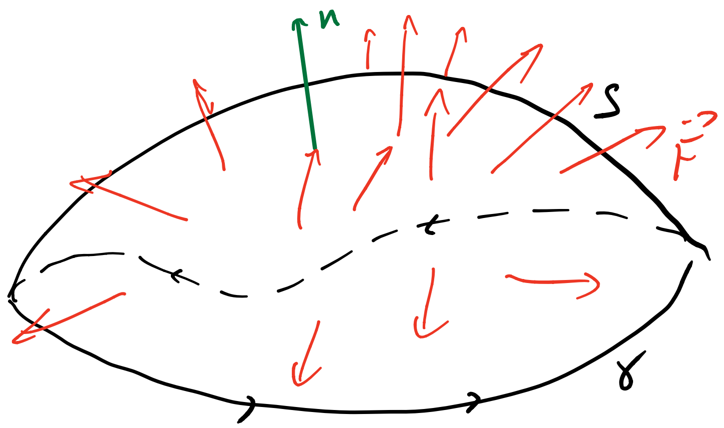
13. Stokes' theorem

1. Stokes' theorem



Let S be a piecewise-smooth surface in $\mathbb{R}^3$ oriented by a unit normal n and γ be the boundary of S positively oriented with respect to the normal n (if the thumb of the right hand points in the direction of n, then the other fingers in the direction γ').

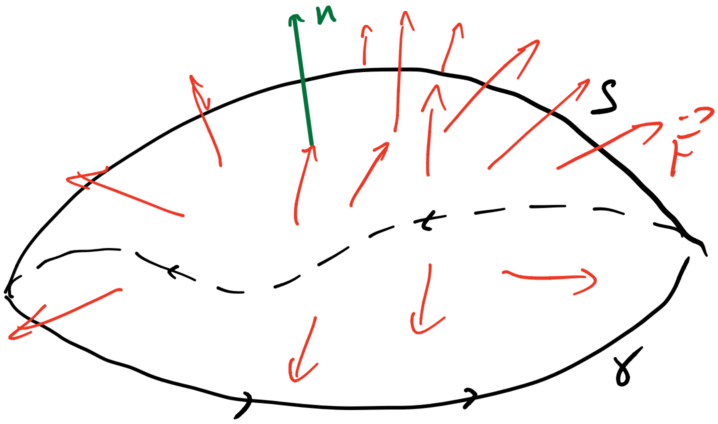
Let $\vec{F}$ be a continuously differentiable vector field on S. We recall

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Theorem 13.1 (Stokes) Under the assumption above

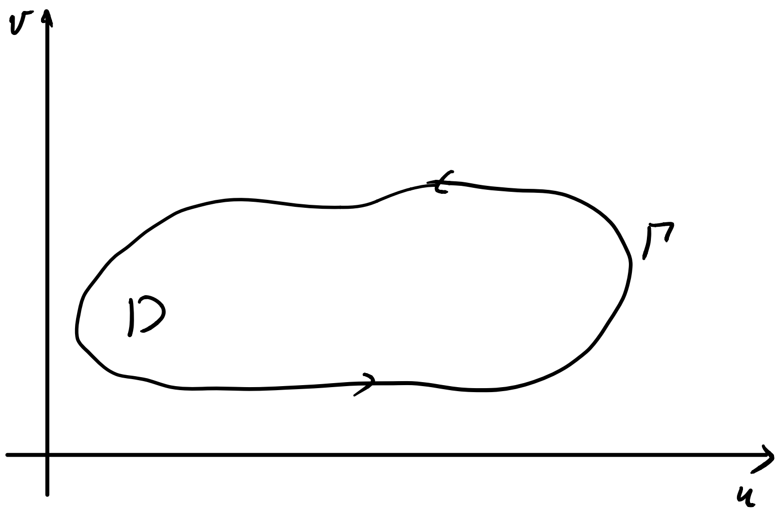
$$\boxed{\int_\gamma \vec{F} \cdot ds = \iint_S \operatorname{curl} \vec{F} \cdot dS.}$$

Remark 13.1 a) Let S be parametrized by $\{ r(u,v),\ (u,v) \in \bar{D} \}$ and let Γ be the positively oriented boundary of $D \subseteq \mathbb{R}^2$. Then



$\gamma = r(\Gamma)$ and is positively oriented with respect to the normal

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\| \vec{r}_u \times \vec{r}_v \|} .$$



if $\vec{F} = (P, Q, R)$, then Stokes' theorem can be equivalently stated as

$$\int_\gamma P\,dx + Q\,dy + R\,dz = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy\,dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz\,dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy \qquad (13.1)$$

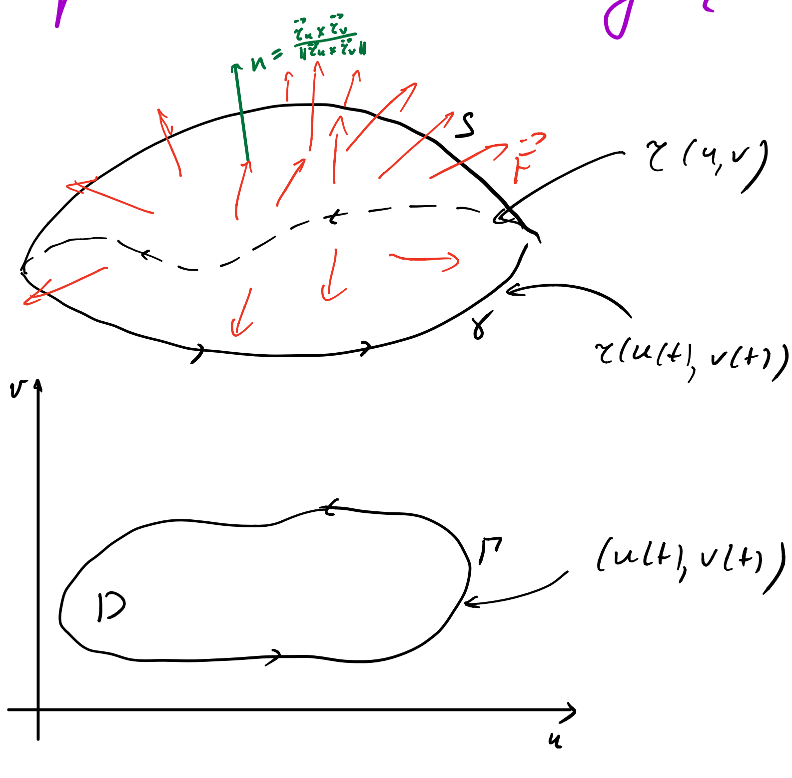
b) Let $\vec{n} = (n_x, n_y, n_z)$. Then

$$\iint_S \operatorname{curl} \vec{F} \cdot ds = \iint_S (\operatorname{curl} \vec{F} \cdot \vec{n})\, dS =$$

$$= \iint_S \begin{vmatrix} n_x & n_y & n_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS.$$

Proof of Th 13.1 As in the proof of Gauss–Ostrogradskii theorem, it is suffices to prove the theorem for the fields $(P, 0, 0)$, $(0, Q, 0)$, $(0, 0, R)$. We will prove it for the case $\vec{F} = (P, 0, 0)$.

We will prove Stokes' theorem in the form of (13.1). Let S be parametrized by $\{ r(u,v), (u,v) \in D \}$ and let Γ be the positively oriented boundary of $D \subseteq \mathbb{R}^2$ parametrized by $\{ (u(t), v(t)), 0 \le t \le T \}$.



Then $\gamma = r(\Gamma)$ is the boundary of S parametrized by $\{ r(u(t), v(t)), 0 \le t \le T \}$ and positively oriented with respect to $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$

So, we compute

$$\int_\gamma \vec{F}\cdot ds = \int_\gamma P\,dx$$

$$= \int_0^T P(r(u(t),v(t)))\cdot \frac{\partial x(u(t),v(t))}{\partial t}\,dt$$

$$= \int_0^T P(r(u(t),v(t)))\left(\frac{\partial x}{\partial u}u'(t) + \frac{\partial x}{\partial v}v'(t)\right)dt$$

$$= \int_\Gamma P\frac{\partial x}{\partial u}\,du + P\frac{\partial x}{\partial v}\,dv \qquad \text{(Green's theorem)}$$

$$= \iint_D \left[\frac{\partial}{\partial u}\left(P\frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(P\frac{\partial x}{\partial u}\right)\right]du\,dv$$

$$= \iint_D \left[\frac{\partial P}{\partial z}\frac{\partial(z,x)}{\partial(u,v)} - \frac{\partial P}{\partial y}\frac{\partial(x,y)}{\partial(u,v)}\right]du\,dv$$

$$= \iint_S \frac{\partial P}{\partial z}\,dz\,dx - \frac{\partial P}{\partial y}\,dx\,dy$$

$$= \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}. \qquad \blacksquare$$

Example 13.1 Let γ be the curve of intersection of paraboloid

$$x^2+y^2+z=3$$

and the plain

$$x+y+z=2$$

oriented positively with respect to the vector $(1,1,1)$.

Let S be the surface in the plane spanned by γ oriented by the unit normal

$$\vec{n}=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right).$$

Note that γ is positively oriented with respect to $\vec{n}$.

We aim to compute

$$I=\int_{\gamma}(y^2-z^2)\,dx+(z^2-x^2)\,dy+(x^2-y^2)\,dz.$$

We are going to use the Stokes theorem.

We compute for $P=y^2-z^2$, $Q=z^2-x^2$, $R=x^2-y^2$

$$\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}=-2(y+z)$$

$$\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}=-2(x+z)$$

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=-2(x+y)$$

By Stokes' theorem (γ is positively oriented with respect to $\vec{n}$)

$$I = \iint_S (\operatorname{curl} \vec{F} \cdot \vec{n})\, dS = -\frac{4}{\sqrt{3}} \iint_S (x+y+z)\, dS$$

Since S is a subset of the plain

$$x+y+z=2$$

$$I = -\frac{8}{\sqrt{3}} \iint_S dS$$

The surface S can be parametrized as

$$z = 2-x-y, \qquad (x,y) \in \bar{D}.$$

By Example 10.3 f)

$$\sqrt{EG-F^2} = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{3}$$

Therefore

$$I = -8 \text{ Area}(D)$$

The boundary of D is the projection of γ on xy-coordinate plane. To find its equation, we eliminate the dependency on z from the system of equations

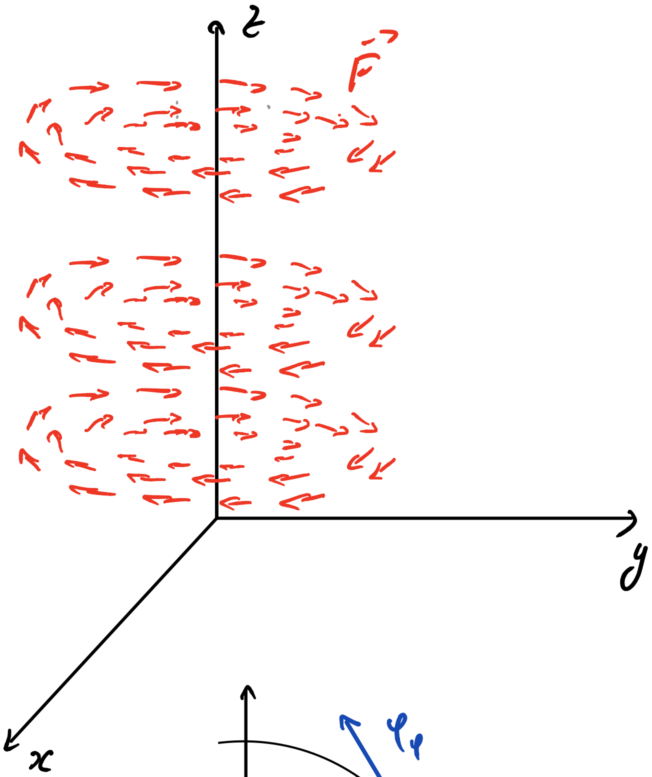
$$\begin{cases} x^2+y^2+z=3 \\ x+y+z=2 \end{cases}$$

to get $\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{3}{2}$.
which is a circle of radius $\sqrt{\frac{3}{2}}$.
Thus, Area $(D) = \frac{3}{2}\pi$ and

$$I = -12\pi.$$

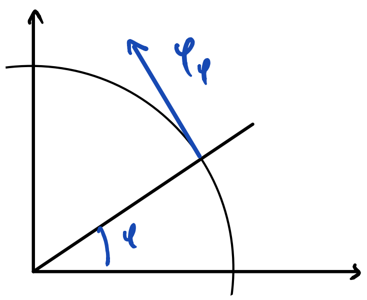
2. Physical meaning of the curl.

Suppose that the entire space, regarded as a rigid body, is rotating with constant angular speed ω about a fixed axis (say the z-axis)



Let us find the curl of the field $\vec{F}$ of linear velocities of the points of space.
In cylindrical coordinates (r, φ, z) we have the simple expression

$$\vec{F}(r, \varphi, z) = \omega r e_\varphi$$

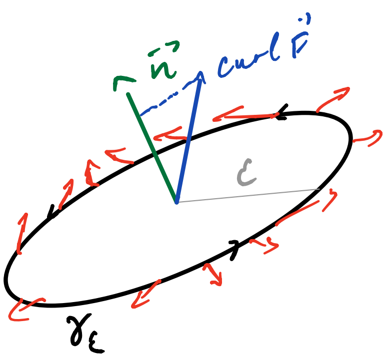


A simple computations shows that

$$\operatorname{curl} \vec{F} = 2\omega e_z.$$

where $e_z = (0,0,1)$.
That is, curl $\vec{F}$ is a vector directed along the axis of rotation. Its magnitude 2ω equals the angular velocity of the rotation, up to the coefficient 2, and the direction of the vector determines the direction of rotation.

Locally the curl of a vector field at a point characterizes the degree of vorticity of the field in a neighborhood of the point. Indeed, let $\vec{n}$ be a unit vector and γ_ε be a circle of radius ε centered at $p \in \mathbb{R}^3$, lying in the plain perpendicular to $\vec{n}$ and positively oriented with respect to $\vec{n}$.



Then the projection of curl $\vec{F}$ on $\vec{n}$ can be computed using Stockes' theorem

$$\operatorname{curl} \vec{F}(p)\cdot\vec{n} = \lim_{\varepsilon\to 0} \frac{1}{\pi\varepsilon^2} \int_{\gamma_\varepsilon} \vec{F}\cdot ds,$$

where $\int_{\gamma_\varepsilon} \vec{F}\cdot ds$ is the circulation of $\vec{F}$ along γ.

The value of curl $\vec{F}(r)\cdot\vec{n}$ is maximal in the direction of $\vec{n}$ coinciding with direction of curl $\vec{F}$.

3. Solenoidal vector fields

Def 13.1 A vector field $\vec{F}$ in $\mathbb{R}^3$ is solenoidal, or divergence free, in $V \subseteq \mathbb{R}^3$ if $\operatorname{div} F = 0$ in V.

Examples 13.2 a) Coulomb's force

$$\vec{F} = -\left(\frac{x}{\|r\|^3}, \frac{y}{\|r\|^3}, \frac{z}{\|r\|^3}\right), \quad r = \sqrt{x^2+y^2+z^2}.$$

b) Velocity field of incompressible fluid.

c) magnetic field.

The following characterization of solenoidal fields follows from Gauss-Ostrogradskii theorem

Prop 13.1 Let V be a simply connected domain in $\mathbb{R}^3$ and $\vec{F}$ be a smooth vector field on $\overline{V}$. Then, $\vec{F}$ is solenoidal in V iff for any solid $\tilde{V} \subset V$ with smooth boundary $\hat{S}$, the flux of $\vec{F}$ through $\hat{S}$ is zero.