20. Taylor Series. Further properties of holomorphic functions

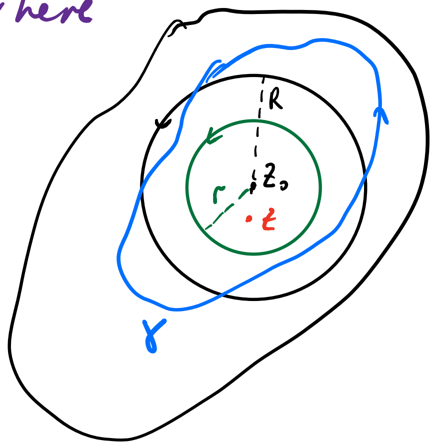
1. Differentiability of Taylor series

We recall from the previous lecture that any holomorphic function can be expanded into a Taylor series

Theorem 19.1 Let f be holomorphic in U and $z_0 \in U$. Then the function f may be represented as a sum

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$

inside any disk $B_R = \{|z - z_0| < R\} \subset U$, where



$$c_n = \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}$$

$$\gamma_r = \{ z : |z - z_0| = r \}.$$

Remark 20.1 Let γ be any simple positively oriented path around the point z_0. Then

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^{n+1}}$$

The statement of Remark 20.1 immediately follows from the Cauchy theorem (see Prop. 18.3)

We next discuss the radius of convergence of the power series. We will assume further that c_n, $n \geq 0$, are any complex numbers.

Theorem 20.1 (The Cauchy - Hadamard formula)
Let the coefficients of the power series

$$\sum_{n=0}^{\infty} c_n (z-a)^n \qquad (20.1)$$

satisfy

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} = \frac{1}{R},$$

with $0 \leq R \leq +\infty$. Then the series (20.1) converges at all z such that

$$|z-a| < R$$

and diverges at all z such that

$$|z-a| > R.$$

Proof. Let $|z-a| < R$. We want to show that

$$\sum_{n=0}^{\infty} |c_n (z-a)^n|$$

converges.

This will imply the convergence of

$$\sum_{n=0}^{\infty} c_n (z-a)^n.$$

(check this as an exercise)

We remark that by the root test (see Math 1 Th 20.2), the series

$$\sum_{n=0}^{\infty} |c_n (z-a)^n| = \sum_{n=0}^{\infty} |c_n|\,|z-a|^n$$

converges if

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|\,|z-a|^n} < 1.$$

But

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|\,|z-a|^n} = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}\,|z-a| =$$

$$= \frac{1}{R} \underbrace{|z-a|}_{<R} < 1.$$

Next, let $|z-a| > R$. This implies that

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|\,|z-a|^n} > 1.$$

Consequently, there exists a subsequence such that

$$\sqrt[n_k]{|c_{n_k}|\,|z-a|^{n_k}} > 1 \quad \forall\, k \geq 1.$$

So $|c_{n_k}||z-a|^{n_k} > 1 \Rightarrow |c_{n_k}||z-a|^{n_k} \not\to 0$

Hence, $c_{n_k}(z-a)^{n_k} \not\to 0$. This implies that the series

$$\sum_{n=1}^{\infty} c_n (z-a)^n$$

diverges.

$\blacksquare$

Theorem 20.1 implies that the set

$$B_R = \{ z : |z-a| < R \}$$

is the domain of converges of (20.1)

Theorem 20.2 The sum of a power series

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$$

is holomorphic in its domain of convergence. Moreover,

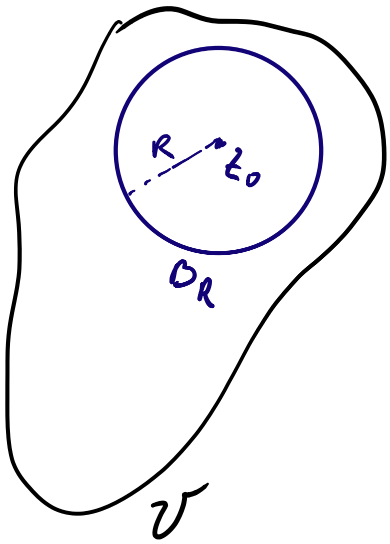
$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-a)^{n-1}$$

2. Properties of holomorphic functions

Theorem 20.3 Let f be a holomorphic function in an open subset $U \subseteq \mathbb{C}$.

Then f' is also holomorphic in U.

Proof. Let $z_0 \in U$. The function f can be represented as a sum of a Taylor series in B_R, by Theorem 19.1. Then by Theorem 18.2, the series is holomorphic. Consequently f is holomorphic in B_R. This impies the statement. ∎



Theorem 20.4 Any holomorphic function f in U has derivatives of all orders in U which are also holomorphic in U.

Proof The theorem is a direct consequence of Theorem 20.3.

Theorem 20.5 Let a function f have a representation

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \qquad (20.2)$$

in a disk $B_R = \{|z - z_0| < R\}$. Then the coefficients c_n are determined uniquely as

$$c_n = \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, \ldots$$

Proof Inserting $z = z_0$ into (20.2) we have

$$f(z_0) = c_0$$

Differenting (20.2) termwise we obtain

$$f'(z) = c_1 + 2c_2(z - z_0) + \dots$$

So $f'(z_0) = c_1$

We remark that we may differentiate (20.2) termwise according to Theorem 20.2

Similarly

$$f^{(n)}(z) = n!\, c_n + \tilde{c}_1 (z - z_0) + \dots$$

So, $f^{(n)}(z_0) = n!\, c_n$. ∎

Cauchy integral formula for derivatives of holomorphic function f in U:

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}, \quad n = 1, 2, \dots}$$

where γ is a simple positive oriented path in U around z.

This formula follows from Theorem 18.1 and Theorem 20.5.

Example 20.1

$$e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots \quad z \in \mathbb{C}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \quad z \in \mathbb{C}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \quad z \in \mathbb{C}$$

$$\frac{1}{1-z} = 1 + z + z^2 + \dots \quad |z| < 1.$$

3. Zeros of holomorphic functions

Def 20.1 A zero of a function f is a point $a \in \mathbb{C}$ where f vanishes, that is $f(a) = 0$.

We are going to show that a holomorphic function has only zeros which are isolated.

Theorem 20.6 Let a point $a \in \mathbb{C}$ be a zero of a function f that is holomorphic at this point. Let also f is not equal identically to zero in neighborhood of a. Then there exists $n \in \mathbb{N}$ so that

$$f(z) = (z-a)^n \varphi(z),$$

where φ is holomorphic at a and $\varphi(z) \neq 0$ $\forall z$ from a neighborhood of a.

Proof f can be represented by a power series in a neighborhood of a

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n .$$

Since $f(a) = 0 \Rightarrow c_0 = 0$.

Let $n \in \mathbb{N}$ be such that $c_n \neq 0$, and $c_0 = c_1 = \dots = c_{n-1} = 0$.

Then

$$f(z) = c_n (z-a)^n + c_{n+1} (z-a)^{n+1} + \dots,$$

$$c_n \neq 0$$

Hence

$$f(z) = (z-a)^n \big(\underbrace{c_n + c_{n+1}(z-a) + \dots}_{=: \varphi(z)} \big).$$

Note φ is holomorphic at a, by

Theorem 20.2. Moreover, $\varphi(a) = c_n \neq 0$ and φ is continuous at a. Hence $\varphi(z) \neq 0$ $\forall z$ from a neighborhood of a.

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