On asymptotic behavior of the modified Arratia flow

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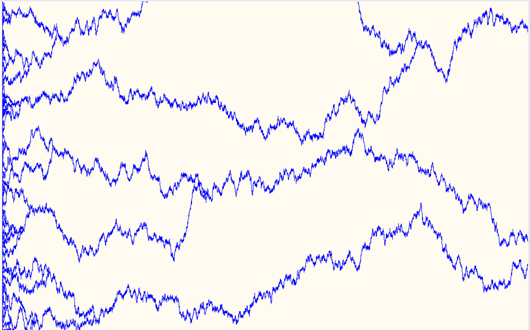
Arratia flow

The system of Brownian particles on the real line that

1. start from all points of $\mathbb{R}$;
2. move independently up to the moment of meeting;
3. coalesce;
4. the diffusion is unchangeable and equals 1. (**R.A. Arratia '79**)

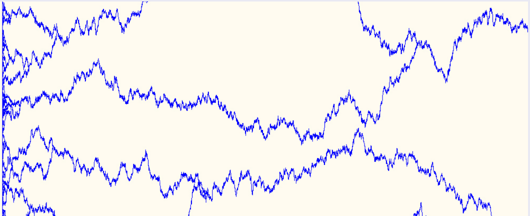


Arratia flow

The Arratia flow, the mathematical description

$\{x(u,t),\ t \in [0,T], u \in \mathbb{R}\}$ in $D(\mathbb{R}, C[0,T])$ such that

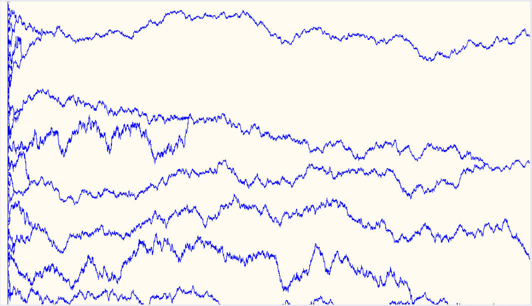
1. $x(u,\cdot)$ is a continuous square integrable martingale with respect to the joint filtration;
2. $x(u,0) = u, \quad u \in \mathbb{R}$;
3. $x(u,t) \le x(v,t), \quad u < v$;
4. $\langle x(u,\cdot)\rangle_t = t$;
5. $\langle x(u,\cdot), x(v,\cdot)\rangle_t = 0, \quad t < \tau_{u,v} = \inf\{t : x(u,t) = x(v,t)\}$.



Main object

We consider the system of diffusion particles on the real line which

1. start from some set of points;
2. move independently up to the moment of the meeting and then coalesce;
3. masses add after coalescing;
4. the diffusion is inversely proportional to the mass;



Continuum system

Theorem 1 (K. '14)

There exists a random element $\{y(u,t),\ u \in [0,1],\ t \in [0,T]\}$ in $D([0,1], C[0,T])$ such that

1°) $y(u,\cdot)$ is a continuous square integrable martingale with respect to

$$\mathcal{F}_t = \sigma(y(u,s),\ u \in [0,1],\ s \le t), \quad t \in [0,T];$$

2°) $y(u,0) = u$, $u \in [0,1]$;

3°) $y(u,t) \le y(v,t)$, $u < v$;

4°) $\langle y(u,\cdot)\rangle_t = \int\limits_0^t \frac{ds}{m(u,s)}, \qquad m(u,t) = \mathrm{Leb}\{v : \ y(v,t) = y(u,t)\};$

5°) $\langle y(u,\cdot), y(v,\cdot)\rangle_t = 0, \quad t \le \tau_{u,v} = \inf\{t : \ y(u,t) = y(v,t)\} \wedge T.$

The reason of existence

The main estimation

$$\mathbb{P}\{m(u,t) < r\} \le \frac{2}{\sqrt{2\pi}} \int_0^{\frac{r^{3/2}}{\sqrt{t}}} e^{-\frac{x^2}{2}}\,dx \le \frac{2r\sqrt{r}}{\sqrt{2\pi t}}.$$

The mass growth

For $\beta \in (0, \frac{3}{2})$

$$\mathbb{E}\frac{1}{m^\beta(u,t)} = \int_1^{+\infty} \mathbb{P}\left\{m(u,t) < \frac{1}{r^{1/\beta}}\right\} dr \le \frac{C}{\sqrt[3]{t^\beta}}.$$

(If particles start from interval $[0, b]$, $b \ge 1$, then the constant C is independent of b.)

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Connection with Wasserstein diffusion

Wasserstein diffusion (M.-K. von Renesse, K.-T. Sturm '09)

$$d\mu_t = \beta\Delta\mu_t dt + \hat{\Gamma}(\mu_t)dt + \mathrm{div}(\sqrt{\mu}dW_t),$$

with

$$\langle f, \hat{\Gamma}(\nu)\rangle = \sum_{I\in \mathsf{gaps}(\nu)} \left[\frac{f''(I_+) + f''(I-)}{2} - \frac{f'(I_+) - f'(I_+)}{|I|}\right].$$

Evolution of particle mass in modified Arratia flow

Let $\mu_t := \mathrm{Leb} \circ y(\cdot, t)^{-1}$ then

$$d\mu_t = \Gamma(\mu_t)dt + \mathrm{div}(\sqrt{\mu}dW_t),$$

where

$$\langle f, \Gamma(\nu)\rangle = \sum_{x\in supp(\nu)} f''(x).$$

Large deviations

LDP (K., Max von Renesse '15)

The family of processes $\{y(\varepsilon\cdot)\}_{\varepsilon>0}$ satisfies a large deviations principle with the rate function

$$I(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}(t)\|^2_{L_2(du)}dt, & \varphi \in H, \\ +\infty, & \text{otherwise.} \end{cases}$$

$$H = \{\varphi \in C([0,T], L_2^{\uparrow}([0,1],du)) : \ \varphi(0) = \mathrm{id} \text{ and } \\ t \to \varphi(t) \in L_2([0,1],du) \text{ is absolutely continuous}\}.$$

Varadhan's formula for the modified Arratia flow

$$\mathbb{P}\{y(t) = g\} \sim e^{-\frac{\|\mathrm{id}-g\|^2}{2t}}, \quad t \to 0.$$

For $\mu_t := \mathrm{Leb} \circ y(\cdot,t)^{-1}$, $\mathbb{P}\{\mu_t = \nu\} \sim e^{-\frac{\mathcal{W}(\mathrm{Leb},\nu)^2}{2t}}, \quad t \to 0.$

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The aim of talk

Asymptotic behavior of particles

To find $\varphi(t)$ and $\psi(t)$, $t \geq 0$, such that

$$\varlimsup_{t\to 0+} \frac{|y(u,t) - y(u,0)|}{\varphi(t)} = 1 \quad \text{a.s.},$$

$$\varlimsup_{t\to 0+} \frac{m(u,t)}{\psi(t)} = 1 \quad \text{a.s.}$$

Asymptotic behavior of Arratia flow x

Law of the iterated logarithm

$$\overline{\lim_{t\to 0+}} \frac{|x(0,t)|}{\sqrt{2t\ln\ln\frac{1}{t}}} = 1 \quad \text{a.s.}$$

LIL for the cluster size

$$\overline{\lim_{t\to 0+}} \frac{\nu(t)}{\sqrt{2t\ \ln\ln\frac{1}{t}}} \ge 1 \quad \text{a.s.}$$

$$\overline{\lim_{t\to 0+}} \frac{\nu(t)}{2\sqrt{t\ \ln\ln\frac{1}{t}}} \le 1 \quad \text{a.s.,}$$

where $\quad \nu(t) = \mathrm{Leb}\{u:\ x(u,t) = x(0,t)\}.$

(A.A. Dorogovtsev, A.V. Gnedin, M.B. Vovchanskii '12)

Asymptotic behavior of the modified Arratia flow

Asymptotic behavior (K. '16)

For all $\varepsilon > 0$ and $u \in [0,1]$

$$\mathbb{P}\left\{\lim_{t\to 0} \frac{m(u,t)}{\sqrt[3]{t}\left(\ln\frac{1}{t}\right)^{1+\epsilon}} = 0\right\} = \mathbb{P}\left\{\overline{\lim_{t\to 0}} \frac{m(u,t)}{\sqrt[3]{t}\left(\ln\frac{1}{t}\right)^{-1-\epsilon}} = +\infty\right\} = 1,$$

$$\mathbb{P}\left\{\lim_{t\to 0} \frac{|y(u,t)-u|}{\sqrt[3]{t}\left(\ln\frac{1}{t}\right)^{\frac{1}{2}+\epsilon}} = 0\right\} = \mathbb{P}\left\{\overline{\lim_{t\to 0}} \frac{|y(u,t)-u|}{\sqrt[3]{t}\left(\ln\frac{1}{t}\right)^{-\frac{1}{2}-\epsilon}} = +\infty\right\} = 1.$$

$$\mathbb{E}m(u,t) \le C\sqrt[3]{t}, \quad \mathbb{E}\frac{1}{m(u,t)} \le \frac{C}{\sqrt[3]{t}}, \quad t \in (0,T].$$

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$$\mathbb{P}\left\{\lim_{t\to 0}\frac{m(u,t)}{\sqrt[3]{t}\left(\ln\frac{1}{t}\right)^{1+\epsilon}}=0\right\}=1.$$

$$\mathbb{E}m(u,t)\le C\sqrt[3]{t}.$$

Let $\theta > 0$, $\lambda < 1$ and $t_n = \lambda^n$, $n \in \mathbb{N}$.

$$\mathbb{P}\{m(u,t) > \theta\varphi(t), \quad \text{for some } t \in (t_{n+1}, t_n]\}$$

$$\le \mathbb{P}\{m(u,t_n) > \theta\varphi(t_{n+1})\} \le \frac{1}{\theta\varphi(t_{n+1})}\mathbb{E}m(u,t_n)$$

$$\le \frac{Ct_n^{\frac{1}{2\alpha+1}}}{\theta t_{n+1}^{\frac{1}{2\alpha+1}}\left(\ln\frac{1}{t_{n+1}}\right)^{1+\epsilon}} = \frac{C}{\theta\lambda^{\frac{1}{2\alpha+1}}\left(\ln\frac{1}{\lambda}\right)^{1+\epsilon}(n+1)^{1+\epsilon}}.$$

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$$y(u,t) - u = w\left(\int_0^t \frac{ds}{m(u,s)}\right) \quad + \quad \text{LIL for Wiener process.}$$

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Rescaling property

$$y_\rho(u,t) = \frac{1}{\rho} y(u\rho, t\rho^3), \quad u \in [0, 1/\rho], \quad \rho > 0.$$

y_ρ satisfies the same properties as y, but is defined for $u \in [0, 1/\rho]$, i.e. it describes the system of particles starting from $[0, 1/\rho]$. Moreover

$$m_\rho(u,t) = \mathrm{Leb}\{v: \ y_\rho(u,t) = y_\rho(v,t)\} = \frac{1}{\rho} m(u\rho, t\rho^3).$$

If $\mathbb{E}m_\rho(0,T) \le C$ for all $\rho \in (0,1]$, then

$$\mathbb{E}m(0,T\rho^3) \le \rho \mathbb{E}m_\rho(0,T) \le C\rho.$$

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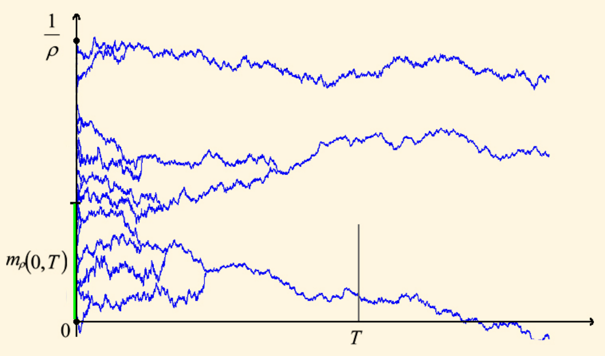
Estimation $\mathbb{E}m_\rho(0,T) \le C$



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Estimation $\mathbb{E}m_\rho(0,T) \le C$

Set $M_\rho(u,t) = y_\rho(u,t) - y_\rho(0,t)$.

Using the Paley-Zygmund inequality, we can estimate

$$\begin{aligned}\mathbb{P}\{y_\rho(0,T) = y_\rho(u,T)\} &= 1 - \mathbb{P}\{M_\rho(u,T) > 0\} \\ &\le 1 - \frac{\left(\mathbb{E}M_\rho(u,T)\right)^2}{\mathbb{E}M_\rho^2(u,T)} = 1 - \frac{u^2}{\mathbb{E}M_\rho^2(u,T)} \\ &= 1 - \frac{u^2}{\operatorname{Var} M_\rho(u,T) + u^2} = \frac{\operatorname{Var} M_\rho(u,T)}{\operatorname{Var} M_\rho(u,T) + u^2}.\end{aligned}$$

Estimation $\mathbb{E}m_\rho(0,T) \le C$

$$\operatorname{Var} M_\rho(u,T) = \mathbb{E}\left(M_\rho(u,T) - u\right)^2 \le \int_0^T \mathbb{E}\frac{1}{m_\rho(u,s)}ds + \int_0^T \mathbb{E}\frac{1}{m_\rho(0,s)}ds \le C_1,$$

where we used

$$\mathbb{E}\frac{1}{m_\rho(u,s)} < \frac{C}{\sqrt[3]{s}}, \quad s \in (0,T].$$

Thus

$$\mathbb{P}\{y_\rho(0,T) = y_\rho(u,T)\} \le \frac{\operatorname{Var} M_\rho(u,T)}{\operatorname{Var} M_\rho(u,T) + u^2} \le \frac{C_1}{u^2}.$$

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Description of the model

Let μ be a probability measure on $\mathbb{R}$, $A = \operatorname{supp}\mu$.

$z(\xi,t)$ is the position of particles at time t which starts from $\xi \in A$.

$z(\xi,t)$, $\xi \in A$, $t \in [0,T]$, should satisfy the same condition as y:

1) $z(\xi,\cdot)$ is a continuous square integrable martingale;
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Description of the model

Let μ be a probability measure on $\mathbb{R}$, $A = \operatorname{supp}\mu$.

$z(\xi, t)$ is the position of particles at time t which starts from $\xi \in A$.

$z(\xi, t)$, $\xi \in A$, $t \in [0, T]$, should satisfy the same condition as y:

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Alternative description

Let μ be the push forward of the Lebesgue measure on $(0,1)$

$$\mu = \mathrm{Leb}\big|_{(0,1)} \circ g^{-1},$$

where $g : (0,1) \to \mathbb{R}$ is a non-decreasing right-continuous function.

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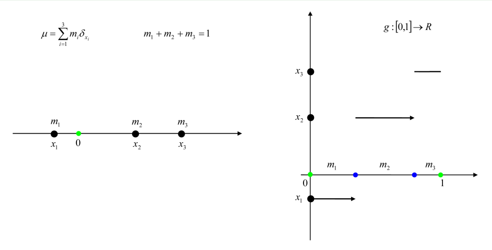
Example



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Theorem 3 (K. '16)

Let $\|g\|_{L_{2+\varepsilon}} < \infty$ (it's equivalent to $\int_{\mathbb{R}} |\xi|^{2+\varepsilon}\mu(d\xi) < \infty$). Then there exists $y(u,t)$, $u \in (0,1)$, $t \in [0,T]$, in $D([0,1], C[0,T])$ satisfying

1°) $y(u,\cdot)$ is a continuous square integrable martingale;

2°) $y(u,0) = g(u)$, $u \in (0,1)$;

3°) $y(u,t) \le y(v,t)$, $u < v$;

4°) $\langle y(u,\cdot)\rangle_t = \int\limits_0^t \frac{ds}{m(u,s)}$, $\quad m(u,t) = \mathrm{Leb}\{v : \ y(v,t) = y(u,t)\}$;

5°) $\langle y(u,\cdot), y(v,\cdot)\rangle_t = 0$, $\quad t \le \tau_{u,v} = \inf\{t : \ y(u,t) = y(v,t)\} \wedge T$.

Alternative description

Set $A = \{g(u),\ u \in (0,1)\}$ and

$$z(g(u),\cdot) = y(u,\cdot), \quad u \in (0,1).$$

Since $g(u) = g(v)$ implies $y(u,\cdot) = y(v,\cdot)$, the process z is well-defined.

If $\xi = g(u)$,

$$z(\xi,0) = z(g(u),0) = y(u,0) = g(u) = \xi$$

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Alternative description

Asymptotic behavior (K. '16)

Let $g'(u) \sim |u-u_0|^{\alpha-1}$, $\alpha > \frac{1}{2}$. Then for all $\varepsilon > 0$

$$\mathbb{P}\left\{\lim_{t\to 0} \frac{m(u_0,t)}{t^{\frac{1}{2\alpha+1}}\left(\ln\frac{1}{t}\right)^{1+\epsilon}} = 0\right\} = \mathbb{P}\left\{\lim_{t\to 0} \frac{m(u_0,t)}{t^{\frac{1}{2\alpha+1}}\left(\ln\frac{1}{t}\right)^{-1-\epsilon}} = +\infty\right\} = 1,$$

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Remark

If μ has the density h and $h(\xi) \sim |\xi-\xi_0|^{\beta}$, where $\xi_0 = g(u_0)$ and $-1 < \beta < 1$, then $\alpha = \frac{1}{1+\beta}$ and

$$\frac{1}{2\alpha+1} = \frac{1+\beta}{3+\beta}, \quad \frac{\alpha}{2\alpha+1} = \frac{1}{3+\beta}.$$

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Thank you for your attention!