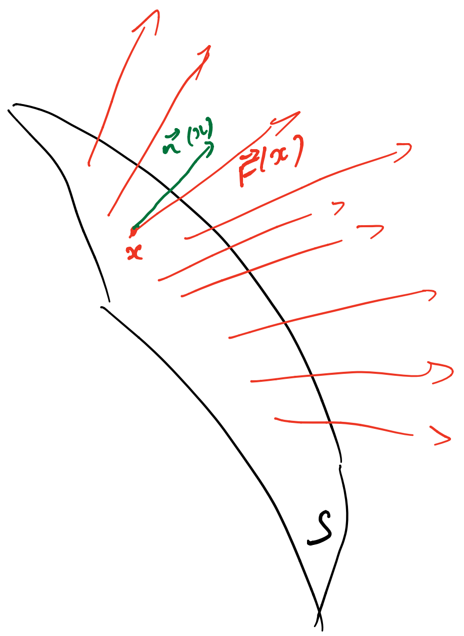
11. Surface integral of a vector field

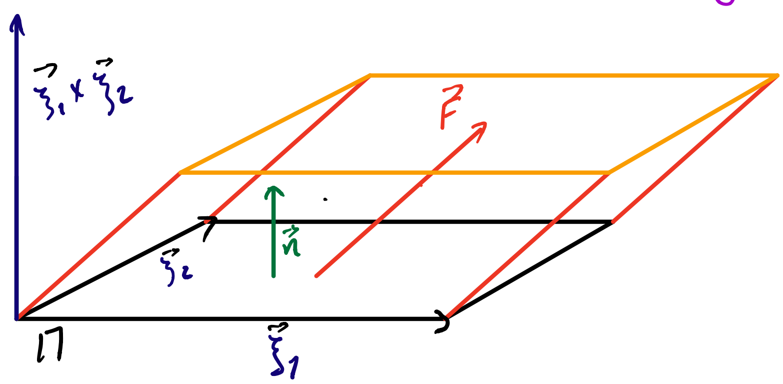
1. Flux across a surface.

Suppose there is a steady flow of liquid in a domain G and that $x \mapsto \vec{F}(x)$ is the velocity field of this flow. Assume that S is a smooth surface in G. Let $x \to \vec{n}(x)$ be a field of normal vectors to S.



We ask how to determine the (volumetric) outflow or flux of fluid across the surface S. More precisely, we ask how to find the volume of fluid that flows across the surface S per unit time in the direction indicated by the orienting field of normals to the surface.

We remark that if the velocity field of the flow is constant $\vec{F}$, then the flow per unit time across a parallelogram Π equals the volume of the parallelepiped constructed on the vectors $\vec{F}, \vec{\xi}_1, \vec{\xi}_2$.

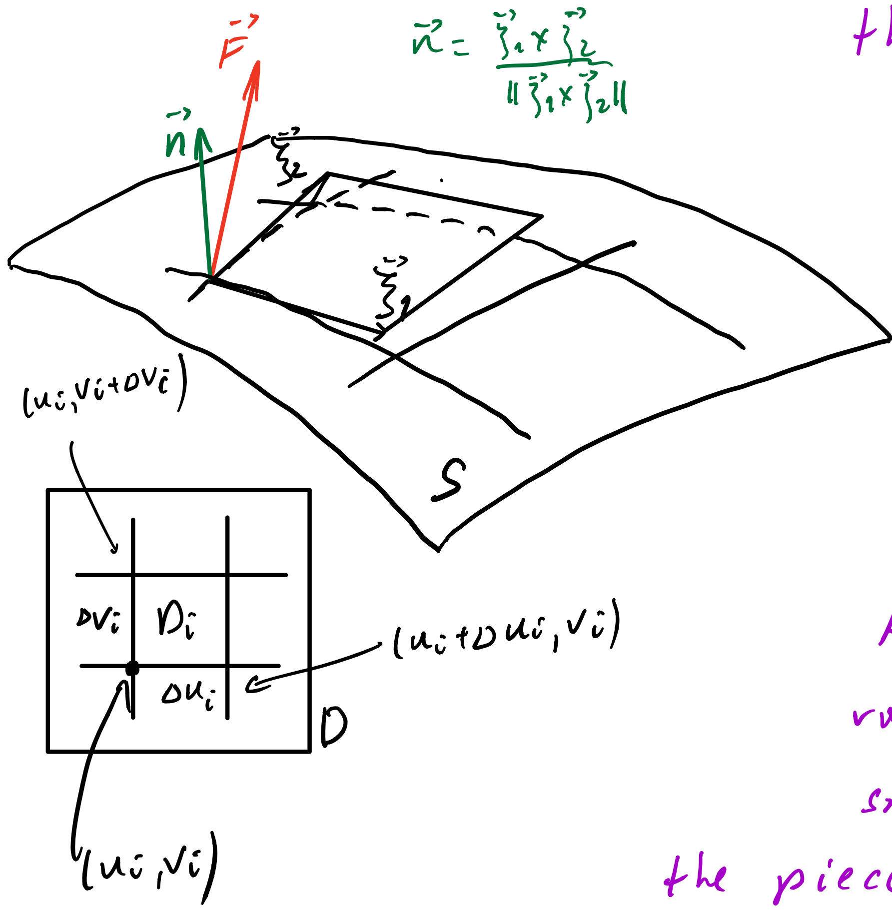


The volume is equal $\vec{F}\cdot(\vec{\xi}_1\times\vec{\xi}_2) = (\vec{F}, \vec{\xi}_1, \vec{\xi}_2)$ that is the triple product of $\vec{F}, \vec{\xi}_1, \vec{\xi}_2$.
If the orientation is opposite to the direction $\vec{F}$, then the flow equals $-(\vec{F}, \vec{\xi}_1, \vec{\xi}_2)$.

Now, let S admits a smooth parametrization

$$S = \{ \tau = \tau(u,v) : (u,v)\in D \}.$$

In order to define the flux of the flow across S, we fix a partition $\{D_i\}$ and appoximate the immage $\tau(D_i)$ by the parallelogram spanned by the images



$$\vec{\xi}_1 = \tau_u(u_i,v_i)\Delta u_i$$

$$\vec{\xi}_2 = \tau_v(u_i,v_i)\Delta v_i$$

Assume that $\vec{F}(x)$ varies by only the small amount inside the piece of surface $\tau(D_i)$ and replacing $\tau(D_i)$ by this parallelogram, we may assume that the flux ΔF_i accross the piece $\tau(D_i)$ of the surface is equall (with some small error) to the flux of a constant velocity field

$$\vec{F}(x_i, y_i, z_i) = \vec{F}(r(u_i, v_i))$$

across the parallelogram spanned by the vectors $\vec{\xi}_1, \vec{\xi}_2$. So

$$\Delta \mathcal{F}_i \approx \left(\vec{F}(x_i, y_i, z_i), \vec{\xi}_1, \vec{\xi}_2\right) =$$

$$= \left(\vec{F}(r(u_i, v_i)), \vec{r}_u(u_i, v_i), \vec{r}_v(u_i, v_i)\right) \Delta u_i \Delta v_i$$

Summing the elementary fluxes, we obtain

$$\mathcal{F} = \sum_i \Delta \mathcal{F}_i$$

$$\approx \sum_i \left(\vec{F}(r(u_i, v_i)), \vec{r}_u(u_i, v_i), \vec{r}_v(u_i, v_i)\right) \Delta u_i \Delta v_i$$

Hence, we can define

$$\boxed{\mathcal{F} = \iint_D \vec{F}(r(u,v)) \cdot \left(\vec{r}_u(u,v) \times \vec{r}_v(u,v)\right) du\, dv}$$

– flux of $\vec{F}$ across S in the direction of $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$. (unit vector with direction $\vec{r}_u \times \vec{r}_v$)

Remark 11.1 Using the definition of surface integral of a scalar field, we have

$$\mathcal{F} = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \|\vec{r}_u \times \vec{r}_v\| \, du\,dv =$$

$$= \iint_S \vec{F} \cdot \vec{n} \, dS \qquad (11.1)$$

2. Definition of surface integral of a vector field

Let $S = \{ r = r(u,v),\ (u,v) \in \bar{D} \}$ be a smooth (differentiable) surface in $\mathbb{R}^3$.

- S is orientable if the unit normal
$$\frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$$
is continuous in D.

- if n is a fixed continuous unit normal to S on D, then we say that S is oriented by the normal n.

So, let S be a smooth surface oriented by a unit normal $\vec{n}$ and $\vec{F} = (P, Q, R)$ be a vector field defined on S.

Def 11.1 The integral of $\vec{F}$ over S is denoted by and defined as

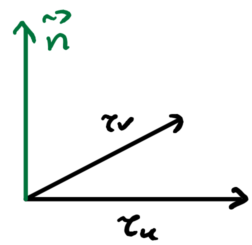
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n})\, dS,$$

where the right hand side is the surface integral of scalar field $\vec{F} \cdot \vec{n}$ over S.

Remark 11.2 If S is oriented by the normal $\vec{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$, then by Def 10.5 (or by (11.1))

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S (\vec{F} \cdot \vec{n})\, dS = \iint_D \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\|r_u \times r_v\|} \|r_u \times r_v\| du\,dv$$

$$= \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\,dv =$$

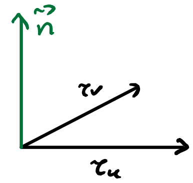


$$= \iint_D \begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ z_v & y_v & z_v \end{vmatrix} du\,dv$$

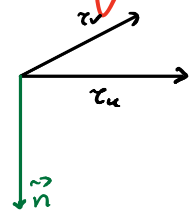
Remark 11.3 The identity

$$\begin{vmatrix} P & Q & R \\ x_u & y_u & z_u \\ x_v & y_v & z_v \end{vmatrix} = P \frac{\partial(y,z)}{\partial(u,v)} + Q \frac{\partial(z,x)}{\partial(u,v)} + R \frac{\partial(x,y)}{\partial(u,v)}$$

motivates the following alternative notation for the integral of F over S when S is oriented by the normal $\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$



$$\iint_S \vec{F} \cdot dS = \iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$



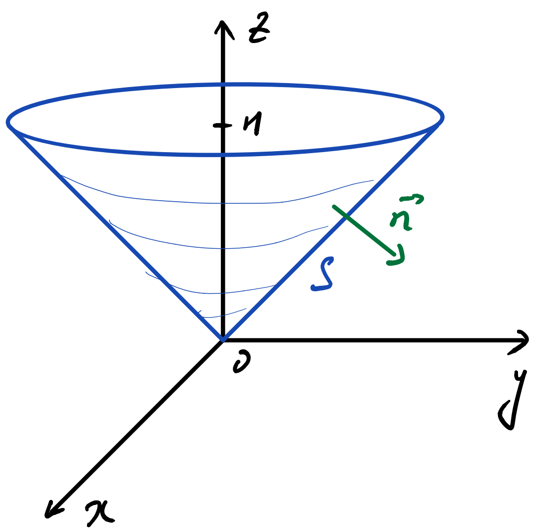
If S is oriented by $\vec{n} = -\frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ then

$$\iint_S \vec{F} \cdot dS = -\iint_S P\,dy\,dz + Q\,dz\,dx + R\,dx\,dy$$

Example 11.1 Compute $I = \iint_S z\,dx\,dy$, where S is the upper part of the lateral surface of the cone

$$z = \sqrt{x^2+y^2}, \quad 0 \le z \le H$$

oriented outwards.



We take the parametrization

$$\begin{cases} x = u \\ y = v \\ z = \sqrt{u^2+v^2} \end{cases} \quad (u,v) \in D,$$

where $D = \{(x,y) : x^2+y^2 \le H^2\}$

We note, that the vectors

$$\vec{r}_u = \left(1, 0, \frac{u}{\sqrt{u^2+v^2}}\right)$$

and

$$\vec{r}_v = \left(0, 1, \frac{v}{\sqrt{u^2+v^2}}\right)$$

points inwards, which is opposite to the orientation of S. So,

$$I = -\iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv =$$

$= \vec{F} = (0, 0, z)$

$$= -\iint_D \begin{vmatrix} 0 & 0 & \sqrt{u^2+v^2} \\ 1 & 0 & \frac{u}{\sqrt{u^2+v^2}} \\ 0 & 1 & \frac{v}{\sqrt{u^2+v^2}} \end{vmatrix} du\, dv =$$

$$= -\iint_D \sqrt{u^2+v^2}\, du\, dv = -\int_0^{2\pi} d\varphi \int_0^H r \cdot r\, dr =$$

$$\left\{ \begin{array}{l} u = r\cos\varphi \\ v = r\sin\varphi \end{array} \right.$$

$0 \le r \le H$

$0 \le \varphi \le 2\pi$

$J = r$

$$= -\frac{2}{3}\pi H^3$$

Example 11.2 Compute $I = \iint_S \frac{dy\,dz}{x} + \frac{dz\,dx}{y} + \frac{dx\,dy}{z}$, where S is part of ellipsoid

$$\begin{cases} x = a\cos u\cos v \\ y = b\sin u\cos v \\ z = c\sin v \end{cases} \qquad D:\ \begin{matrix} \frac{\pi}{4} \le u \le \frac{\pi}{3} \\ \frac{\pi}{6} \le v \le \frac{\pi}{4} \end{matrix}$$

oriented outward.

First, we compute

$$\vec{r}_u = (-a\sin u\cos v,\ b\cos u\cos v,\ 0)$$

$$\vec{r}_v = (-a\cos u\sin v,\ -b\sin u\sin v,\ c\cos v)$$

The vector $\vec{r}_u \times \vec{r}_v$ is also oriented outward

(say the vector $\vec{r}_u(0,0)\times\vec{r}_v(0,0) = (0, b, 0)\times(0,0,c) =$

$$= \begin{vmatrix} i & j & k \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = i\cdot bc + 0j + 0k = (bc, 0, 0)$$

– normal vector to $(a, 0, 0)$)

So,

$$I = \iint_D \begin{vmatrix} \frac{1}{a\cos u\cos v} & \frac{1}{b\sin u\cos v} & \frac{1}{c\sin v} \\ -a\sin u\cos v & b\cos u\cos v & 0 \\ -a\cos u\sin v & -b\sin u\sin v & c\cos v \end{vmatrix} du\,dv =$$

$$= p\iint_D \cos v\,du\,dv = p\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} du \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos v\,dv =$$

$$= p\,\frac{\pi}{12}\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right),\ \text{where } p = \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$