№ 9 Path Independence of line integrals

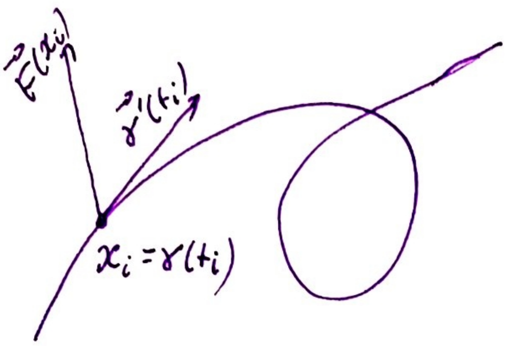
1. Work of a Vector Field.

Let S be a subset of $\mathbb{R}^d$. We say that S is a domain if S is open and connected (every two points of S can be connected by a curve). Let $\vec{F}(x)$ be a continuous force field acting in the domain S.

The displacement of a test particle in the field is accompanied by work. We ask how we can compute the work done by the field in moving a unit test particle along a given trajectory (a differentiable curve γ).

In a constant field $\vec{F}$ the displacement by a vector $\vec{\xi}$ is associated with an amount of work $\langle \vec{F}, \vec{\xi} \rangle = \vec{F} \cdot \vec{\xi}$.

Let $\gamma = \gamma(t)$, $t \in [a, b]$, be a differentiable curve with a natural parametrisation.



we consider a partition $a = t_0 < t_1 < \dots < t_n = b$.
Then the work ΔA_i corresponding to the (time) interval $[t_{i-1}, t_i]$ is

$$\Delta A_i \approx \vec{F}(x_i) \cdot \vec{\gamma}'(t_i) \Delta t_i$$

The total work done by a vector field $\vec{F}$ is given by

$$A = \sum_{i=1}^{n} \Delta A_i \approx \sum_{i=1}^{n} \vec{F}(\underset{\gamma(t_i)}{\underbrace{x_i}}) \cdot \vec{\gamma}'(t_i) \Delta t_i .$$

Passing to the limit, we obtain

$$A = \int_a^b \vec{F}(\gamma(t)) \gamma'(t)\, dt = \int_\gamma \vec{F}(x)\, d\vec{s} .$$

– work of a vector field F along the curve γ.

In the next section we will discuss vector fields where the work depends only on the initial and the end point of γ

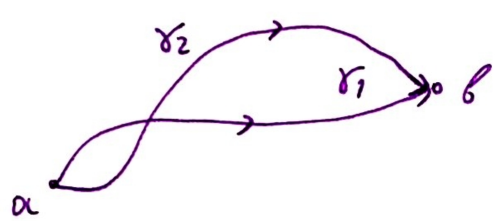
2. Conservative and potential vector fields

We recall from Lecture 8 that

- A vector field $\vec{F}: S \to \mathbb{R}^d$ is called <u>conservative</u> if for any points a, b in S, and any curves γ_1 and γ_2 from a to b

$$\int_{\gamma_1} \vec{F}\cdot ds = \int_{\gamma_2} \vec{F}\cdot ds \left(= \int_a^b \vec{F}\cdot ds \right)$$

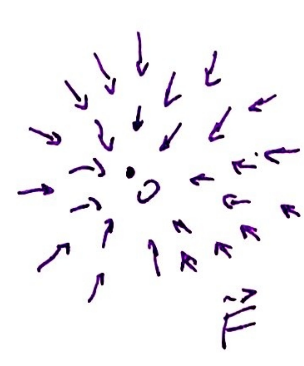
notation



- A vector field $\vec{F}: S \to \mathbb{R}^d$ is called a <u>gradient</u> vector field or a <u>potential</u> field if there exists a continuously differentiable function $\varphi: S \to \mathbb{R}^d$ (called a potential) such that $\vec{F} = \operatorname{grad} \varphi = \nabla \varphi$.

<u>Ex.</u> $\varphi(x) = \dfrac{\alpha}{\|x\|}$, $x \in \mathbb{R}$

$$\vec{F}(x) = \nabla \varphi = -\alpha \frac{x}{\|x\|^3}$$



<u>Proposition 9.1</u> Let $\vec{F}$ be a continuous vector field in a domain $S \subseteq \mathbb{R}^d$. The following statements are equivalent

i) $\vec{F}$ is a potential vector field

ii) for any closed curve γ in S

$$\int_\gamma \vec{F} \cdot ds = 0$$

iii) $\vec{F}$ is conservative in S.

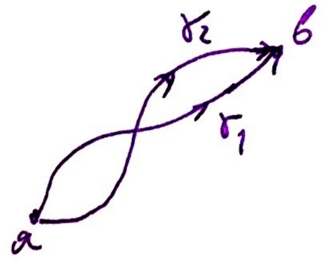
<u>Proof</u> i) $\Rightarrow$ ii) let $\gamma = \gamma(t)$, $t \in [a, b]$ be a closed curve. Then

$$\int_\gamma \vec{F} \cdot ds = \int_a^b \vec{F}(\gamma(t)) \cdot \gamma'(t)\, dt = \int_a^b \nabla\varphi(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_a^b \frac{d}{dt}\left(\varphi(\gamma(t))\right) dt = \varphi(\gamma(b)) - \varphi(\gamma(a)) = 0,$$

since $\gamma(a) = \gamma(b)$.

ii) $\Rightarrow$ iii). We take two curves γ_1, γ_2



Consider $\gamma = \gamma_1 \cup \gamma_2^R$, the concatenation of γ_1 and the reversal of γ_2.

⑤ By ii)

$$0 = \int_{\gamma} \vec{F} \cdot ds = \int_{\gamma_1} \vec{F} \cdot ds + \int_{\gamma_2^R} \vec{F} \cdot ds =$$

$$= \int_{\gamma_1} \vec{F} \cdot ds - \int_{\gamma_2} \vec{F} \cdot ds .$$

iii) => i) Let $a \in S$ be fixed. For any $x \in S$ we define

$$\varphi(x) = \int_{\gamma} \vec{F} \cdot ds = \int_{a}^{x} \vec{F} \cdot ds \qquad (9.1)$$

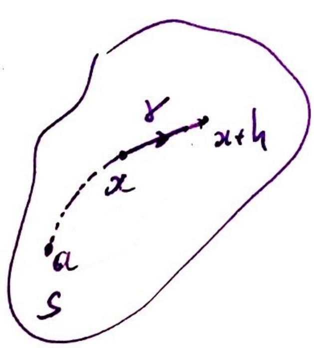
where γ is any curve connecting a and x.

We show that

$$\nabla \varphi(x) = \vec{F}(x).$$

We have to show, that

$$\lim_{h \to 0} \frac{|\varphi(x+h) - \varphi(x) - \vec{F}(x) \cdot h|}{\|h\|} = 0.$$



So, $\varphi(x+h) - \varphi(x) = \int_{x}^{x+h} \vec{F} \cdot ds$

we take $\gamma(t) = x + th$, $t \in [0,1]$.
Then γ is a curve which connects x and $x+h$

So $\varphi(x+h)-\varphi(x)=\int_0^1 \vec{F}(\gamma(t))\cdot\gamma'(t)\,dt=$

$$=\int_0^1 \vec{F}(x+th)\cdot h\,dt=\vec{F}(x+\theta h)\cdot h$$

by the mean value theorem, where $\theta\in[0,1]$.

Hence, $\dfrac{|\varphi(x+h)-\varphi(x)-\vec{F}(x)\cdot h|}{\|h\|}=$

$$=\frac{|\vec{F}(x+\theta h)\cdot h-\vec{F}(x)\cdot h|}{\|h\|}\underset{\text{Cauchy-Schwarz ineq.}}{\leq}$$

$$\leq\frac{\|\vec{F}(x+\theta h)-\vec{F}(x)\|\cdot\|h\|}{\|h\|}=\|\vec{F}(x+\theta h)-\vec{F}(x)\|\xrightarrow[h\to 0]{}0.$$

<u>Remark 9.1</u> Formula (9.1) can be used for the computation of a potential of a potential vector field.

<u>Ex 9.1</u> The field $\vec{F}(x,y)=(y,x)$ is a potential field with

$$\varphi(x,y)=x\cdot y\quad,(x,y)\in\mathbb{R}^2.$$

Hence, by Prop. 9.1. the integral

$$\int_\gamma y\,dx+x\,dy$$

depends only on the initial and the end point of γ.

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How we can check which field is potential or conservative?

3. Curl-free vector fields (case $\mathbb{R}^3$)

Def 9.1 The curl of a vector field $\vec{F} = (P, Q, R)$ in S is the vector field

$$\text{curl } F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Def 9.2 A vector field $\vec{F}$ is called irrotational or curl-free if

$$\text{curl } \vec{F} = 0 \quad \text{in } S$$

Prop 9.2 Let $\vec{F}$ be a continuously differentiable vector field in a domain $S \subseteq \mathbb{R}^3$.

1) If $\vec{F}$ is conservative in S, then $\vec{F}$ is curl-free in S.

2) If $\vec{F}$ is curl-free and S is simply connected (any closed curve in S with start a and end a can be continuously transformed into a), then F is conservative.

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<u>Proof.</u> 1) Since $\vec{F}$ is conservative, then there exists $\varphi : S \to \mathbb{R}$ such that

$$\vec{F} = \nabla \varphi,$$

by Prop. 9.1. Then

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = \frac{\partial}{\partial y}\frac{\partial \varphi}{\partial z} - \frac{\partial}{\partial z}\frac{\partial \varphi}{\partial y} = 0.$$

Similarly $\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$

2) We only prove in the case

$$\vec{F} = (P, Q, 0)$$

and curves $\gamma(t) = (x(t), y(t), 0)$, $t \in [a,b]$.

Since $\operatorname{curl} \vec{F} = 0$, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ in S.

By Green's formula, for any closed γ in S surrounding the set $D \subset \mathbb{R}^2$,

$$\int_\gamma \vec{F} \cdot ds = \int_\gamma P\,dx + Q\,dy =$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0.$$

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