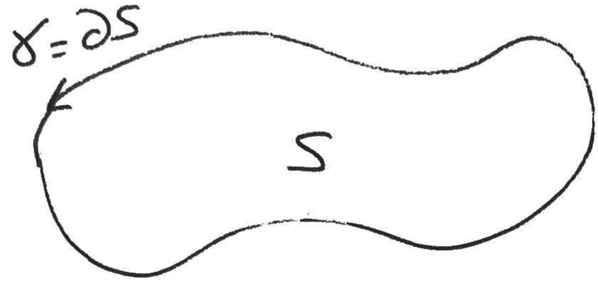
8| Green's formula

$\boxed{\mathbb{R}^2}$



γ positively oriented if the set S stays on the left when traveling along γ.

Theorem (Green's formula)

> Let $\vec{F} = (P, Q)$ where P and Q are continuously differentiable functions on $\overline{S}$.
>
> Then $$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_\gamma \vec{F} \cdot ds .$$

1. Examples and proof of Green's formula

Example:

S = unit ball of $\mathbb{R}^2$

$= \{(x,y) : x^2 + y^2 \leq 1\}$

γ = unit circle of $\mathbb{R}^2$ oriented counterclockwise $\longrightarrow$ correct orientation!

Compute $I = \int_\gamma \left(x^2 y\, dx - x y^2\, dy\right)$.

$\hookrightarrow$ we apply Green's formula to $P(x,y) = x^2 y$ and $Q(x,y) = -xy^2$

$$I = \int_\gamma \vec{F} \cdot ds = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

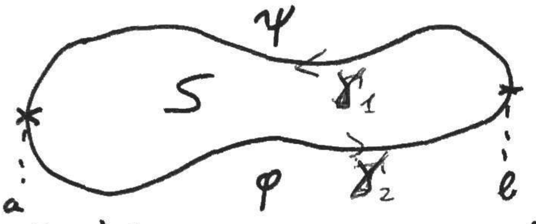
$$= -\iint_S (y^2 + x^2)\, dx\,dy \underset{\text{polar coordinates}}{=} -\int_0^1 \int_0^{2\pi} r^2\, dr\, r\, d\theta$$

$$= -2\pi \times \frac{1}{4} = -\frac{\pi}{2} .$$

Proof of Green's formula:

Let us prove the formula in the case where

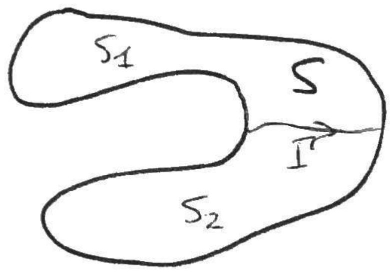
* $Q \equiv 0$: $\vec{F} = (P, 0)$
* S is of the form



$$S = \{(x,y) : a \le x \le b,\ \varphi(x) \le y \le \psi(x)\}$$

⇝ for general Q: same proof with $\vec{F} = (0, Q)$

⇝ for general S:



$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_{S_1} (-)\,dx\,dy + \iint_{S_2} (-)\,dx\,dy$$

and $\int_\gamma \vec{F}\cdot ds = \int_{\gamma_1} \vec{F}\cdot ds + \int_{\gamma_2} \vec{F}\cdot ds$

because $\int_\Gamma \vec{F}\cdot ds = -\int_{\Gamma^R} \vec{F}\cdot ds$.

Let us compute

$$\iint_S \left(\frac{-\partial P}{\partial y}\right) dx\,dy = -\int_a^b \int_{\varphi(x)}^{\psi(x)} \frac{\partial P}{\partial y}(x,y)\,dy\,dx$$

$$= -\int_a^b \big(P(x,\psi(x)) - P(x,\varphi(x))\big)\,dx$$

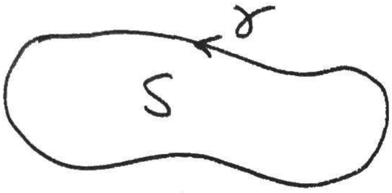
$$\int_\gamma \vec{F}\cdot ds = \int_{\gamma_1} \vec{F}\cdot ds + \int_{\gamma_2} \vec{F}\cdot ds = -\int_{\gamma_1^R} \vec{F}\cdot ds + \int_{\gamma_2} \vec{F}\cdot ds$$

$$= -\int_a^b P(t,\psi(t))\cdot 1\,dt + \int_a^b P(t,\varphi(t))\cdot 1\,dt$$

Thus $\displaystyle\iint_S \left(-\frac{\partial P}{\partial y}\right) dx\,dy = \int_\gamma \vec{F}\cdot ds.$

$\square$

Example: area of a set bounded by a curve



area of S : $\mu(S) = \displaystyle\iint_S dx\,dy.$

For every $\vec{F} = (P, Q)$ <u>such that</u> $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$ everywhere in S

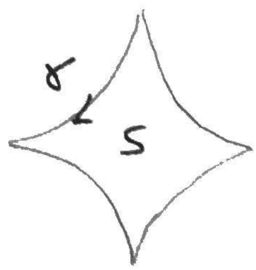
we have $\mu(S) = \displaystyle\int \vec{F}\cdot ds.$

- $P(x,y) = 0$ and $Q(x,y) = x$
- $P(x,y) = -y$ and $Q(x,y) = 0$
- $P(x,y) = -\dfrac{y}{2}$ and $Q(x,y) = \dfrac{x}{2}$

$$\boxed{\mu(S) = \frac{1}{2}\int_\gamma x\,dy - y\,dx}.$$

→ Astroid in $\mathbb{R}^2$

$$\gamma(t) = \begin{cases} x(t) = a\cos^3 t \\ y(t) = a \sin^3 t \end{cases} \qquad t \in [0, 2\pi)$$



$$\mu(S) = \frac{1}{2}\int_\gamma x\,dy - y\,dx = \frac{1}{2}\int_0^{2\pi} \big(x(t)\,y'(t) - y(t)\,x'(t)\big)dt$$

$$= \frac{1}{2}\int_0^{2\pi} 3a^2 \left(\cos^3 t \cos t \sin^2 t - \cos^2 t(-\sin t)\sin^3 t\right) dt$$

$$= \frac{3a^2}{2}\int_0^{2\pi} \cos^2 t \sin^2 t \,\underbrace{\left(\cos^2 t + \sin^2 t\right)}_{=1} dt$$

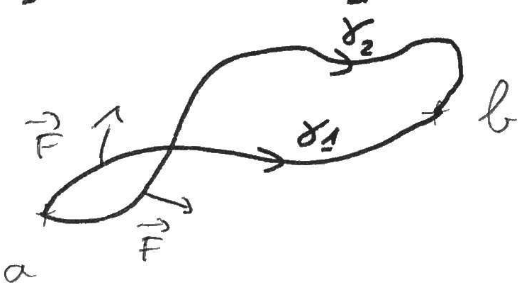
$$= \frac{3a^2}{2} \cdot \frac{\pi}{4} = \frac{3a^2\pi}{8}$$

Exercice: $\int_0^{2\pi} \cos^2 t \sin^2 t \, dt = \frac{\pi}{4}$.

2. Conservative vector fields

Def: A vector field $\vec{F} : S \subset \mathbb{R}^n \to \mathbb{R}^n$ is called conservative if for any points a, b in S, and any curves γ_1 and γ_2 from a to b,

$$\int_{\gamma_1} \vec{F} \cdot ds = \int_{\gamma_2} \vec{F} \cdot ds$$



Def: A vector field $\vec{F} : S \subset \mathbb{R}^n \to \mathbb{R}^n$ is called gradient vector field or potential field if there exists a continuously differentiable function $\varphi : S \to \mathbb{R}^n$ such that $\vec{F} = \operatorname{grad} \varphi = \vec{\nabla} \varphi$.

Ex: The gravitational force

$$\vec{F}(x) = -\alpha \frac{x}{\|x\|^3} \quad \text{with} \quad \alpha = \gamma m M$$

is a gradient vector field with potential

$$\varphi(x) = \frac{\alpha}{\|x\|}.$$

since $\operatorname{grad}\left(f(\|x\|)\right) = f'(\|x\|) \frac{x}{\|x\|}$.