27 Laplace equation

1 Laplace equation in a disc

We consider a domain $D \subseteq \mathbb{R}^d$ and let $\Delta u := \sum_{k=1}^{d} \frac{\partial^2 u}{\partial x_k^2}$.

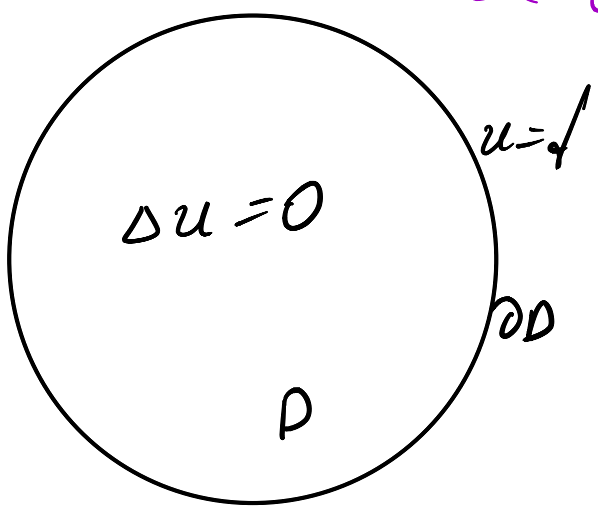
The equation

$$\Delta u = 0$$
$$u\big|_{\partial D} = f$$

is called the Laplace equation

In this section we will consider the case

$$D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}.$$



So, we consider the equation

(27.1) $\Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad x \in D$

(27.2) $u(x,y) = f(x,y), \quad (x,y) \in \partial D = \{(x,y) : x^2+y^2 = 1\}.$

The idea of solution of the Laplace equation on the disc is to rewrite it in polar coordinates and then use the method of separation of variables. So, we change the variables in (27.1), (27.2).

$$\begin{cases} x = r\cos\varphi, & 0 \le r \le 1, \\ y = r\sin\varphi & -\pi \le \varphi \le \pi. \end{cases}$$

Set

$$v(r,\varphi) := u(r\cos\varphi, r\sin\varphi)$$

Then,

$$\frac{\partial v}{\partial r} = \underline{u_x \cos\varphi + u_y \sin\varphi}$$

$$\frac{\partial^2 v}{\partial r^2} = (u_{xx}\cos\varphi + u_{xy}\sin\varphi)\cos\varphi + (u_{xy}\cos\varphi + u_{yy}\sin\varphi)\sin\varphi$$

$$= \underline{u_{xx}\cos^2\varphi} + \underline{2u_{xy}\cos\varphi\sin\varphi} + \underline{u_{yy}\sin^2\varphi}$$

$$\frac{\partial v}{\partial \varphi} = -u_x r\sin\varphi + u_y r\cos\varphi$$

$$\frac{\partial^2 v}{\partial \varphi^2} = (-u_{xx} r\sin\varphi + u_{xy} r\cos\varphi)(-r\sin\varphi)$$

$$+\left(-u_{xy}\, r\sin\varphi + u_{yy}\, r\cos\varphi\right) r\cos\varphi$$

$$-u_x\, r\cos\varphi - u_y\, r\sin\varphi =$$

$$= r^2\left(\underline{u_{xx}\sin^2\varphi} - \underline{2u_{xy}\cos\varphi\sin\varphi} + \underline{u_{yy}\cos^2\varphi}\right) - r\left(\underline{u_x\cos\varphi + r\sin\varphi}\right)$$

Hence

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \varphi^2} =$$

$$= u_{xx} + u_{yy} = \Delta u$$

So, equation (27.1) has the form in the new coordinates

$$\boxed{v_{rr}(r,\varphi) + \frac{1}{r}v_r(r,\varphi) + \frac{1}{r^2}v_{\varphi\varphi}(r,\varphi) = 0, \quad r\in(0,1),\ \varphi\in(-\pi,\pi)} \tag{27.3}$$

Let,

$$F(\varphi) := f(\cos\varphi, \sin\varphi), \quad \varphi\in[-\pi,\pi]$$

Then

$$\boxed{v(1,\varphi) = F(\varphi), \quad \varphi\in[-\pi,\pi].} \tag{27.4}$$

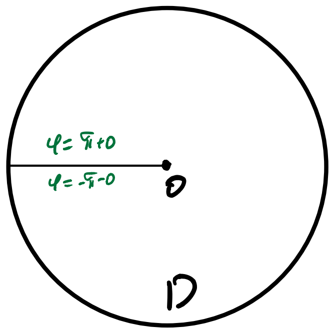
is the boundary condition for v.

We note, that by continuity

$$U(r,\pi-0)=U(r,-\pi+0)$$

$$U_\varphi(r,\pi-0)=U_\varphi(r,-\pi+0) \qquad (27.5)$$

$$\lim_{r\to 0+} U(r,\varphi) \text{ exists.}$$



We next find a solution to (27.3)-(27.5).
We use the method of separation of variables.

1) Find solution in the form

$$U(r,\varphi)=v(r)w(\varphi) \to (27.3)$$

$$v''(r)w(\varphi)+\frac{1}{r}v'(r)w(\varphi)+\frac{1}{r^2}v(r)w''(\varphi)=0 \quad \Big/ : vw\left(\frac{1}{r^2}\right)$$

Then

$$-r^2\frac{v''(r)}{v(r)}-r\frac{v'(r)}{v(r)}=\frac{w''(\varphi)}{w(\varphi)}=-\lambda$$

Hence,

$$\boxed{\begin{gathered} w''(\varphi)+\lambda w(\varphi)=0, \\ \varphi\in(-\pi,\pi) \end{gathered}} \qquad (27.6)$$

From (27.5)

$$\boxed{\begin{aligned} w(-\pi) &= w(\pi) \\ w'(-\pi) &= w'(\pi) \end{aligned}} \qquad (27.7)$$

2) We find nonzero solutions to the Sturm-Liouville problem (27.6), (27.7)
Write the characteristic equation

$$\mu^2 + \lambda = 0 \Rightarrow \mu = \pm\sqrt{-\lambda}.$$

a) $\lambda > 0$.

$$w(\varphi) = c_1 e^{\sqrt{-\lambda}\varphi} + c_2 e^{-\sqrt{-\lambda}\varphi}.$$

From (27.7)

$$c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi} = c_2 e^{-\sqrt{-\lambda}\pi} + c_2 e^{\sqrt{-\lambda}\pi}$$

$$\Rightarrow (c_1 - c_2) e^{\sqrt{-\lambda}\pi} = (c_1 - c_2) e^{-\sqrt{-\lambda}\pi} \Rightarrow c_1 = c_2$$

From the second condition:

$$c_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}\pi} - c_2\sqrt{\lambda}\, e^{-\sqrt{-\lambda}\pi} = c_1\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\pi} - c_2\sqrt{-\lambda}\, e^{\sqrt{-\lambda}\pi}$$

$$\Rightarrow c_1 + c_2 = 0 \Rightarrow c_1 = c_2 = 0.$$

We obtain only zero solution

b) $\lambda = 0$

$$w(\varphi) = c_1 \varphi + c_2$$

From (27.7):

$$c_1 \pi + c_2 = -c_1 \pi + c_2 \Rightarrow c_1 = 0$$

$$w'(\varphi) = c_1 = 0$$

Hence, $\boxed{w_0(\varphi) = 1, \qquad \lambda_0 = 0}$

is a nonzero solution to (27.6), (27.7).

c) $\lambda > 0 \Rightarrow \mu = \pm i\sqrt{\lambda}$

$$w(\varphi) = C_1 \cos\sqrt{\lambda}\,\varphi + C_2 \sin\sqrt{\lambda}\,\varphi$$

From (27.7)

$$C_1 \cos\sqrt{\lambda}\,\pi + C_2 \sin\sqrt{\lambda}\,\pi = C_1 \cos\sqrt{\lambda}\,\pi - C_2 \sin\sqrt{\lambda}\,\pi$$

$$\Rightarrow \quad 2C_2 \sin\sqrt{\lambda}\,\pi = 0$$

$$w'(\varphi) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\varphi + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\varphi$$

So, $-C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\pi =$

$$= C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\varphi + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\,\pi$$

$$\Rightarrow \quad 2C_1\sqrt{\lambda}\sin\sqrt{\lambda}\,\pi = 0.$$

This implies that

$$\sin\sqrt{\lambda}\,\pi = 0$$

Hence

$$\sqrt{\lambda}\,\pi = \pi n \,, \quad n = 1, 2, 3, \ldots$$

Consequently,

$$\boxed{\begin{array}{l} \sqrt{\lambda_n} = n \,, \quad \lambda_n = n^2 \,, \quad n = 1, 2, 3, \ldots \\ w_n(\varphi) = a_n \cos n\varphi + b_n \sin n\varphi \end{array}}$$

3) Find a solution to (27.6) in the form

$$v(\tau, \varphi) = \sum_{n=0}^{\infty} v_n(\tau)\, w_n(\varphi) \to (27.6)$$

We obtain

$$\sum_{n=0}^{\infty} v_n''(\tau)\, w_n(\varphi) + \frac{1}{\tau} \sum_{n=0}^{\infty} v_n'(\tau)\, w_n(\varphi) +$$

$$+ \frac{1}{\tau^2} \sum_{n=0}^{\infty} v_n(\tau)\, \underbrace{w_n''(\varphi)}_{= -n^2 w_n(\varphi)} = 0$$

So, we obtain the equation

(27.8)
$$\boxed{\begin{array}{c} v_n''(\tau) + \frac{1}{\tau} v_n'(\tau) - \frac{n^2}{\tau^n} v_n(\tau) = 0 \\ \tau \in (0,1). \end{array}}$$

We find a general solution to (27.8).

a) $n=0$

$$v''(r) + \frac{1}{r} v'(r) = 0$$

Set $g(r) := v'(r)$. Then

$$g'(r) + \frac{1}{r} g(r) = 0.$$

It is the ODE with separable variables

$$g(r) = \frac{c}{r}.$$

Then
$$v(r) = c \ln r + \tilde{c}$$

From the third equality of (27.5) $c=0$

and

$$\boxed{v_0(r) = 1}$$

b) $n = 1, 2, \dots$

Then $\quad v(r) = r^n, \quad v(r) = r^{-n}$

are solutions to (27.8). Again, only $v(r) = r^n$ satisfies (27.5)

Hence
$$\boxed{v_n(r) = r^n, \; n = 1, 2, \dots}$$

Consequently,

$$U(r,\varphi) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\varphi + b_n \sin n\varphi\right) r^n.$$

4) We find coefficients a_n, b_n from (27.4).

$$U(1,\varphi) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\varphi + b_n \sin n\varphi\right) = F(\varphi).$$

Using orthogonality of $\{\sin n\varphi, \cos n\varphi\}$ for different n, we have

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(\psi) \cos n\psi \, d\psi, \quad n = 0, 1, 2, \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(\psi) \sin n\psi \, d\psi. \quad n = 1, 2, \ldots$$

So, $U(r,\varphi) = \frac{1}{\pi}\left[\frac{1}{2} \int_{-\pi}^{\pi} F(\psi)\, d\psi + \right.$

$$\left. + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} F(\psi) \left(\cos n\psi \cos n\varphi + \sin n\psi \sin n\varphi \right) r^n d\psi \right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} F(\psi) \underbrace{\left(\frac{1}{2} + \sum_{n=1}^{\infty} \cos n(\varphi - \psi) r^n \right)}_{=: I} d\psi \right].$$

We simplify

$$I = \frac{1}{2} + \frac{1}{2} \sum_{n=1}^{\infty} r^n \left(e^{in(\varphi-\psi)} + e^{-in(\varphi-\psi)} \right) =$$

$$= \frac{1}{2} \left\{ 1 + \sum_{n=1}^{\infty} \left(r e^{i(\varphi-\psi)} \right)^n + \sum_{n=1}^{\infty} \left(r e^{-i(\varphi-\psi)} \right)^n \right\}$$

$$= \frac{1}{2} \left\{ 1 + \frac{r e^{i(\varphi-\psi)}}{1 - r e^{i(\varphi-\psi)}} + \frac{r e^{-i(\varphi-\psi)}}{1 - r e^{-i(\varphi-\psi)}} \right\} =$$

$$= \frac{1}{2} \frac{1 - r^2}{1 - 2 r \cos(\varphi-\psi) + r^2}$$

We have obtained

$$\boxed{v(r,\varphi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\psi) \frac{1-r^2}{1 - 2 r \cos(\varphi-\psi) + r^2} d\psi}$$

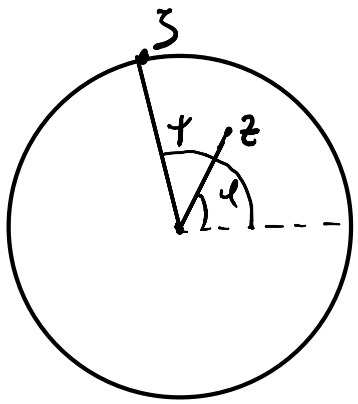
Let come back to the old variables (x,y).

Let $z := (x,y) = (r \cos\varphi, r \sin\varphi)$

$\zeta := (\xi, \eta) = (\cos\psi, \sin\psi)$

Then $\quad \|z\|^2 = r^2$

$\|z-\zeta\|^2 = 1 - 2 r \cos(\varphi-\psi) + r^2$



S_0,

$$\boxed{u(x,y) = u(z) = \frac{1}{2\pi} \int\limits_{\|\zeta\|=1} \frac{1-\|z\|^2}{\|z-\zeta\|^2} f(\zeta)\, dS(\zeta).}$$