26. Wave equation

1. Heat equation on $[0,\ell]$

Last time we started solving a heat equation on $[0,\ell]$ using the method of separation of variables.

Ex 26.1 (Ex 25.1 from lecture 25)

We consider the equation

$$(26.1) \quad u_t = a^2 u_{xx} + \overbrace{\cos \frac{3\pi}{2\ell} x}^{=:f}, \quad t>0, \ x\in(0,\ell)$$

$$(26.2) \quad \begin{cases} u_x(t,0)=0, \\ u(t,\ell)=0, \end{cases} \quad t\geq 0 \quad \text{- boundary cond.}$$

$$(26.3) \quad u(0,x) = A(\ell - x), \quad x\in[0,\ell]$$

$\leftarrow$ initial condition

1) We first found a solution to (26.1) in the form

$$u(x,t) = X(x)T(t) \longrightarrow (26.1) \text{ (with } f=0)$$

We obtained

$$T'(t)X(x) = a^2 T(t) X''(x) \quad /: a^2TX$$

$$\frac{T'(t)}{a^2T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

This gives the equation

$$X''(x) + \lambda X(x) = 0 \qquad (26.4)$$

From boundary conditions (26.2) we have

$$X'(0) = 0, \quad X(\ell) = 0 \qquad (26.5)$$

Nonzero solutions to Sturm-Liouville problem (26.4)-(26.5) are

$$\boxed{X_n(x) = \cos\frac{\pi(1+2n)}{2\ell}x, \quad n = 0, 1, 2, \dots}$$

and

$$\boxed{\sqrt{\lambda_n} = \frac{\pi(1+2n)}{2\ell}, \quad n = 0, 1, 2, \dots}$$

2) Find solution to (26.1)-(26.3) in the form

$$u(t,x) = \sum_{n=0}^{\infty} T_n(t) X_n(x)$$

$$= \sum_{n=0}^{\infty} T_n(t) \cos\frac{\pi(1+2n)}{2\ell}x. \to (26.1)$$

So,

$$\sum_{n=0}^{\infty} T_n'(t) X_n(x) = \sum_{n=0}^{\infty} a^2 T_n(t) \overbrace{X_n''(x)}^{= -\lambda_n X_n(x)} + f(t,x)$$

If we can write

$$f(t,x) = \sum_{n=0}^{\infty} f_n(t) X_n(x), \text{ then}$$

$$\sum_{n=0}^{\infty} T_n'(t) X_n(x) = \sum_{n=0}^{\infty} a^2 T_n(t)(-\lambda_n) X_n(x) + \sum_{n=0}^{\infty} f_n(t) X_n(x)$$

So, $$T_n'(t) + a^2 \lambda_n T_n(t) = f_n(t) \quad (26.6)$$

– equation for T_n.

Next we plug in u into (26.3):

$$\sum_{n=0}^{\infty} T_n(0) X_n(x) = A(l-x) = \sum b_n X_n(x)$$

consequently

$$T_n(0) = b_n \quad (26.7)$$

4) Find coefficients b_n and functions $f_n(t)$, using the formula

$$b_n = \frac{1}{\|X_n\|^2} \int_0^l \varphi(x) X_n(x)\, dx,$$

where $\|X_n\|^2 = \int_0^l X_n^2(x)\, dx.$

(The same formula valid for d_n)

So $$\|X_n\|^2 = \int_0^l \cos^2 \frac{\pi(1+2n)}{2l} x\, dx = \frac{l}{2}.$$

So, $$b_n = \frac{2A}{l} \int_0^l (l-x) \cos \frac{\pi(1+2n)}{2l} x\, dx =$$

$$= \dots = \frac{8A}{\pi^2(1+2n)^2}$$

For d_n, we remark that

$$f(t,x) = \cos \frac{3\pi}{2l} = X_1(x).$$

So, $d_1(t) = 1, \quad d_n(t) = 0, \; n \neq 1.$

5) Find T_n from (26.6), (26.7).

a) $n \neq 1$

$$T_n'(t) + a^2\lambda_n T_n(t) = 0$$

$$T_n(0) = b_n$$

$$\Rightarrow T_n(t) = b_n e^{-a^2\lambda_n t} = \frac{8A}{\pi^2(1+2n)^2} e^{-a^2 \frac{\pi^2(1+2n)^2}{4l^2} t}$$

b) $n = 1$

$$T_1'(t) + a^2\lambda_1 T_1(t) = 1$$

$$T_1(0) = b_1$$

So, $$T(t) = b_n e^{-a^2\lambda_n t} + \frac{1}{a^2\lambda_1} =$$

$$= \frac{8A}{9\pi^2} e^{-a^2\pi^2 \frac{9}{4l^2} t} + \frac{4l^2}{a^2\pi^2 9}$$

Hence

$$u(t,x) = \left(\frac{8A}{9\pi^2} e^{-a^2\pi^2 \frac{9}{4l^2} t} + \frac{4l^2}{a^2\pi^2 9}\right)\cos\frac{3\pi}{2l}x$$

$$+ \sum_{\substack{n=0 \\ n\neq 1}}^{\infty} \frac{8A}{\pi^2(1+2n)^2} e^{-a^2 \frac{\pi^2(1+2n)^2}{4l^2} t} \cos\frac{\pi(1+2n)}{2l} x$$

is a solution to (26.1)–(26.3).

2. Wave equation on $\mathbb{R}$. D'Alambert's formula.

In this section, we solve the wave equation on $\mathbb{R}$:

$$(26.8) \quad u_{tt} = a^2 u_{xx}$$

$$(26.9) \quad u(x,0) = f(x) \text{ — initial position}$$

$$(26.10) \quad u_t(x,0) = g(x) \text{ — initial velocity}$$

In order to derive a formula for solution to (26.8)–(26.10) we first find general solution to (26.8).

a) General solution (26.8).

Let u be a solution to (26.8). We consider a new function

$$w = u_t + a u_x$$

and show that w solves the transport equation

$$w_t - a w_x = 0.$$

Indeed,

$$w_t - a w_x = \underline{u_{tt}} + \underline{a u_{xt}} - \underline{a u_{tx}} - \underline{a^2 u_{xx}} = 0$$

Moreover, the equation

$$u_{tt} = a^2 u_{xx} \qquad (26.8)$$

is equivalent to

$$\begin{cases} u_t + a u_x = w \\ w_t - a w_x = 0, \end{cases} \qquad (26.11)$$

that is, if w, u satisfies (26.11), then u solves (26.8). An exsample of solution to (26.11) is

$$w = 0,$$

then

$$u_t + a u_x = 0 \quad \text{- transport eq.}$$

we know from Lecture 24

$$\boxed{u(t,x) = p(x - at)}$$

Similarly, (26.8) is equivalent to

$$\begin{cases} u_t - a u_x = v \\ v_t + a v_x = 0 \end{cases}$$

For $v = 0$,

$$\boxed{u(t,x) = q(x + at)}$$

Adding two solutions, we have

$$\boxed{u(t,x) = p(x-at) + q(x+at),} \quad (26.12)$$

where p, q are twice diff. functions from $\mathbb{R}$ to $\mathbb{R}$.

Exercise 26.1 Show that any solution to (26.8) can be written in the form (26.12).

b) D'Alomber's formula

We find functions p, q in (26.12) from initial conditions (26.9), (26.10).

So

$$u(t,x) = p(x-at) + q(x+at) \rightarrow (26.9)$$

$$u_t(x,t) = -ap'(x-at) + aq'(x+at)$$

Then

$$u(x,0) = p(x) + q(x) = f(x) \quad (26.13)$$

$$u_t(x,0) = -ap'(x) + aq'(x) = g(x)$$

We integrate the second equation. Let

$$G'(x) = g(x)$$

$$-a p(x) + a q(x) = G(x) \qquad (26.14)$$

(26.13), (26.14) give

$$p(x) = \frac{1}{2} f(x) - \frac{1}{2a} G(x),$$

$$q(x) = \frac{1}{2} f(x) + \frac{1}{2a} G(x).$$

Hence,

$$\boxed{u(t,x) = \frac{1}{2}\left(f(x-at) + f(x+at)\right) + \frac{1}{2a}\int_{x-at}^{x+at} g(y)\,dy.}$$

Ex 26.2 We solve the equation

$$u_{tt} = u_{xx}$$

$$u(0,x) = \sin x$$

$$u_t(0,x) = x + \cos x$$

$$u(t,x) = \frac{1}{2}\left(\sin(x-t) + \sin(x+t)\right) +$$

$$+ \frac{1}{2}\int_{x-at}^{x+at} (y + \cos y)\,dy =$$

$$= \sin\frac{x+t+x-t}{2}\cos\frac{x+t-(x-t)}{2} +$$

$$+\frac{1}{2}\left(\frac{y^2}{2}+\sin y\right)\Big|_{x-at}^{x+at} =$$

$$= \sin x \cos t + \frac{1}{4}\left((x+t)^2-(x-t)^2\right)+$$

$$+\frac{1}{2}\left(\sin(x+t)-\sin(x-t)\right) =$$

$$= xt + 2\sin t \cos x .$$

For more details about the wave equation see Section 2.2 p.45 [Tikhonov, Samarskii].