A particle model for Wasserstein type diffusion

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Cambridge – 2018

Joint work with

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Alexander von Humboldt
Stiftung/Foundation

UNIVERSITÄT LEIPZIG

Dean-Kawasaki equation and Wasserstein diffusion

In some space of probability measures, we consider the equation

$$d\mu_t = \Gamma(\mu_t)dt + \mathrm{div}(\sqrt{\mu_t}dW_t).$$

We mean that here that

for each test function φ $\langle \varphi, \mu_t \rangle - \int_0^t \langle \varphi, \Gamma(\mu_s) \rangle ds$ is a martingale with q.v.

$$\int_0^t \langle |\nabla \varphi|^2, \mu_s \rangle ds$$

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Some partial examples

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in the space of probability measures $\mathcal{P}_2(S)$ with the finite second moment.

1. **Wasserstein Diffusion** (von Renesse, Sturm '09): $S = [0,1]$ or is a circle

$$\langle \varphi, \Gamma(\mu_t)\rangle = \beta\langle \Delta\varphi, \mu_t\rangle + \sum_{I \in \mathrm{gaps}\,(\mu_t)} \left[\frac{\varphi''(I_+) + \varphi''(I_-)}{2} - \frac{\varphi'(I_+) - \varphi'(I_-)}{|I|}\right].$$

2. **Modified Arratia flow** (K., von Renesse '15),
Coalescing-Fragmentating Wasserstein Dynamics (K., von Renesse '17): $S = \mathbb{R}$

$$\langle \varphi, \Gamma(\mu_t)\rangle = \sum_{x \in \mathrm{supp}\,\mu_t} \varphi''(x),$$

or shortly

$$\Gamma(\mu_t) = \frac{1}{2}\Delta\mu_t^*$$

Note: We have non-uniqueness in this case

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Connection with the Geometry of Wasserstein space

Wasserstein metric on $\mathcal{P}_2(S)$:

$$d_{\mathcal{W}}^2(\nu_1, \nu_2) = \inf \left\{ \mathbb{E}\|\eta_1 - \eta_2\|_S^2 : \ \eta_i \sim \nu_i, \ i = 1, 2 \right\}$$

For the mentioned processes the **Varadhan formula** holds:

$$\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d_{\mathcal{W}}^2(\mu_0, \nu)}{2t}}, \quad t \ll 1$$

Today: We will discus a construction of a solution to the equation

$$d\mu_t = \Gamma(\mu_t)dt + \mathrm{div}(\sqrt{\mu_t} dW_t)$$

for some Γ, using a particle approach.

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Lift of n-particle motion

$w_1, \ w_2$ – independent Brownian particles on $\mathbb{R}$ with diffusion rates a_1 and a_2

$$\mu_t := m_1\delta_{w_1(t)} + m_2\delta_{w_2(t)}$$

The Ito formula gives that $\langle \varphi, \mu_t\rangle = m_1\varphi(w_1(t)) + m_2\varphi(w_2(t))$ is a semimartingale with q.v.

$$\int_0^t \left(m_1^2\dot\varphi(w_1(s))^2 a_1 + m_2^2\dot\varphi(w_2(s))^2 a_2\right) ds = \int_0^t \langle \dot\varphi^2, \mu_s\rangle ds,$$

if $a_1 = \frac{1}{m_1}$ and $a_2 = \frac{1}{m_2}$.

The diffusion rate of each particle has to be inversely proportional to its mass!

$$\mu_t^n := \sum_{k=1}^n m_k\delta_{w_k(t)}$$

diverges, if $\mu_0^n \to \mathrm{Leb}|_{[0,1]}$, since the diffusion rate of each particle tends to infinity.

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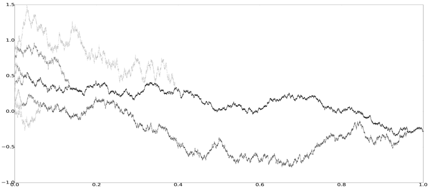
Modified Arratia flow on the real line (K., von Renesse '15)

To overcome the problem with the increasing of the diffusion rates of particles, we can coalesce particles after their meeting.



Grayscale colour coding is for atom mass.

Physical interpretation:

1. each particle has a mass that obeys the conservation law;
2. diffusion rate of each particle inversely depends on its mass;
3. particles move independently and coalesce after meeting.

In this case, $\mu^n_\cdot \to \mu_\cdot$ and

$$d\mu_t = \frac{1}{2}\Delta\mu_t^* dt + \mathrm{div}(\sqrt{\mu_t} dW_t) \quad \text{in} \quad \mathcal{P}_2(\mathbb{R})$$

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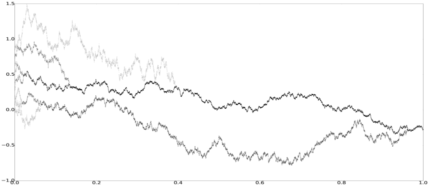

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Can we replace the coalescing of particles by something else?

So, we assume that:

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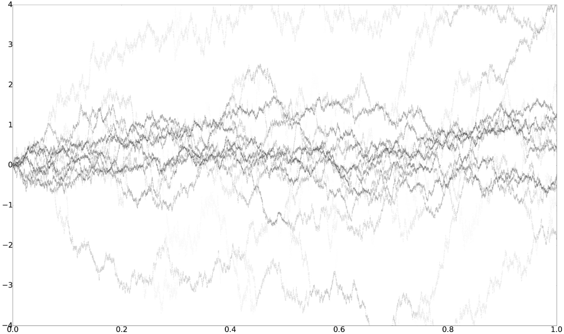
3. particles move independently and can reflect from each other.



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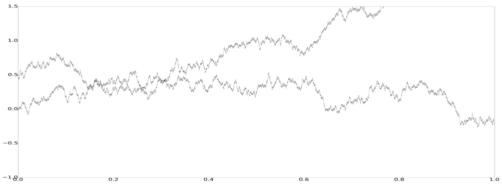
Two particle model

$x_1(t) \le x_2(t)$ denote the positions of particles at time $t \ge 0$
$m_1 = m_2 = \frac{1}{2}$ particle mass at start (the total mass is always 1)



Let w_1, w_2 be two indep. Brownian motions with $\mathrm{Var}\,(w_i(t)) = \frac{1}{m_i} t = 2t$

$$dx_i(t) = \mathbb{I}_{\{x_1(t)<x_2(t)\}} dw_i(t) + \mathbb{I}_{\{x_1(t)=x_2(t)\}} d\frac{w_1(t)+w_2(t)}{2}$$

$$+\lambda_i \mathbb{I}_{\{x_1(t)=x_2(t)\}} dt, \quad \lambda_1 \le \lambda_2$$

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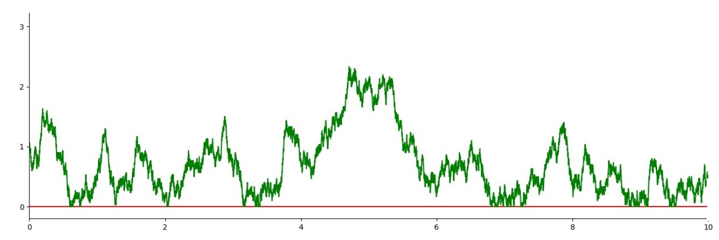
Sticky-reflected Brownian motion on $\mathbb{R}_+$



$$dy(t) = \mathbb{I}_{\{y(t)>0\}} dw(t) + \lambda \mathbb{I}_{\{y(t)=0\}} dt, \quad \lambda \geq 0$$

(Engelbert, Peskir '14)

Let w_1

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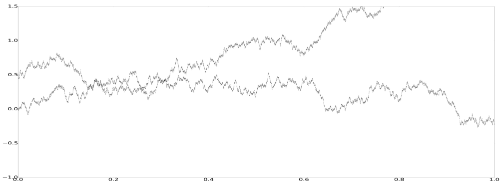

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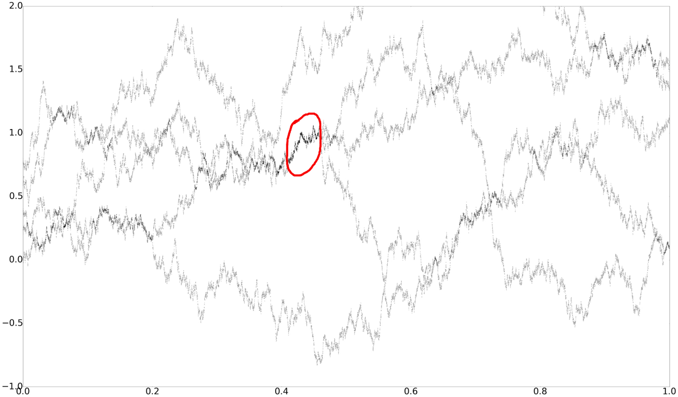
Three and more particles

Can three or more particles occupy the same position and which therms should contain the equation?



The equation should contain:

$$\mathbb{I}_{\{x_2(t)=x_3(t)=x_4(t)\}} d\frac{w_2(t)+w_3(t)+w_4(t)}{3}$$

where $\mathrm{Var}\left(w_i(t)\right) = \frac{1}{m_i} t = 5t$

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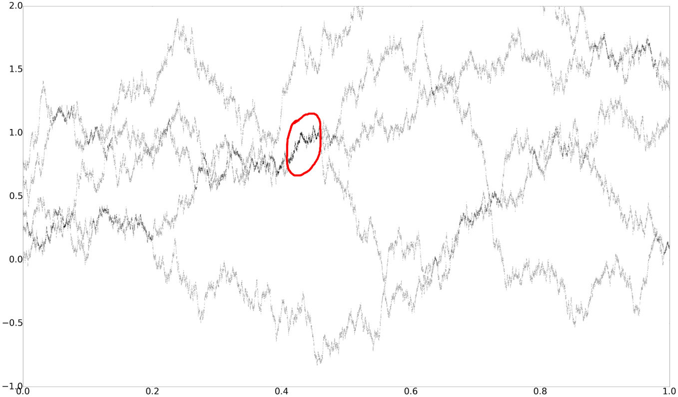

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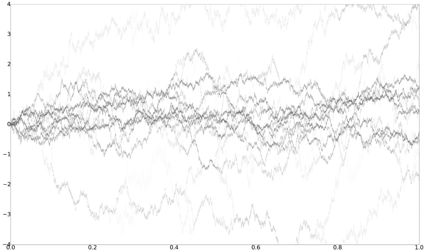
Uncountable number of particles. Diffusion term



Let $X(u,t)$ denote the position of the particle labeld by $u \in [0,1]$ at time $t \geq 0$.

$$X(u,t) \leq X(v,t), \quad u \leq v$$

$$dX(u,t) = d\frac{1}{m(u,t)} \int_{\pi(u,t)} W_t + \text{drift term}$$

where $\pi(u,t) = \{v : \ X(u,t) = X(v,t)\}$ and $m(u,t) = \mathrm{Leb}\{\pi(u,t)\}$

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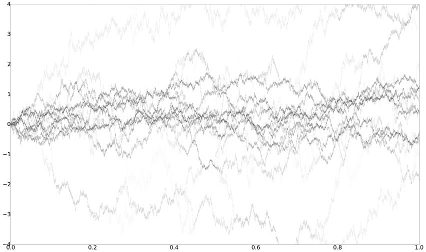

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$\xi(u)$ – an interaction potential of the particle u,

where $\xi : [0,1] \to \mathbb{R}$, $\xi(u) \le \xi(v)$, $u \le v$.

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1. If $\xi = const$, then particles coalesce.
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3. If $\xi = \xi_1 \mathbb{I}_{[0,1/2)} + \xi_2 \mathbb{I}_{[1/2,1]}$ and $X(u,0) = x_1(0)\mathbb{I}_{[0,1/2)} + x_2(0)\mathbb{I}_{[1/2,1]}$, then the equation corresponds its two dimensional analog:

$$dx_i(t) = \mathbb{I}_{\{x_1(t)<x_2(t)\}} dw_i(t) + \mathbb{I}_{\{x_1(t)=x_2(t)\}} d\frac{w_1(t)+w_2(t)}{2} + \lambda_i \mathbb{I}_{\{x_1(t)=x_2(t)\}} dt$$

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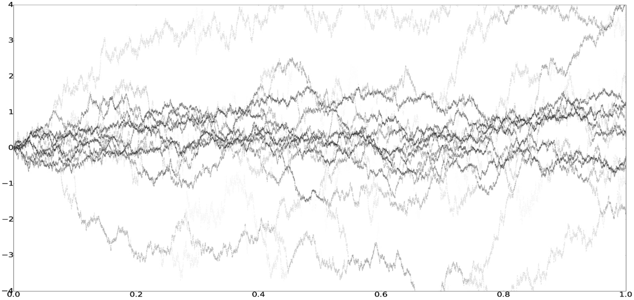

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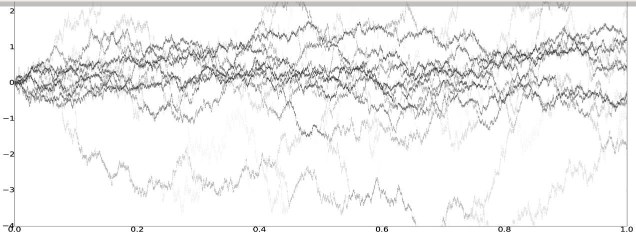
$$dX(u,t) = d\frac{1}{m(u,t)}\int_{\pi(u,t)} W_t + \left(\xi(u) - \frac{1}{m(u,t)}\int_{\pi(u,t)} \xi\right) dt$$

Let $N(t)$ denote a number of particles at time t

Is $N(t)$ finite or infinite?



The equation in $L_2[0,1]$

$$dX(u,t) = d\frac{1}{m(u,t)}\int_{\pi(u,t)} W_t + \left(\xi(u) - \frac{1}{m(u,t)}\int_{\pi(u,t)} \xi\right) dt$$

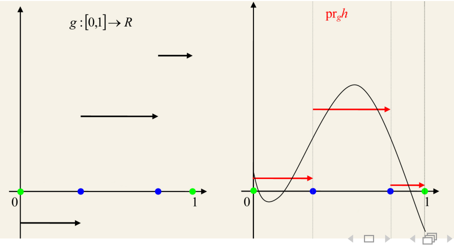
$\pi(u,t) = \{v : \; X(u,t) = X(v,t)\}$ and $m(u,t) = \mathrm{Leb}\{\pi(u,t)\}$

This family of equations can be rewritten as a one equation but in infinite dimensional space $L_2[0,1]$:

$$dX_t = \mathrm{pr}_{X_t} dW_t + (\xi - \mathrm{pr}_{X_t}\xi)dt \quad \text{in } L_2^{\uparrow}[0,1]$$

where $X_t := X(\cdot, t)$ and pr_g is the projection in $L_2[0,1]$ onto

$$L_2(g) = \{f : f \;\text{ is }\; \sigma(g) - \text{measurable}\}$$



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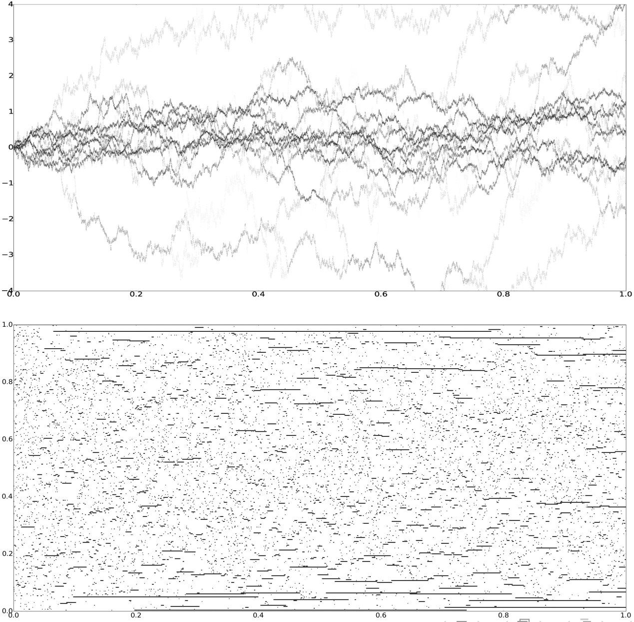

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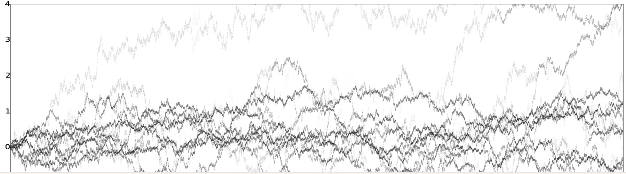
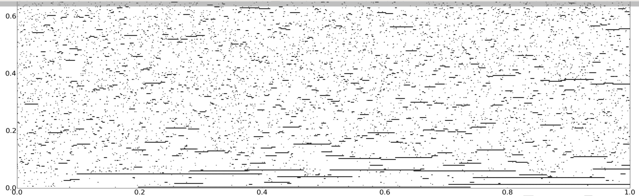


$$dX_t = \mathrm{pr}_{X_t} dW_t + (\xi - \mathrm{pr}_{X_t}\xi)dt$$

$\xi(u) = u$

$N(t) = \|\mathrm{pr}_{X_t}\|_{HS}^2$ denotes a number of particles at time t.

- $\int_0^T N(t)dt < \infty$ a.s.
- $\{t : \; N(t) = \infty\}$ is dense in $[0, \infty)$.



Reversible case.

Dirichlet form approach.

The main results

Theorem (K., von Renesse '17)

There exists a σ-finite measure Ξ on $L_2^{\uparrow}[0,1]$ and a Markov process X_t in $L_2^{\uparrow}[0,1]$ such that

- The measure Ξ is invariant for X_t and $\operatorname{supp}\Xi = L_2^{\uparrow}(\xi)$
- X_t solves
$$dX_t = \mathrm{pr}_{X_t} dW_t + (\xi - \mathrm{pr}_{X_t}\xi)dt \quad \text{in } L_2^{\uparrow}[0,1]$$
- The process $\mu_t = X(\cdot,t)|_{\#}\mathrm{Leb}|_{[0,1]}$, that describes the evolution of particle mass, solves the equation
$$d\mu_t = \frac{1}{2}\triangle\mu_t^* dt + \mathrm{div}(\sqrt{\mu_t}dW_t), \quad \text{in } \mathcal{P}_2(\mathbb{R}),$$
where $\mu_t^* = \sum_{x\in\operatorname{supp}\mu_t}\delta_x$
- The Varadhan formula
$$\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{d_{\mathcal{W}}^2(\mu_0,\nu)}{2t}}, \quad t \ll 1,$$
holds, if ξ strictly increases, where $d_{\mathcal{W}}$ denotes the Wasserstein distance

Note: X_t solves the equation only for $\mathcal{E}$-q.e. $X_0 \in \operatorname{supp}\Xi = L_2^{\uparrow}(\xi)$.

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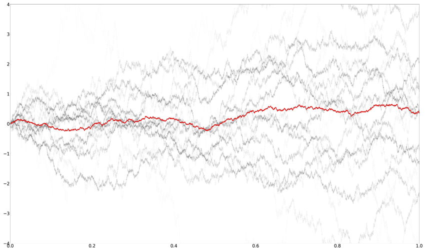
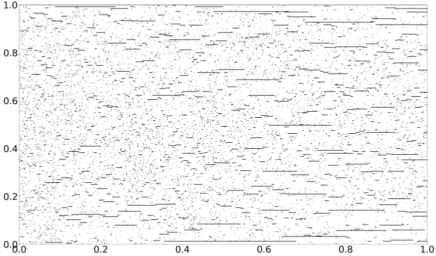
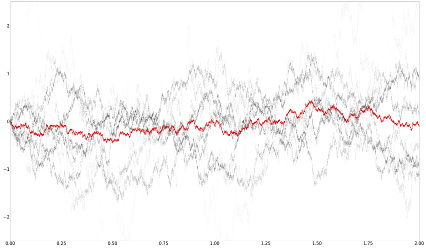
Other cases of drift therm

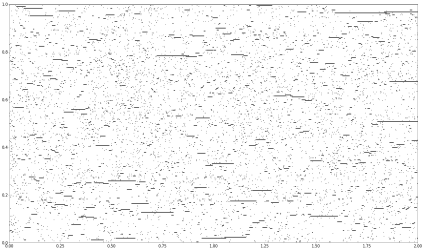
Drift term: $(\xi - \mathrm{pr}_{X_t}\xi)dt$, Invariant measure: Ξ





Drift term: $(\xi - \mathrm{pr}_{X_t}\xi - X_t)dt$, Invariant measure: $e^{-\frac{\|g\|_2^2}{2}}\Xi(dg)$ – *finite measure*





Invariant measure and the Differential operator

- **Invariant measure:**

$$\Xi = \sum_{n=1}^{\infty} \Xi^n,$$

where Ξ^n: $n-1$ jumps are distributed according

$$d\nu_\xi^n = \prod_{k=1}^{n} (q_k - q_{k-1}) d\xi(q_1) \ldots \xi(q_{n-1}),$$

n-values are distributed according to $\mathrm{Leb}_{x_1 \le \ldots \le x_n}$

- **Space of “smooth” functions:**

$$\mathcal{FC} = \Big\{ U = u((h_1, \cdot), \ldots, (h_k, \cdot))\varphi(\| \cdot \|_{L_2}^2) \Big\};$$

- **Differential operator:** $\mathrm{D}U(g) = \mathrm{pr}_g \nabla^{L_2} U(g) \in L_2[0,1]$;

(Ex. $\mathrm{D}u((h,g)) = u'((h,g))\mathrm{pr}_g h, \quad \mathrm{D}\|g\|_{L_2}^2 = 2g$)

Integration by parts (K., von Renesse '17)

Let $U, V \in \mathcal{FC}$. Then

$$\int_{L_2^\uparrow} (\mathrm{D}U(g), \mathrm{D}V(g))\Xi(dg) = -\int_{L_2^\uparrow} LU(g)V(g)\Xi(dg)$$
$$-\int_{L_2^\uparrow} V(g)(\nabla^{L_2}U(g), \xi - \mathrm{pr}_g\xi)\Xi(dg).$$

(Ex. $Lu((h,g)) = u''((h,g))\|\mathrm{pr}_g h\|_{L_2}^2, \quad L\|g\|_{L_2}^2 = 2\#g$)

Dirichlet form:

$$\mathcal{E}(U,V) = \frac{1}{2}\int_{L_2^\uparrow(\xi)} (\mathrm{D}U(g), \mathrm{D}V(g))\Xi(dg), \quad U, V \in \mathcal{FC}$$

Theorem (K., von Renesse '17)

$\mathcal{E}$ is a closable bilinear form on $L_2(L_2^\uparrow, \Xi)$, its closure is a quasi-regular local symmetric Dirichlet form and $\|\cdot\|_{L_2}$ is its intrinsic metric.

Integration by parts and Dirichlet form

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$$\int_{L_2^\uparrow} (\mathrm{D}U(g), \mathrm{D}V(g))\Xi(dg) = -\int_{L_2^\uparrow} LU(g)V(g)\Xi(dg)$$
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General case.

Finite particle approximation.

Weak existence and some regularity

Theorem (K. '17)

1. For each $\xi \in L_\infty^\uparrow[0,1]$ and $X_0 \in L_2^\uparrow[0,1]$ (not only from $L_2^\uparrow(\xi)$) the equation

$$dX_t = \mathrm{pr}_{X_t} dW_t + (\xi - \mathrm{pr}_{X_t}\xi)dt \quad \text{in } L_2^\uparrow[0,1] \tag{1}$$

has a weak martingale solution.

2. If g, ξ are piecewise $\frac{1}{2}$+-Hölder continuous then there exists $X = \{X(u,t),\ u \in [0,1],\ t \geq 0\}$ in the Skorohod space $D([0,1], C([0,\infty)))$ s.t.:
   1. $X(\cdot,0) = g$;
   2. $X(u,t) \leq X(v,t)$ if $u < v$;
   3. $M(u,t) := X(u,t) - g(u) - \int_0^t (\xi(u) - \left(\mathrm{pr}_{X(\cdot,s)}\xi\right)(u))ds$ is a continuous square integrable martingale;
   4. $[M(u,\cdot), M(v,\cdot)]_t = \int_0^t \frac{\mathbb{I}_{\{X(u,s)=X(v,s)\}}}{m(u,s)} ds$, where $m(u,t) := \mathrm{Leb}\{v : \ X(u,t) = X(v,t)\}$.

   Moreover, the process $X(\cdot,t)$, $t \geq 0$, is a weak solution of (1).

3. $\mu_t = X(\cdot,t)|_{\#}\mathrm{Leb}|_{[0,1]}$ solves

$$d\mu_t = \frac{1}{2}\triangle\mu_t^* dt + \mathrm{div}(\sqrt{\mu_t} dW_t), \quad \text{in } \mathcal{P}_2(\mathbb{R}).$$

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Thank you for your attention!

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