Large Deviation Principle for Modified Arratia flow

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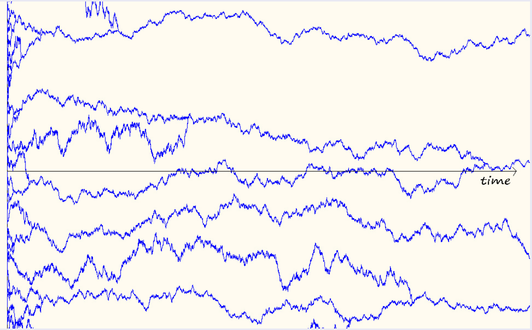
joint work with Max von Renesse

Main object

Modified Arratia flow

1. system of diffusion particles on the rial line;
2. particles move independently up to the moment of the meeting and coalesce;
3. particles have mass which obeys the conservation law;
4. diffusion of every particle is inversely proportional to the mass.



Comparing with Arratia flow

Arratia flow

- diffusion of particles is not changed (Brownian particles);
- adding new particles into the system does not influence on the motion of other particles;
- infinite system can be completely describe be description of its finite subsystems.

Modified Arratia flow

- diffusion of particles is changed (not Brownian particles);
- adding new particles into the system influences on the motion of other particles;
- infinite system cannot be completely describe be description of its finite subsystems.

Aim of investigation

We are going to study relationship between the modified Arratia flow and the geometry of state space (in sense of large deviation)

Varadhan's formula (examples)

w is a Wiener process in $\mathbb{R}^n$, $w(0) = x$

$$\lim_{t \to 0+} t \ln \mathbb{P}\{w(t) \in G\} = \frac{d_E(x, G)^2}{2}$$

Varadhan '1967

ξ is a diffusion process with the generator $\mathcal{L}f(x) = \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x)$

$$\lim_{t \to 0+} t \ln \mathbb{P}\{\xi(t) \in G\} = \frac{d_{\mathcal{L}}(x, G)^2}{2},$$

where $d_{\mathcal{L}}(x, y) = \inf\limits_{f(0)=x,\ f(T)=y} \int_0^T \sqrt{\dot{f}(s) a^{-1}(f(s)) \dot{f}(s)} ds.$

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Finite system of particles

Let particles start from $x_1^0 \le \ldots \le x_n^0$ with masses $m_1^0, \ldots, m_n^0$ and $x_k(t)$ denote the position of particles at time t.

Modified Arratia flow

1. $x_k(\cdot)$ is a continuous square integrable martingale with respect to the filtration
$$\mathcal{F}_t = \sigma(x_i(s),\ s \le t,\ i = 1, \ldots, n);$$
2. $x_k(0) = x_k^0$;
3. $x_k(t) \le x_l(t)$, $k < l$;
4. $\langle x_k(\cdot)\rangle_t = \int\limits_0^t \frac{1}{m_k(s)} ds$
where $m_k(t) = \sum_{\{i:\ \exists s \le t\ \ x_i(s) = x_k(s)\}} m_i^0$;
5. $\langle x_k(\cdot), x_l(\cdot)\rangle_t = 0, \quad t < \tau_{k,l}$,
where $\tau_{k,l} = \inf\{t :\ \ x_k(t) = x_l(t)\}$.

LDP for finite system

$$H_{x^0} = \{f = (f_k) : [0,1] \to \mathbb{R}^n_\uparrow : \ f_k(0) = x_k^0, \ \int_0^1 (f_k'(t))^2 dt < \infty, \ \forall k\}$$

LDP (K. '14)

$\{x(\varepsilon\cdot), \ \varepsilon > 0\}$ satisfies LDP in the space $\mathrm{C}\left([0,1];\mathbb{R}^n_\uparrow\right)$ with the good rate function

$$I_{x^0}(f) = \begin{cases} \frac{1}{2}\int\limits_0^1 \sum_{k=1}^n (f_k'(t))^2 m_k dt, & f \in H_{x^0}, \\ +\infty & \text{otherwise,} \end{cases}$$

i.e.

$$-\inf_{A^\circ} I_{x^0} \le \overline{\lim_{\varepsilon \to 0}}\, \varepsilon \ln \mathbb{P}\{x(\varepsilon\cdot) \in A\} \le -\inf_{\overline{A}} I_{x^0}.$$

Varadhan's formula for finite system

Taking $A = \{f(1) \in G \subset \mathbb{R}^n_\uparrow\}$, we get the following results

Lemma

$$\lim_{t \to 0+} t \ln \mathbb{P}\{x(t) \in G\} = \frac{d_{m^0}(x^0, G)^2}{2},$$

where $d_{m^0}(y, z) = \left(\sum_k (y_k - z_k)^2 m_k^0\right)^{1/2}$ is a distance on $\mathbb{R}^n_\uparrow$.

Or considering the evolution of particle masses

$$\nu(t) = \sum_k m_k^0 \delta_{x_k(t)}$$

for each measurable set $G \subset \mathcal{H}_{m^0} = \{\sum_k m_k^0 \delta_{a_k}, \ a \in \mathbb{R}^n\}$

$$\lim_{t \to 0+} t \ln \mathbb{P}\{\nu(t) \in G\} = \frac{\mathcal{W}(\nu(0), G)^2}{2},$$

where $\mathcal{W}(\mu, \nu) = \inf_{\sigma \in C(\mu,\nu)} \int_{\mathbb{R}^2} |u - v|^2 \sigma(du, dv)$ is a Wasserstein distance.

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Estimation of diffusion

Suppose that $m_k^0 = \frac{1}{n}$, $x_k^0 = \frac{k-1}{n}$, $k = 1, \ldots, n$

The mass growth (K. '14)

For $\beta \in \left(0, \frac{3}{2}\right)$ and $t \in [0, 1]$

$$\mathbb{E}\frac{1}{m_k^\beta(t)} \leq \frac{C}{\sqrt[3]{t^\beta}},$$

where C is independent of n.

Continuum system

Existence (K. '14)

There exists a random element $\{y(u,t),\ u \in [0,1],\ t \in [0,T]\}$ in the Skorohod space $D([0,1], C[0,T])$ such that

1°) $y(u,\cdot)$ is a continuous square integrable martingale with respect to

$$\mathcal{F}_t = \sigma(y(u,s),\ u \in [0,1],\ s \le t), \quad t \in [0,T];$$

2°) $y(u,0) = u$;

3°) $y(u,t) \le y(v,t)$, $u < v$;

4°) $\langle y(u,\cdot)\rangle_t = \int\limits_0^t \frac{ds}{m(u,s)}$,
where $m(u,t) = \lambda\{v :\ \exists s \le t\ y(v,s) = y(u,s)\}$;

5°) $\langle y(u,\cdot), y(v,\cdot)\rangle_t = 0, \quad t \le \tau_{u,v}$,
where $\tau_{u,v} = \inf\{t :\ y(u,t) = y(v,t)\} \wedge T$.

Large deviations

$L_2(\mu) = L_2([0,1],\mu)$, $\mu(du) = \kappa(u)du$, where $\kappa : [0,1] \to [0,1]$

$$\kappa(u) = \begin{cases} u^\beta, & u \in [0,1/2] \\ (1-u)^\beta, & u \in (1/2,1], \end{cases}$$

for some fixed $\beta > 1$, and

$$H = \{\varphi \in C([0,T], L_2^{\uparrow}([0,1],du)) : \ \varphi(0) = \mathrm{id} \text{ and } \\ t \to \varphi(t) \in L_2([0,1],du) \text{ is absolutely continuous}\},$$

LDP (K., Max von Renesse '15)

The family of processes $\{y(\varepsilon\cdot)\}_{\varepsilon>0}$ satisfies a large deviations principle in the space $C([0,T], L_2^{\uparrow}(\mu))$ with the good rate function

$$I(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}(t)\|^2_{L_2(du)}dt, & \varphi \in H, \\ +\infty, & \text{otherwise.} \end{cases}$$

Varadhan's formula for the modified Arratia flow

Set

- $\mu_t := y(\cdot, t)_{\#}\mathrm{Leb}, \quad t \in [0, T]$;
- $\iota : D^{\uparrow}([0,1]) \ni g \mapsto g_{\#}\mathrm{Leb} \in \mathcal{P}(\mathbb{R})$;
- τ_μ is the image topology on $\mathcal{P}(\mathbb{R})$ of the $L^2(\mu)$-topology on $\mathcal{P}(\mathbb{R})$ induced from the bijection ι.

A set $A \subset \mathcal{P}(\mathbb{R})$ is called displacement convex if it is the image of a convex subset of $D^{\uparrow}([0,1])$ under the map ι.

(K., Max von Renesse '15)

Let $A \subset \mathcal{P}(\mathbb{R})$ τ_μ-closed with nonempty interior and displacement convex, then

$$\lim_{\varepsilon \to 0} \epsilon \log P(\mu_\varepsilon \in A) = -\frac{(\mathcal{W}(\mathrm{Leb}, A))^2}{2}.$$

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Upper bound

For $\varphi \in C_{\mathrm{id}}([0,T], L_2^{\uparrow}(\mu))$ we show

$$\lim_{r\to 0} \overline{\lim_{\varepsilon\to 0}} \varepsilon \ln \mathbb{P}\{y(\varepsilon\cdot) \in B_r(\varphi)\} \le -I(\varphi).$$

$$M_t^{\varepsilon,h} = \exp\left\{\frac{1}{\varepsilon}\left[\int_0^t (h(s), dy^\varepsilon(s))_{L_2(du)} - \frac{1}{2}\int_0^t \|pr_{y^\varepsilon(s)} h(s)\|^2_{L_2(du)} ds\right]\right\}.$$

$$M_T^{\varepsilon,h} = \exp\left\{\frac{1}{\varepsilon} F(y^\varepsilon, h)\right\},$$

where

$$F(\varphi, h) = (h(T), \varphi(T))_{L_2(du)} - (h(0), \mathrm{id})_{L_2(du)} - \int_0^T (\dot{h}(s), \varphi(s))_{L_2(du)} ds - \frac{1}{2}\int_0^T \|pr_{\varphi(s)} h(s)\|^2_{L_2(du)} ds.$$

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where

$$F(\varphi,h)=(h(T),\varphi(T))_{L_2(du)}-(h(0),\mathrm{id})_{L_2(du)}$$
$$-\int_0^T(\dot h(s),\varphi(s))_{L_2(du)}ds-\frac{1}{2}\int_0^T\|pr_{\varphi(s)}h(s)\|^2_{L_2(du)}ds.$$

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Upper bound

$$\begin{aligned}\mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} &= \mathbb{E}\left[\mathbb{I}_{\{y^\varepsilon \in B_r(\varphi)\}} \frac{M_T^{\varepsilon,h}}{M_T^{\varepsilon,h}}\right] \\ &\leq \exp\left\{-\frac{1}{\varepsilon} \inf_{\psi \in B_r(\varphi)} F(\psi,h)\right\} \mathbb{E} M_T^{\varepsilon,h} \\ &= \exp\left\{-\frac{1}{\varepsilon} \inf_{\psi \in B_r(\varphi)} F(\psi,h)\right\}.\end{aligned}$$

$$\begin{aligned}F(\varphi,h) &= (h(T),\varphi(T))_{L_2(du)} - (h(0),\mathrm{id})_{L_2(du)} \\ &\quad - \int_0^T (\dot{h}(s),\varphi(s))_{L_2(du)} ds - \frac{1}{2}\int_0^T \|pr_{\varphi(s)} h(s)\|^2_{L_2(du)} ds.\end{aligned}$$

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$$\mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} = \mathbb{E}\left[\mathbb{I}_{\{y^\varepsilon \in B_r(\varphi)\}} \frac{M_T^{\varepsilon,h}}{M_T^{\varepsilon,h}}\right]$$
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$$\|pr_{\varphi(s)} h(s)\|_{L_2(du)}^2 \leq \|h(s)\|_{L_2(du)}^2$$

$$\overline{\lim_{\varepsilon \to 0}}\, \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \leq -\inf_{\psi \in B_r(\varphi)} F(\psi, h) \leq -\inf_{\psi \in B_r(\varphi)} \Phi(\psi, h),$$

Upper bound

$$\mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} = \mathbb{E}\left[\mathbb{I}_{\{y^\varepsilon \in B_r(\varphi)\}} \frac{M_T^{\varepsilon,h}}{M_T^{\varepsilon,h}}\right]$$

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$$\|pr_{\varphi(s)} h(s)\|^2_{L_2(du)} \leq \|h(s)\|^2_{L_2(du)}$$

$$\overline{\lim_{\varepsilon \to 0}}\, \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \leq - \inf_{\psi \in B_r(\varphi)} F(\psi, h) \leq - \inf_{\psi \in B_r(\varphi)} \Phi(\psi, h),$$

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$$\mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} = \mathbb{E}\left[\mathbb{I}_{\{y^\varepsilon \in B_r(\varphi)\}} \frac{M_T^{\varepsilon,h}}{M_T^{\varepsilon,h}}\right]$$
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$$\Phi(\varphi, h) = (h(T), \varphi(T))_{L_2(du)} - (h(0), \mathrm{id})_{L_2(du)} - \int_0^T (\dot{h}(s), \varphi(s))_{L_2(du)} ds - \frac{1}{2} \int_0^T \|h(s)\|^2_{L_2(du)} ds$$

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Proposition

For each $\varphi \in C_{\mathrm{id}}([0,T], L_2^\uparrow(\mu))$

$$\sup_h \Phi(\varphi, h) = I(\varphi).$$

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For each $\varphi \in C_{\mathrm{id}}([0,T], L_2^{\uparrow}(\mu))$

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Lower bound

We find $\mathcal{R} \subset C_{\mathrm{id}}([0,T], L_2^{\uparrow}(\mu))$ such that for each $\varphi \in \mathcal{R}$,

$$\lim_{r\to 0} \varliminf_{\varepsilon\to 0} \varepsilon \ln \mathbb{P}\{y^{\varepsilon} \in B_r(\varphi)\} \geq -I(\varphi),$$

and for each φ satisfying $I(\varphi) < \infty$, there exists $\{\varphi_n\} \subset \mathcal{R}$ such that

$$\varphi_n \to \varphi$$

and

$$I(\varphi_n) \to I(\varphi).$$

$\mathcal{R} = \{\varphi \in C([0,T], L_2^{\uparrow\uparrow}(du)) : \ \varphi$ is enough regular$\}$.

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Lower bound

Take $h = \dot{\varphi}$ and define the new probability measure $\mathbb{P}^{\varepsilon,h}$ with density

$$\frac{d\mathbb{P}^{\varepsilon,h}}{d\mathbb{P}} = M_T^{\varepsilon,h} = \exp\left\{\frac{1}{\varepsilon}\left[\int_0^t (h(s), dy^\varepsilon(s))_{L_2(du)} - \frac{1}{2}\int_0^t \|pr_{y^\varepsilon(s)}h(s)\|^2_{L_2(du)}ds\right]\right\}.$$

The random element y^ε in $D([0,1], C[0,T])$ satisfies (w.r.t. $\mathbb{P}^{\varepsilon,h}$)

1. $$\eta^\varepsilon(u,\cdot) = y^\varepsilon(u,\cdot) - \int_0^\cdot \left(\mathrm{pr}_{y^\varepsilon(s)}h(s)\right)(u)ds$$
   is a continuous local square integrable $(\mathcal{F}_{\varepsilon t})$-martingale;
2. $y^\varepsilon(u,0) = u$;
3. $y^\varepsilon(u,t) \le y^\varepsilon(v,t)$ for $u < v$;
4. $\langle \eta^\varepsilon(u,\cdot), \eta^\varepsilon(v,\cdot)\rangle_t = \varepsilon \int_0^t \frac{\mathbb{I}_{\{\tau^\varepsilon_{u,v} \le s\}}ds}{m^\varepsilon(u,s)}$.

$\lim_{\varepsilon\to 0} \mathbb{P}^{\varepsilon,\dot{\varphi}}\{y^\varepsilon \in B_r(\varphi)\} = 1$ for all $r > 0$.

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Lower bound

$$\mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} = \mathbb{E}^{\varepsilon,h} \frac{\mathbb{I}_{\{y^\varepsilon \in B_r(\varphi)\}}}{M_T^{\varepsilon,h}}$$

$$\geq \exp\left\{-\frac{1}{\varepsilon} \sup_{\psi \in B_r(\varphi) \cap L_2([0,T],L_2(du))} F(\psi,h)\right\} \mathbb{P}^{\varepsilon,h}\{y^\varepsilon \in B_r(\varphi)\},$$

Thus

$$\varliminf_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \geq - \sup_{\psi \in B_r(\varphi) \cap L_2([0,T],L_2(du))} F(\psi,h).$$

$$\lim_{r \to 0} \varliminf_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \geq -F(\varphi,h) = -I(\varphi).$$

Lower bound

$$\mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} = \mathbb{E}^{\varepsilon,h} \frac{\mathbb{I}_{\{y^\varepsilon \in B_r(\varphi)\}}}{M_T^{\varepsilon,h}}$$

$$\geq \exp\left\{-\frac{1}{\varepsilon} \sup_{\psi \in B_r(\varphi) \cap L_2([0,T], L_2(du))} F(\psi, h)\right\} \mathbb{P}^{\varepsilon,h}\{y^\varepsilon \in B_r(\varphi)\},$$

Thus

$$\varliminf_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \geq -\sup_{\psi \in B_r(\varphi) \cap L_2([0,T], L_2(du))} F(\psi, h).$$

$$\lim_{r \to 0} \varliminf_{\varepsilon \to 0} \varepsilon \ln \mathbb{P}\{y^\varepsilon \in B_r(\varphi)\} \geq -F(\varphi, h) = -I(\varphi).$$

Lower bound

$$F(\varphi, h) = (h(T), \varphi(T))_{L_2(du)} - (h(0), \mathrm{id})_{L_2(du)} - \int_0^T (\dot{h}(s), \varphi(s))_{L_2(du)} ds - \frac{1}{2} \int_0^T \|pr_{\varphi(s)} h(s)\|^2_{L_2(du)} ds.$$

If $\varphi(s) \in L_2^{\uparrow\uparrow}(du)$ then

$$\|pr_{\psi_n(s)} h(s)\|^2_{L_2(du)} \to \|pr_{\varphi(s)} h(s)\|^2_{L_2(du)} = \|h(s)\|^2_{L_2(du)}$$

Thank you!