21. The Laurent series

1. Uniqueness of holomorphic functions

We recall that a point $a \in \mathbb{C}$ is called a zero of a function f if

$$f(a) = 0.$$

Last time we have proved the following theorem

Theorem 20.6 Let a point $a \in \mathbb{C}$ be a zero of a function f that is holomorphic at this point. Let also f is not equal identically to zero in neighborhood of a. Then there exists $n \in \mathbb{N}$ so that

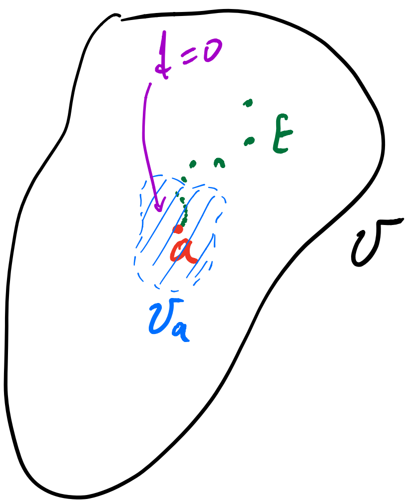
$$f(z) = (z-a)^n \varphi(z),$$

where φ is holomorphic at a and $\varphi(z) \neq 0$ $\forall z$ from a neighborhood of a

We use this theorem for proof of uniqueness theorem

Theorem 21.1 (Uniqueness)
Let f_1, f_2 be holomorphic in a connected open set $U \subset \mathbb{C}$. Then if $f_1 = f_2$ on $E \subseteq U$ that has a limit point in U then $f_1(z) = f_2(z)$ for all $z \in U$.

Proof The function $f = f_1 - f_2$ is holomorphic in U. We only have to prove that $f = 0$ on U. Let a be a limit point of E. Remark that $a \in U$. Since the function f is continuous on U and a is a limit point of E (that is $\exists z_n \in E$ s.t. $z_n \to a$) we have



$$f(a) = \lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} 0 = 0.$$

By theorem 20.6 $f = 0$ on a neighborhood U_a of a.

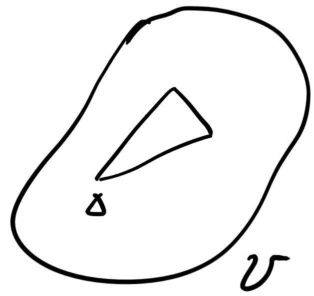
Repeating the same procedure for any limit point of U_a, we obtain that $f=0$ on U.

$\blacksquare$

In this section, we also formulate

useful statements without proof.

Theorem 21.2 (Morera) If a function f is continuous in U and



$$\int_{\partial\Delta} f(z)\,dz = 0$$

for any triangle $\Delta \subseteq U$, then f is holomorphic.

Theorem 21.3 (Weierstrass) If the series

$$f(z) = \sum_{n=0}^{\infty} f_n(z)$$

of holomorphic functions in U converges uniformly on any compact subset of U, then f is also holomorphic and

$$f^{(m)}(z) = \sum_{n=0}^{\infty} f_n^{(m)}(z)$$

for any $m \in \mathbb{N}$.

2. The Laurent series

Theorem 21.4 (Laurent) Any holomorphic function f in an annulus

$$V = \{z \in \mathbb{C} : r < |z-a| < R\}$$

may be represented in this annulus as a sum

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n, \qquad (21.1)$$

where

$$(21.2) \quad c_n = \frac{1}{2\pi i} \int_{\gamma_\rho} \frac{f(\xi)}{(\xi - a)^{n+1}}, \quad n = 0, \pm 1, \pm 2, \dots$$

$$\gamma_\rho = \{ z : |z-a| = \rho \}, \qquad r < \rho < R$$

Proof The proof is similar to the proof of Theorem 19.1 and can be found in [Shabat p. 75] ∎

Def 21.1 The series (21.1) with the coefficients determined by (21.2) is called the Laurent series of the function f in the annulus V.
The term

$$\sum_{n=0}^{\infty} c_n (z-a)^n = c_0 + c_1 (z-a) + \dots$$

is called its regular part and

$$\sum_{n=-\infty}^{-1} c_n (z-a)^n = \frac{c_{-1}}{z-a} + \frac{c_{-2}}{(z-a)^2} + \dots$$

is called the principal part.

Let us understand how the domain of convergence of

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

can be defined.

By the Cauchy - Hadamard formula
The series

$$\sum_{n=0}^{\infty} c_n (z-a)^n$$

converges in the disk $\{|z-a|<R\}$,
where

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} .$$

Next we consider the series

$$\sum_{n=-\infty}^{-1} c_n (z-a)^n$$

We replace $w := \frac{1}{z-a}$. So, we obtain

$$\sum_{n=-\infty}^{-1} c_n (z-a)^n = c_{-1} \frac{1}{z-a} + c_{-2} \frac{1}{(z-a)^2} + \dots =$$

$$= c_{-1} w + c_{-2} w^2 + c_{-3} w^3 + \dots$$

Hence the last series converges for all $|w| < \rho$, where

$$\frac{1}{\rho} = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_{-n}|} .$$

or for $|z-a| > \frac{1}{\rho} =: r$

Consequently, the domain of convergence of

$$\sum_{n=-\infty}^{\infty} c_n (z-a)^n$$

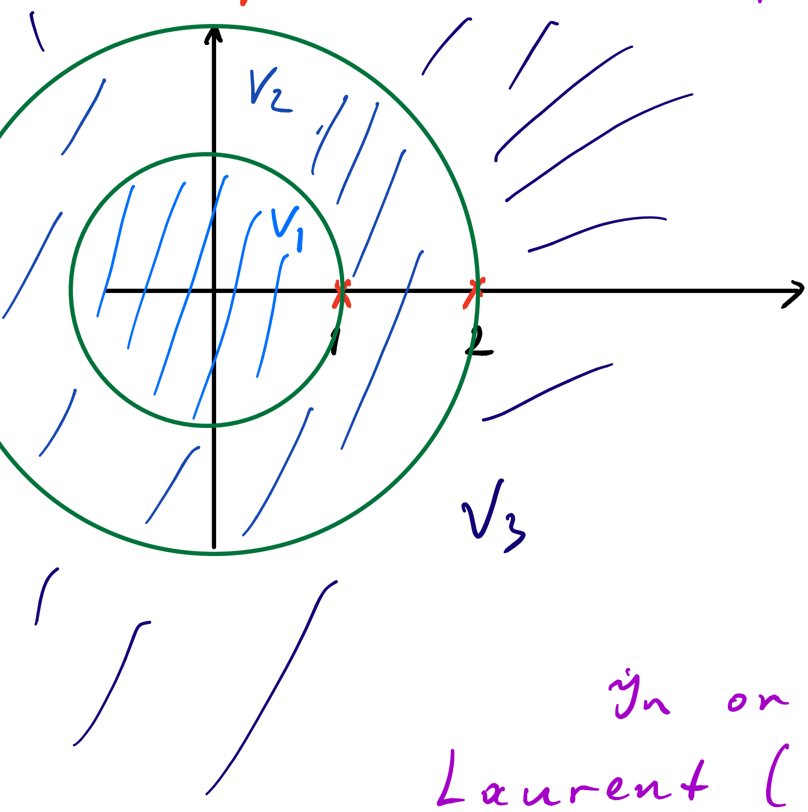
is the annulus

$$V = \{ r < |z-a| < R \},$$

where

$$\boxed{\begin{aligned} r &= \overline{\lim_{n\to\infty}} \sqrt[n]{|c_{-n}|}, \\ \frac{1}{R} &= \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}. \end{aligned}}$$

Example 21.1 The function $f(z) = \frac{1}{(z-1)(z-2)}$



is holomorphic in the disk

$$V_1 = \{ |z| < 1 \}$$

and annuli

$$V_2 = \{ 1 < |z| < 2 \}$$

$$V_3 = \{ 2 < |z| < \infty \}$$

In order to obtain its Laurent (or Taylor) series we represent f as

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}.$$

1) Consider the domain V_1.

$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n =$$

$$= -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}} z^n \quad (\text{conv. for } |z|<2)$$

$$-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad (\text{conv. for } |z|<1)$$

Therefore

$$f(z) = \sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right) z^n, \quad |z|<1.$$

2) Annulus V_2:

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty}\frac{1}{2^{n+1}} z^n \quad (\text{conv. for } |z|<2)$$

$$-\frac{1}{z-1} = -\frac{1}{z}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} =$$

$$= -\sum_{n=-\infty}^{-1} z^n \quad \left(\text{conv. for } \frac{1}{|z|}<1 \text{ or } |z|>1\right)$$

Hence $$f(z) = -\sum_{n=-1}^{-\infty} z^n - \sum_{n=0}^{\infty}\frac{1}{2^{n+1}} z^n.$$

3) Similarly,

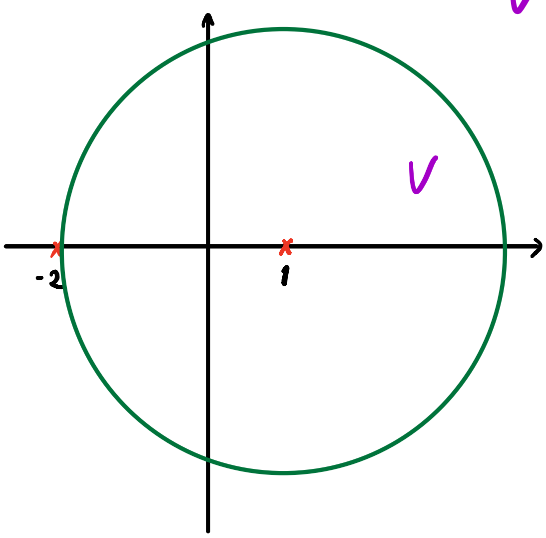
$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{2}\sum_{n=-1}^{-\infty}\frac{1}{2^n} z^n \quad (\text{conv. for } |z|>2)$$

So, $$f(z) = \sum_{n=-1}^{-\infty}\left(\frac{1}{2^{n+1}}-1\right) z^n.$$

Example 20.2 We write the expansion of

$$f(z) = \frac{1}{(1-z)(2+z)}$$

in the annulus



$$V = \{ 0 < |z-1| < 3 \}$$

$$f(z) = \frac{1}{(1-z)(2+z)} =$$

$$= -\frac{1}{(z-1)(z-1+3)} =$$

$$= -\frac{1}{z-1} \cdot \frac{1}{3} \frac{1}{1+\frac{z-1}{3}} =$$

$$= -\frac{1}{z-1} \cdot \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{3^n} =$$

$$= \sum_{n=-1}^{\infty} \frac{(-1)^{n+1}}{3^{n+1}} (z-1)^n, \quad z \in V.$$

Example 21.3 Let $f(z) = \frac{1}{(z-i)^3}$.

We want to write the Laurent series for f with $a = 0$. We will use the formula

$$(1+z)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!} z^n, \quad |z|<1.$$

So, $\frac{1}{(z-i)^3} = (z-i)^{-3} = (-i)^{-3}\left(\frac{z}{-i}+1\right)^{-3} =$

$$= \frac{1}{i}\left(\frac{z}{-i}+1\right)^{-3} = \frac{1}{i}\left(1+\sum_{n=1}^{\infty}\frac{(-3)(-4)\ldots(-3-n+1)}{n!}\left(\frac{z}{-i}\right)^n\right)$$

$$= \frac{1}{i}\left(1+\sum_{n=1}^{\infty}\cancel{(-1)^n}\,\frac{3\cdot4\cdot\ldots\cdot(n+2)}{n!}\,\frac{z^n}{\cancel{(-1)^n}\,i^n}\right) =$$

$$= \frac{1}{i}\left(1+\sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{2}\,\frac{z^n}{i^n}\right) =$$

$$= \frac{1}{i}\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}\,\frac{z^n}{i^n} = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2i^{n+1}}z^n,$$

$|z|<1.$