Problem sheet 6

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Solutions will be collected during the lecture on Monday May 27.

1. **[2+3 points]** Find the eigenvalues and the eigenvectors of linear maps specified in some basis on a real vector space by the following matrices:

$$a)\ \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}; \quad b)\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. **[2+4+2 points]** Determine which of the following matrices of linear maps on a real vector space can be reduced to diagonal form by going over to a new basis. Find that basis and the corresponding matrix:

$$a)\ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}; \quad b)\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

c) Find a matrix Q such that the matrix $Q^{-1}AQ$ has a diagonal form, where A is the matrix from b).

3. **[2 points]** Let i, j, k be an orthonormal basis with right-hand orientation. Let u and v are vectors with coordinates $(1, 2, 1)$ and $(1, 0, 1)$ in this basis, respectively. Compute $u \cdot v$ and $u \times v$.

4. **[2 points]** Show that the function $\langle \cdot, \cdot \rangle : \mathrm{C}([0,1]) \times \mathrm{C}([0,1]) \to \mathrm{C}([0,1])$ defined as

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx, \quad f, g \in \mathrm{C}([0,1]),$$

is an inner product on $\mathrm{C}([0,1])$.

5. **[3 points]** Let the functions $\|\cdot\|_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\|\cdot\|_\infty : \mathbb{R}^2 \to \mathbb{R}$ be defined for any vector $(x, y) \in \mathbb{R}^2$ as

$$\|(x, y)\|_1 = |x| + |y|,$$
$$\|(x, y)\|_\infty = \max\{|x|, |y|\}.$$

Prove that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms.

6. **[2 points]** Prove that for any real numbers $x_1, x_2, \dots, x_n$,

$$(x_1 + \cdots + x_n)^2 \leq n\left(x_1^2 + \cdots + x_n^2\right).$$

(*Hint:* Use the Cauchy-Schwarz Inequality)

7. **[2+3 points]** a) Let $u, v \in V$ be orthogonal, i.e. $\langle u, v \rangle = 0$, and let $\|u\| = \sqrt{\langle u, u \rangle}$ as usual. Prove that

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2. \quad \textbf{Pythagoras Theorem}$$

b) Prove that for any $u, v \in V$

$$\|u + v\|^2 + \|u - v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right). \quad \textbf{Parallelogram law}$$