random variables with parameter $p = \frac{1}{2}$. Let also $S_n = \xi_1 + \cdots + \xi_n$. Using Theorem 1.1 from the lecture notes, show that
$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \varepsilon\right\} < \infty,$$
for all $\varepsilon > 0$. Conclude that $\frac{S_n}{n} \to \frac{1}{2}$ a.s. as $n \to \infty$ *(strong low of large numbers)*.

   (*Hint:* Use the Borel-Cantelly lemma to show the convergence with probability 1)

HW2 [3 points] Assume that a random variable ξ has a finite first moment $\mathbb{E}\,\xi = \mu$ and let φ be the comulant generating function associated with ξ. Show that for every $x > \mu$ and all $\lambda < 0$
$$\lambda x - \varphi(\lambda) \leq 0.$$

(*Hint:* Use Jensen's inequality.)

5. Show that the Fenchel-Legendre transform of a convex function f is also convex.

   (*Hint:* Show first that the supremum of convex functions is a convex function. Then note that the function $\lambda x - \varphi(\lambda)$ is convex in the variable x)

HW3 [4 points] Show that the Fenchel-Legendre transform of the comulant generating function associated with $N(0,1)$ coincides with $\frac{x^2}{2}$.

6. Suppose that φ^* is the Fenchel-Legendre transform of the cumulant generating function of a random variable ξ with $\mathbb{E}\,\xi = \mu$. Show that
   (i) $\varphi^*(x) \geq 0$ for all $x \in \mathbb{R}$. (*Hint:* Use the fact that $\varphi(0) = 0$)
   (ii) $\varphi^*(\mu) = 0$. (*Hint:* Use (i) and Jensen's inequality to show that $\varphi^*(\mu) \leq 0$)
   (iii) φ^* increases on $[\mu, \infty)$ and decreases on $(-\infty, \mu]$. (*Hint:* Use the convexity of φ^*)