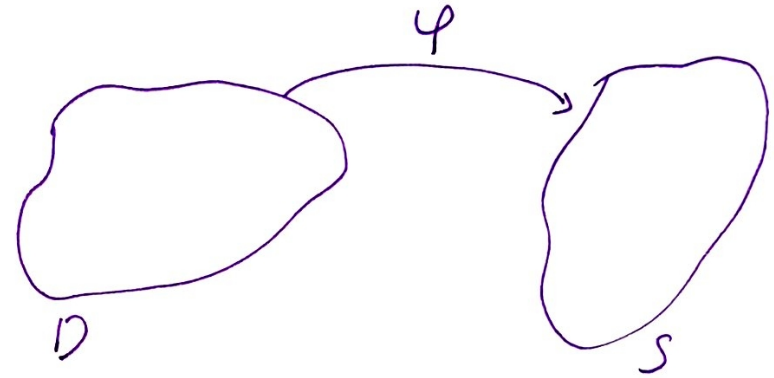
Lecture №4 Change of Variables

1. Heuristic derivation (2d case)

We consider $S \subseteq \mathbb{R}^d$ and $D \subseteq \mathbb{R}^d$ and let $\varphi: D \to S$ be an bijective map



We are interesting if $\int_S f \, dx$ can be rewritten as $\int_D g \, dx$, where g is some function.

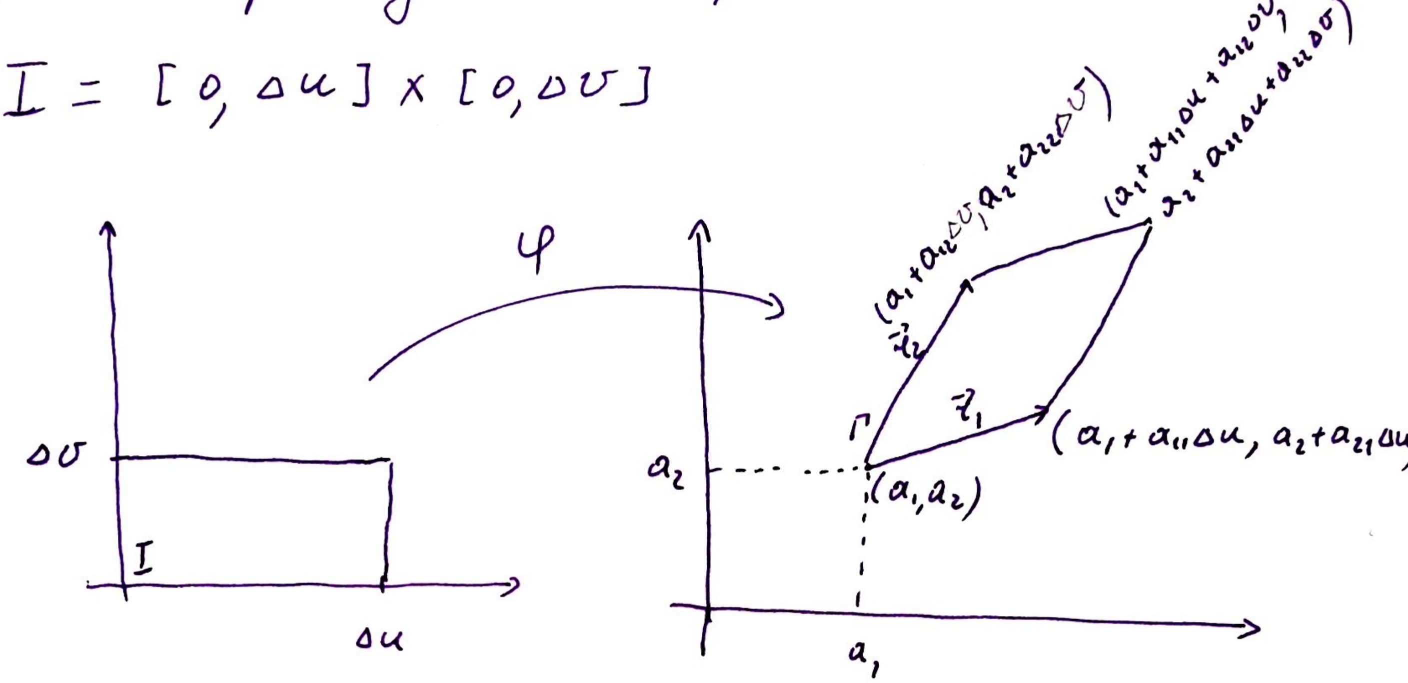
We start from the following problem: Let φ be an affine map:

$$\begin{cases} x(u,v) = a_1 + a_{11} u + a_{12} v \\ y(u,v) = a_2 + a_{21} u + a_{22} v, \quad (u,v) \in \mathbb{R}^2 \end{cases}$$

We are interesting how volume of an interval $I \subseteq D$ is changed under the map φ, i.e. what the volume of $\varphi(I)$ is.

For simplisity of computations we take

$I = [0, \Delta u] \times [0, \Delta v]$



So, the interval I is mapped by φ to the parallelogramm Γ spanned by vectors $\vec{r}_1 = (a_{11}\Delta u, a_{21}\Delta u)$, $\vec{r}_2 = (a_{12}\Delta v, a_{22}\Delta v)$ applied at (a_1, a_2). Hence,

$$\mu(\Gamma) = \| \vec{r}_1 \times r_2 \| = \left| \begin{matrix} a_{11}\Delta u & a_{21}\Delta u \\ a_{12}\Delta v & a_{22}\Delta v \end{matrix} \right| =$$

$$= \left| \begin{matrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{matrix} \right| \Delta u \Delta v = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \cdot |I|$$

Similarly, for any interval I

$$\mu(\Gamma) = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| |I|,$$

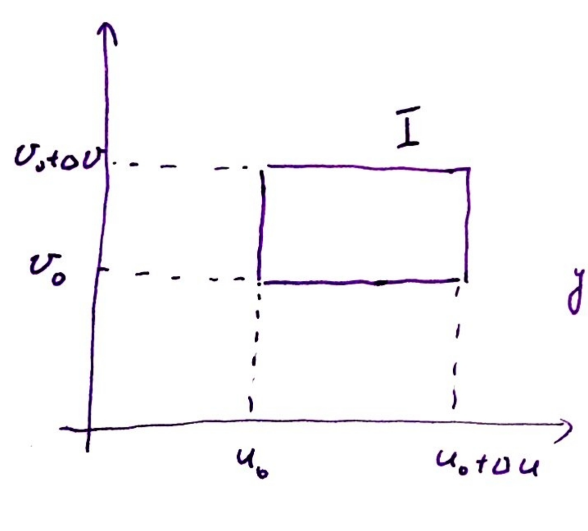
where $\Gamma = \varphi(I)$,

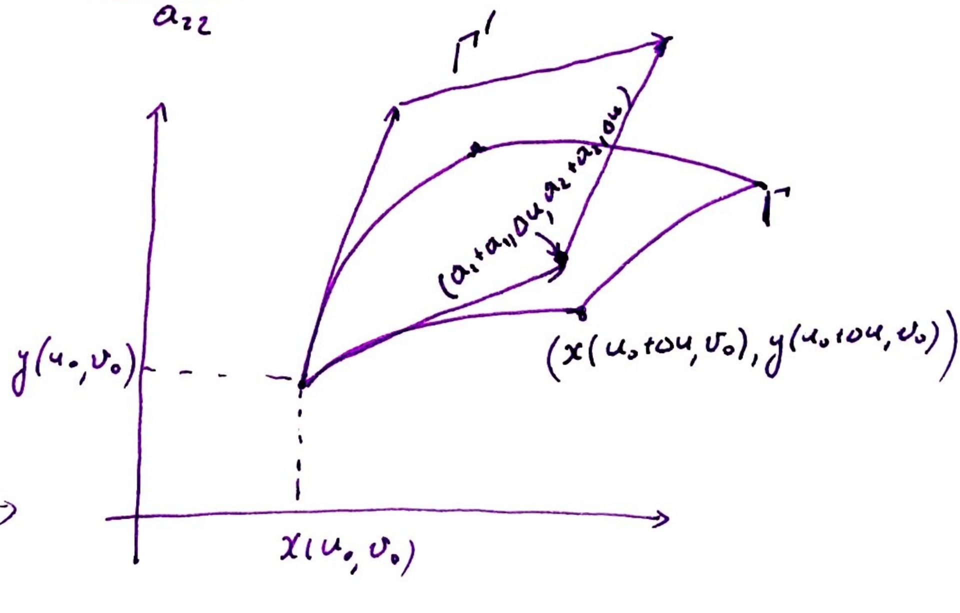
Now, we consider more general case of transformation φ. Let

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

By the Tailor formula

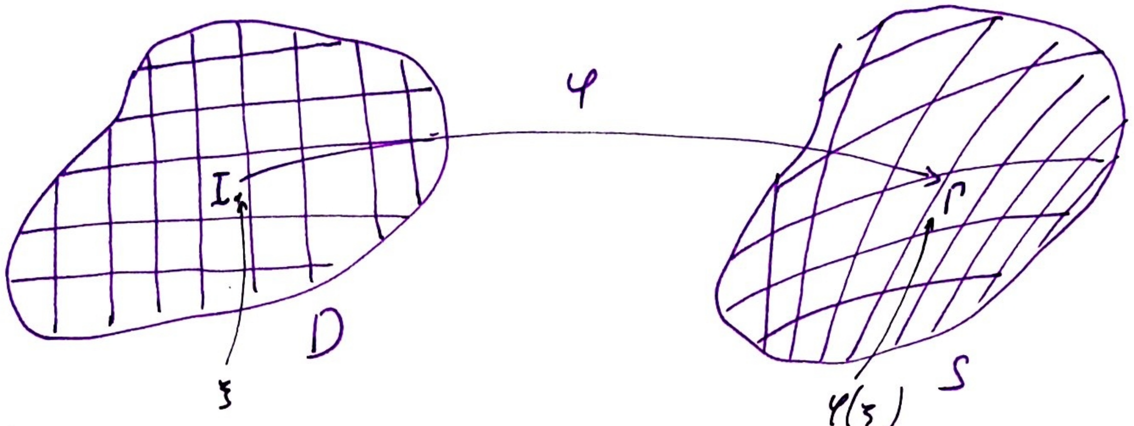
$$\begin{cases} x(u,v) = \overbrace{x(u_0,v_0)}^{a_1} + \overbrace{\frac{\partial x}{\partial u}(u_0,v_0)}^{a_{11}}(u-u_0) + \\ \quad + \overbrace{\frac{\partial x}{\partial v}(u_0,v_0)}^{a_{12}}(v-v_0) + o\left(\sqrt{(\Delta u)^2+(\Delta v)^2}\right) \\ y(u,v) = \overbrace{y(u_0,v_0)}^{a_2} + \overbrace{\frac{\partial y}{\partial u}(u_0,v_0)}^{a_{21}}(u-u_0) + \\ \quad + \underbrace{\frac{\partial y}{\partial v}(u_0,v_0)}_{a_{22}}(v-v_0) + o\left(\sqrt{(\Delta u)^2+(\Delta v)^2}\right) \end{cases}$$





Hence, $\mu(\Gamma) \approx \mu(\Gamma') = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| |I|$.

Next, let us formally obtain the change of variables in $\iint_S f(x,y)\,dx\,dy$.



$$\iint_S f(x,y)\,dx\,dy \approx \sum_i f(\varphi(\xi_i))\,\mu(\Gamma_i) \approx$$

$$\approx \sum_i f(\varphi(\xi_i)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| |I_i| \approx$$

$$\approx \iint_D f(\varphi(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv.$$

2. Change of variables

Th. 4.1 (Change of variables). Let S, D be admissible sets, $\varphi: D \to S$ be a continuously differentiable bijection such that its Jacobian is non-zero in D. Then

$$\int_S f(x)\,dx = \int_D f(\varphi(u)) \left| \frac{\partial(x_1,\dots,x_d)}{\partial(u_1,\dots,u_d)} \right| du.$$

Ex 4.1 Let $S = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2+y^2 \leq 1\}$

We compute

$$\iint_S x \, dx\, dy = \iint_D r\cos\varphi \; r \; dr\, d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} d\varphi \int_0^1 r^2 \cos\varphi \, dr =$$

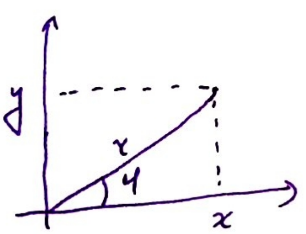
$$= \int_0^{\frac{\pi}{2}} \left(\cos\varphi \; \frac{r^3}{3} \Big|_{r=0}^{r=1} \right) d\varphi =$$

$$= \frac{1}{3} \sin\varphi \Big|_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{3}.$$

$x = r\cos\varphi$

$y = r\sin\varphi$



$D = \{(r,\varphi) : 0 \leq r \leq 1, \; 0 \leq \varphi \leq \frac{\pi}{2}\}$

$$\frac{\partial(x,y)}{\partial(r,\varphi)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} =$$

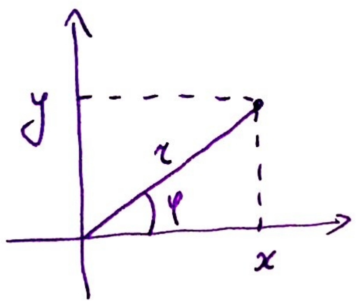
$$= r(\cos^2\varphi + \sin^2\varphi) = r$$

Examples of transformations:

a) Polar coordinates

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}, \quad r > 0, \quad \varphi \in [0, 2\pi)$$

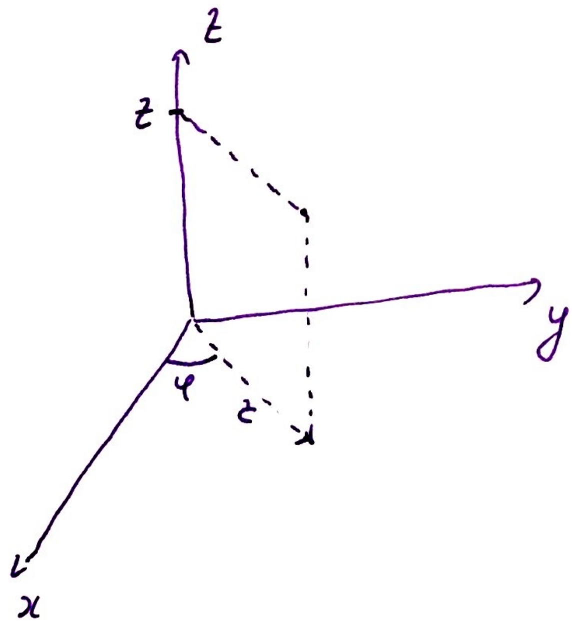
$$\frac{\partial(x,y)}{\partial(r,\varphi)} = r$$



b) Cylindrical coordinates

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \quad r > 0, \quad \varphi \in [0, 2\pi), \quad z \in \mathbb{R}$$

$$\frac{\partial(x,y,z)}{\partial(r,\varphi,z)} = r$$



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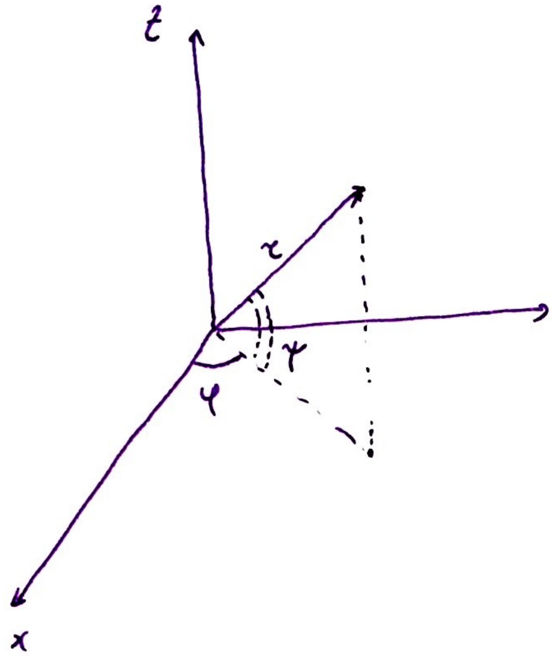
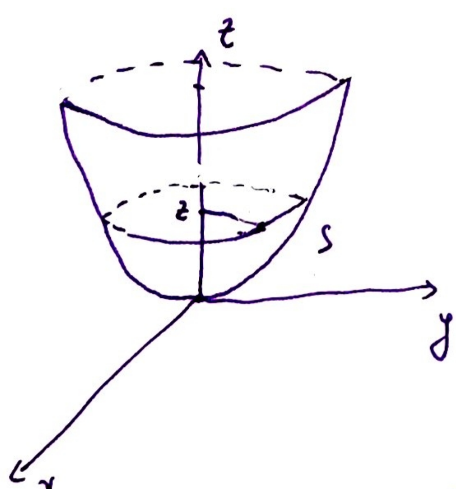
c) Spherical coordinates

$$x = r\cos\varphi\cos\psi \qquad r>0,\ \varphi\in[0,2\pi),\ \psi\in\left[-\tfrac{\pi}{2},\tfrac{\pi}{2}\right]$$
$$y = r\sin\varphi\cos\psi$$
$$z = r\sin\psi \qquad \frac{\partial(x,y,z)}{\partial(r,\varphi,\psi)} = r^2\cos\psi.$$



Ex 4.2 Let S be bounded by graphs of

$z = x^2+y^2$, $z=1$, and $f(x,y,z)=x^2+y^2$.



We use cylindrical coordinates

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \qquad \frac{\partial(x,y,z)}{\partial(r,\varphi,z)} = r$$

Hence, D: $0 \le \varphi < 2\pi$
$0 \le z \le 1$
$0 \le r \le \sqrt{z}$

(8)

Consequently,

$$\iiint_S (x^2+y^2)\,dx\,dy\,dz = \int_0^{2\pi} d\varphi \int_0^1 dz \int_0^{\sqrt{z}} r^2\cdot r\,dr =$$

$$= \frac{1}{4}\,2\pi \int_0^1 r^4\Big|_{r=0}^{r=\sqrt{z}}\,dz = \frac{\pi}{2}\int_0^1 z^2\,dz =$$

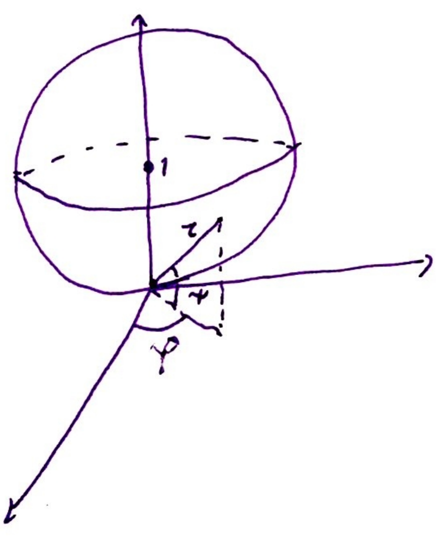
$$= \frac{\pi}{2}\,\frac{z^3}{3}\Big|_0^1 = \frac{\pi}{6}.$$

Ex 4.3 Let S be bounded by $x^2+y^2+z^2=2z$

$$\Downarrow$$

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$

$$x^2+y^2+(z-1)^2=1$$



We use the spherical coordinates:

$$\begin{cases} x = r\cos\varphi\cos\psi \\ y = r\sin\varphi\cos\psi \\ z = r\sin\psi \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(r,\varphi,\psi)} = r^2\cos\psi.$$

Then D: $0 \le \varphi \le 2\pi$

$0 \le \psi \le \frac{\pi}{2}$

$\underbrace{0 \le r \le 2\sin\psi}$ we found from: $x^2+y^2+z^2=2z$

$r^2 = 2r\sin\psi$

$r = 2\sin\psi.$

So, $\iiint_S \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz =$

$$= \iiint_D r\cdot|r^2\cos\psi|\,d\varphi\,d\psi\,dr = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\psi \int_0^{2\sin\psi} r^3\cos\psi\,dr =$$

$$= \frac{2\pi}{4}\int_0^{\frac{\pi}{2}} 16\sin^4\psi\cos\psi\,d\psi = 8\pi\,\frac{\sin^5\psi}{5}\Big|_0^{\frac{\pi}{2}} = \frac{8\pi}{5}.$$