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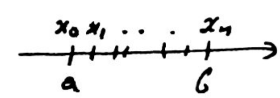
Lecture 1 Riemann integral over n-Dimensional rectangle

1. 1-dim. case (Math 1 Lecture 16)

Let $[a,b] = \{x \in \mathbb{R} : a \le x \le b\}$ be an interval

- A set of points $\{x_0, x_1, \dots, x_n\} =: P$ such that

$$a = x_0 < x_1 < \dots < x_n = b$$



is called a partition of $[a,b]$.

- $\lambda(P) = \max\{\Delta x_k : 1 \le k \le n\}$, $\Delta x_k = x_k - x_{k-1}$

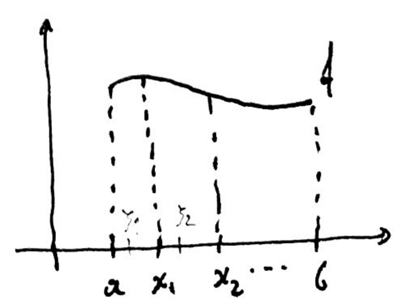
is called a mesh of a partition P

- Let $f : [a,b] \to \mathbb{R}$, $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$, and

$$\xi_k \in [x_{k-1}, x_k]$$

The sum

$$\sigma(f, P, \xi) = \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k$$



is called the Riemann sum.

- A function $f : [a,b] \to \mathbb{R}$ is called

<u>integrable on $[a,b]$</u> if there exists a limit

$$J = \int_a^b f(x)\,dx := \lim_{\lambda(P)\to 0} \sigma(f, P, \xi) \qquad (1)$$

The limit $J = \int_a^b f(x)\,dx$ is called the <u>Riemann integral of f over $[a,b]$</u>.

Limit (1) means:

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall P = \{x_0, \dots, x_n\}$ - partition of $[a,b]$, $\lambda(P) < \delta$

$\forall \xi_k \in [x_{k-1}, x_k]$, $k = 1, \dots, n$

$$|J - \sigma(f, P, \xi)| < \varepsilon.$$

Now we are going to repeat the same for n-dim case.

2. Definition of the integral

First we introduce some definitions.

- The set $I = I_{a,b} = \{x \in \mathbb{R}^d : a_i \le x_i \le b_i,\ i = 1, \dots, d\}$ is called a <u>rectangle</u> or an <u>interval</u> in $\mathbb{R}^d$.
- $|I_{a,b}| = \prod_{i=1}^{n} (b_i - a_i)$ is called <u>volume</u> or <u>measure</u> of the interval $I_{a,b}$.

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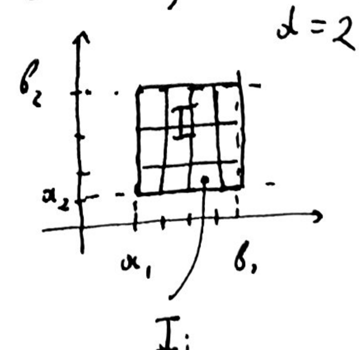
Lemma 1.1 The measure of an interval in $\mathbb{R}^d$ has the following properties.

a) it is homogeneous:
$$|\lambda I_{a,b}| = \lambda^d |I_{a,b}|,$$
where $\lambda > 0$, $\lambda I_{a,b} := I_{\lambda a, \lambda b}$

b) it is additive, i.e. if $I, I_1, \dots, I_n$ are intervals in $\mathbb{R}^d$ such that
$$I = \bigcup_{i=1}^{n} I_i$$
and no two of $I_1, \dots, I_n$ have common interior points, then
$$|I| = \sum_{i=1}^{n} |I_i|.$$

c) if $\quad I \subseteq \bigcup_{i=1}^{n} I_i, \quad I, I_1, \dots, I_n$ - intervals, then
$$|I| \leq \sum_{i=1}^{n} |I_i|$$

Now we introduce partitions of an interval. Let
$$I = \{x \in \mathbb{R}^d : a_i \leq x_i \leq b_i, \ i = 1, \dots, d\}$$



Partitions of the coordinate intervals $[a_i, b_i]$, $i = 1, \dots, d$, induce a partition of the interval I.
$$I = \bigcup_{j=1}^{n} I_j$$
write $\quad P = \{I_1, \dots, I_n\}$

- The quantity $\lambda(P) = \max_{j=1,\dots,n} d(I_j)$,
where $d(I_j) = \max_{x,y \in I_j} \|x-y\|$, $\|x\| = \sqrt{x_1^2 + \dots + x_d^2}$,
is called the mesh of the partition P.

<u>Def. 1.1</u> Let $P = \{I_1, \dots, I_n\}$ be a partition of the interval I, $\xi_i \in I_i$, $i = 1, \dots, n$ and $f : I \to \mathbb{R}$ be a function.
The sum

$$\sigma(f, P, \xi) := \sum_{i=1}^{n} f(\xi_i) \, |I_i|$$

is called the <u>Riemann sum</u> of f.

<u>Def 1.2</u> A function $f : I \to \mathbb{R}$ is called <u>Riemann integrable on I</u> if there exists a limit

$$J = \int_I f(x)\,dx = \int_{a_1}^{b_1} \dots \int_{a_d}^{b_d} f(x_1, \dots, x_d)\, dx_1 \dots dx_d :=$$

$$= \lim_{\lambda(P) \to 0} \sigma(f, P, \xi),$$

i.e. $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall P = \{I_1, \dots, I_n\}$ - part. of I, $\lambda(P) < \delta$

$\forall \xi_i \in I_i, \ i = 1, \dots, d$

$$|J - \sigma(f, P, \xi)| < \varepsilon.$$

We will write shortly $f \in \mathcal{R}(I)$ if f is integrable.

Prop 1.1 (Necessary condition of integrability)
if $f \in \mathcal{R}(I)$, then f is bounded.

Proof idee: if f is unbounded on I, then it is unbounded on some interval I_{i_0} from the partition P.
Let ξ', ξ'' differs only in the choice of the points ξ'_{i_0} and ξ''_{i_0}, then

$$|\sigma(f, P, \xi') - \sigma(f, P, \xi'')| = |f(\xi'_{i_0}) - f(\xi''_{i_0})| \, |I_{i_0}|$$

By changing one of the points ξ'_{i_0} or ξ''_{i_0}, as a result of the unboundedness of f on I_{i_0}, we could make the right hand side arbitrary large. ▪

3. Darboux criterion of Integrability

Let $f: I \to \mathbb{R}$ and $P = \{I_1, \dots, I_n\}$ be a partition of I. We set

$$m_i = \inf_{x \in I_i} f(x), \quad M_i = \sup_{x \in I_i} f(x)$$

Def 1.3 The quantities

$$L(f, P) := \sum_{i=1}^{n} m_i |I_i|, \quad U(f, P) := \sum_{i=1}^{n} M_i |I_i|$$

are called lower and upper Darboux sums of f

Remark 1.1 $L(f,P) \le \sigma(f,P,\xi) \le U(f,P)$.

Def 1.4 $\underline{J} = \sup\limits_P L(f,P)$, $\overline{J} = \inf\limits_P U(f,P)$

are called lower and upper Darboux integrals of f over the interval I.

Remark 1.2 $L(f,P) \le \underline{J} \le \overline{J} \le U(f,P)$.

Th 1.1 (Darboux criterion).

$f \in \mathcal{R}(I)$ iff $\underline{J} = \overline{J}$ and f is bounded on I.

For the proof see Th 3 p.116 [Zorich MA II]

Prop. 1.2 A function $f: I \to \mathbb{R}$ is integrable on I ($f \in \mathcal{R}(I)$) iff $\forall \varepsilon > 0$ $\exists$ P-partition of I such that

$$U(f,P) - L(f,P) < \varepsilon.$$

4. Lebesgue Criterion of Integrability

Def 1.5 A set $E \subseteq \mathbb{R}^d$ has measure zero (in the Lebesgue sense) if for every $\varepsilon > 0$ there exists at most countable system $\{I_i\}$ of d-dim intervals such that

$$E \subseteq \bigcup_i I_i \quad \text{and} \quad \sum_i |I_i| \le \varepsilon.$$

Example 1.1 A set $E = \{a^1, \dots, a^n\}$, $a^i \in \mathbb{R}^d$, has measure zero.

The intervals can be taken as

$$I_i = \left\{ x : a^i_j - \frac{\sqrt[d]{\varepsilon}}{2\sqrt[d]{n}} \le x_j \le a^i_j + \frac{\sqrt[d]{\varepsilon}}{2\sqrt[d]{n}} \right\}.$$

Then $E \subseteq \bigcup_{i=1}^{n} I_i$ and $\sum_{i=1}^{n} |I_i| = \sum_{i=1}^{n} \frac{\varepsilon}{n} = \varepsilon$.

Example 1.2 A set $E = \mathbb{Q}^n$ of rational numbers (coordinates are rational) has measure zero $\boxed{\text{(HW)}}$

Example 1.3 $f : I \to \mathbb{R}$ – continuous function

$$E = \mathrm{Graph}(f) = \{(x, f(x)) \in \mathbb{R}^{d+1} : x \in I\}$$

has measure zero

Lemma 1.2 a) A union of a finite or countable number of sets of measure zero is a set of measure zero $\boxed{\text{(HW)}}$

b) A subset of measure zero is itself of measure zero.

We next formulate a criterion of integrability of $f : I \to \mathbb{R}$.

We say that f is <u>continuous almost everywhere</u> if $D_f = \{x \in I : f \text{ is discontinuous at } x\}$ has measure zero.

(8) **Th 1.2** (Lebesgue's criterion)
$f \in \mathcal{R}(I)$ iff f is bounded and continuous almost everywhere.

Web-page: math.uni-leipzig.de/~konarovskyi
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Literature: V. Zorich "Mathematical Analysis II"

B.V. Shabat "Introduction to Complex Analysis"

A.N. Tikhonov, A.A. Samarskii "Equations of Mathematical Physics".

A. Sapozhnikov "Lecture notes. Part I"