On Conditioning Brownian Particles to Coalesce

Vitalii Konarovskyi* † Victor Marx‡

October 20, 2020

Abstract

We consider a cylindrical Wiener process, interpreted as a system of independent Brownian particles starting from different points of the real line. In this paper, we study the conditional distribution of this system to the event that particles coalesce. After having introduced a notion of conditional distribution to a zero-probability event in a given direction of approximation, we prove that this conditional distribution coincides with the law of a modified massive Arratia flow, defined in [Kon17b]. In the case of finitely many particles, this result is independent of the direction of approximation.

Keywords: Regular conditional probability, modified massive Arratia flow, cylindrical Wiener process, coalescing Brownian motions

AMS MSC 2020: Primary 60E05, 60J90 Secondary 60G15, 60G44, 60K35

Acknowledgement. The authors are very grateful to Max von Renesse for very useful discussions and suggestions. The first author also thanks Andrey A. Dorogovtsev for his helpful comments and the interest to the studying problem

1 Introduction

The original motivation for this paper was the following simple problem. Consider two independent Brownian motions W_1 and W_2, starting at $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$, respectively, with same variance. What is the conditional distribution of (W_1, W_2) to the event that their paths coalesce, *i.e.* that $W_1(t) = W_2(t)$ for every t larger than the first meeting time τ? We will see

*Fakultät für Mathematik und Informatik, Universität Leipzig, Augustusplatz 10, 04109 Leipzig, Germany. E-mail: konarovskyi@gmail.com

†Institute of Mathematics of NAS of Ukraine, Tereschenkivska st. 3, 01024 Kiev, Ukraine

‡Fakultät für Mathematik und Informatik, Universität Leipzig, Augustusplatz 10, 04109 Leipzig, Germany. E-mail: marx@math.uni-leipzig.de

that it should be the law of the process (Y_1, Y_2) that is equal to (W_1, W_2) before time τ and such that $Y_1(t) = Y_2(t) = \frac{W_1(t)+W_2(t)}{2}$ for any $t \geqslant \tau$. We prove in this paper that the conditional distribution of any finite family $(W_1, W_2, \dots, W_n)$ of independent real-valued Brownian motions to the event that the paths coalesce is the law of the modified massive Arratia flow (MMAF), see Definition 1.3 below.

Moreover, that problem turns out to be more challenging in infinite dimension. We still justify that the conditional law of a cylindrical Wiener process in $L_2[0,1]$ starting at some non-decreasing function g to the event of coalescence is the law of a MMAF. But we pay the prize of having to investigate more carefully the notion of conditional law to a zero-probability event, allowing to define it only in some directions of approximation.

1.1 Conditional distribution to a zero-probability event

For the purpose of this paper - but possibly also for quite different uses - we introduce a definition of a conditional distribution along a direction, which extends the commomly-used notion of regular conditional probability (see e.g [IW89, Theorem I.3.3] and [Kal02, Theorem 6.3]).

Let $\mathbf{E}$ be a Polish space, $\mathcal{B}(\mathbf{E})$ denote the Borel σ-algebra on $\mathbf{E}$ and $\mathcal{P}(\mathbf{E})$ be the space of probability measures on $(\mathbf{E}, \mathcal{B}(\mathbf{E}))$ endowed with the topology of weak convergence.

In general, given a random element X in $\mathbf{E}$ and $C \in \mathcal{B}(\mathbf{E})$ such that $\mathbb{P}[X \in C] = 0$, defining the conditional probability $\mathbb{P}[X \in \cdot | X \in C]$ has no sense if we consider $\{X \in C\}$ as an isolated event (see e.g. Borel-Kolmogorov paradox). However, one can make a proper definition if C is given by $C = \mathrm{T}^{-1}(\{z_0\})$, where z_0 belongs to a metric space $\mathbf{F}$ and $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ is some measurable map. Let $p : \mathcal{B}(\mathbf{E}) \times \mathbf{F} \to [0,1]$ be a regular conditional probability of X given $\mathrm{T}(X)$, see Definition A.1 in appendix. By Proposition A.2, $p(\cdot, z)$ is well-defined for $\mathbb{P}^{\mathrm{T}(X)}$-almost every $z \in \mathbf{F}$, where $\mathbb{P}^{\mathrm{T}(X)}$ denotes the law of $\mathrm{T}(X)$. Thus the naive candidate $p(\cdot, z_0)$ to be the conditional distribution of X given $\{\mathrm{T}(X) = z_0\}$ is not well-defined in general. However, it becomes well-defined if e.g. $z \mapsto p(\cdot, z)$ is continuous at z_0, as a map from $\mathbf{F}$ to $\mathcal{P}(\mathbf{E})$, and if z_0 belongs to the support of $\mathbb{P}^{\mathrm{T}(X)}$.

When the continuity of $z \mapsto p(\cdot, z)$ is not obvious, we still can define a value of p at z_0, at least along some given sequence $\{\xi^n\}$ converging to z_0. To make the random element $p(\cdot, \xi^n)$ independent of the version of p, we should assume that the law of ξ^n is absolutely continuous with respect to $\mathbb{P}^{\mathrm{T}(X)}$. Then the value of p at z_0 is defined as the weak limit of $\{p(\cdot, \xi^n)\}_{n \geqslant 1}$, in the following sense:

Definition 1.1. Let $\{\xi^n\}_{n \geqslant 1}$ be a sequence of random elements in $\mathbf{F}$ such that

(B1) for each $n \geqslant 1$, the law of ξ^n is absolutely continuous with respect to the law of $\mathrm{T}(X)$;
(B2) $\{\xi^n\}_{n\geqslant 1}$ converges in distribution to z_0 in $\mathbf{F}$.

A probability measure ν on $(\mathbf{E}, \mathcal{B}(\mathbf{E}))$ is *the value of the conditional distribution of X to the event $\{\mathrm{T}(X) = z_0\}$ along the sequence $\{\xi^n\}$* if for every $f \in \mathcal{C}_b(\mathbf{E})$

$$\mathbb{E}\left[\int_{\mathbf{E}} f(x)p(\mathrm{d}x, \xi^n)\right] \to \int_{\mathbf{E}} f(x)\nu(\mathrm{d}x), \quad n \to \infty, \tag{1}$$

where p is a regular conditional probability of X given $\mathrm{T}(X)$. We denote this measure by $\nu = \mathrm{Law}_{\{\xi^n\}}(X|\mathrm{T}(X) = z_0)$.

In Section 2, we explain that the above definition generalizes the case where p is continuous at z_0 and that it is very close to the intuitive definition of $\mathbb{P}[X \in \cdot\, |X \in C]$ by approximation of the set C. Furthermore, we introduce in Section 2 a method to construct ν.

1.2 Definition of cylindrical Wiener process and of MMAF

We introduce here the main two probabilistic objects appearing in this paper. First, define a cylindrical Wiener process according to [GM11, Definition 2.5]:

Definition 1.2. The process $\mathcal{W}_t,\ t \geqslant 0$, defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ is an *$(\mathcal{F}_t)$-cylindrical Wiener process* (or shortly, *cylindrical Wiener process*) in a Hilbert space H starting at 0 if

i) for each $t \geqslant 0$, $\mathcal{W}_t : H \to L_2(\Omega, \mathcal{F}, \mathbb{P})$ is a linear map;
ii) for any $h \in H$, $\mathcal{W}_t(h)$, $t \geqslant 0$, is an $(\mathcal{F}_t)$-Brownian motion starting at 0;
iii) for any $h_1, h_2 \in H$ and $t \geqslant 0$, $\mathbb{E}\left[\mathcal{W}_t(h_1)\mathcal{W}_t(h_2)\right] = t(h_1, h_2)_H$.

For any $g \in H$, we say that $\mathcal{W}_t,\ t \geqslant 0$, is a *cylindrical Wiener process in H starting at g* if there is a cylindrical Wiener process $\eta_t,\ t \geqslant 0$, in H starting at 0 such that $\mathcal{W}_t(h) = (g, h)_H + \eta_t(h)$, $t \geqslant 0$, $h \in H$.

Second, we introduce the MMAF, already investigated in [Kon10, Kon14, Kon17b, Kon17a, Mar18, KvR19]. Let $D((0,1), \mathcal{C}[0,\infty))$ denote the space of càdlàg functions from $(0,1)$ to $\mathcal{C}([0,\infty), \mathbb{R})$. Let $g : [0,1] \to \mathbb{R}$ be a non-decreasing càdlàg function such that $\int_0^1 |g(u)|^p \mathrm{d}u < \infty$ for some $p > 2$.

Definition 1.3. A random element $\mathcal{Y} = \{\mathcal{Y}(u,t),\ u \in (0,1),\ t \in [0,\infty)\}$ in the space $D((0,1), \mathcal{C}[0,\infty))$ is called *modified massive Arratia flow* (shortly MMAF) starting at g if it satisfies the following properties

(E1) for all $u \in (0,1)$ the process $\mathcal{Y}(u,\cdot)$ is a continuous square-integrable martingale with respect to the filtration

$$\mathcal{F}_t^{\mathcal{Y}} = \sigma(\mathcal{Y}(v,s),\ v \in (0,1),\ s \leqslant t), \quad t \geqslant 0; \tag{2}$$

(E2) for all $u \in (0,1)$, $\mathcal{Y}(u,0) = g(u)$;

(E3) for all $u < v$ from $(0,1)$ and $t \geqslant 0$, $\mathcal{Y}(u,t) \leqslant \mathcal{Y}(v,t)$;

(E4) for all $u,v \in (0,1)$, the joint quadratic variation of $\mathcal{Y}(u,\cdot)$ and $\mathcal{Y}(v,\cdot)$ is

$$\langle \mathcal{Y}(u,\cdot), \mathcal{Y}(v,\cdot)\rangle_t = \int_0^t \frac{\mathbb{1}_{\{\tau_{u,v} \leqslant s\}}}{m(u,s)} \mathrm{d}s, \quad t \geqslant 0,$$

where $m(u,t) = \mathrm{Leb}\,\{v :\ \exists s \leqslant t,\ \mathcal{Y}(v,s) = \mathcal{Y}(u,s)\}$ and $\tau_{u,v} = \inf\{t : \mathcal{Y}(u,t) = \mathcal{Y}(v,t)\}$.

Intuitively, the massive particles $\mathcal{Y}(u,\cdot)$, for each $u \in (0,1)$, evolve like independent Brownian particles with diffusion rates inversely proportional to their masses, until two of them collide. When two particles meet, they coalesce and form a new particle with the mass equal to the sum of masses of the colliding particles.

Moreover, the random element $\mathcal{Y}$ can be identified with an $L_2^{\uparrow}$-valued process $\mathcal{Y}_t$, $t \geqslant 0$, where $L_2^{\uparrow}$ is the subset of $L_2[0,1]$ consisting of all functions which have non-decreasing versions. There exists a cylindrical Wiener process $\mathcal{W}$ in $L_2[0,1]$ starting at g such that

$$\mathcal{Y}_t = g + \int_0^t \mathrm{pr}_{\mathcal{Y}_s} \mathrm{d}\mathcal{W}_s, \quad t \geqslant 0, \tag{3}$$

where for any $f \in L_2^{\uparrow}$, pr_f is the orthogonal projection operator in $L_2[0,1]$ onto the subspace of $\sigma(f)$-measurable functions. Those results will be recalled with further details and references in Section 3.

1.3 Main result

Our main results consists in the construction of the following objects and in the following theorem.

(S1) We start from $\mathcal{Y}$, a MMAF starting at a strictly increasing map g.

(S2) Thus there exists a cylindrical Wiener process $\mathcal{W}$ in $L_2[0,1]$ starting at g satisfying (3). $\mathcal{Y}$ can be seen as the coalescing part of $\mathcal{W}$.

(S3) Given $\mathcal{X} = (\mathcal{Y}, \mathcal{W})$, we decompose $\mathcal{W}$ into $\mathcal{Y}$ and a non-coalescing part $\mathrm{T}(\mathcal{X})$, so that $\mathcal{W}$ is completely determined by $\mathcal{Y}$ and $\mathrm{T}(\mathcal{X})$. We postpone to Section 3.3 the precise definition of the map T. We are interested in the conditional distribution of $\mathcal{X}$ to the event $\{\mathrm{T}(\mathcal{X}) = 0\}$, which is the event where $\mathcal{W}$ coincides with its coalescing part $\mathcal{Y}$.

(S4) For every $n \geqslant 1$, ξ^n is defined as a sequence $\{\xi^n_j\}_{j\geqslant 1}$ of independent Ornstein-Uhlenbeck processes such that $\{\xi^n\}_{n\geqslant 1}$ converges to 0 in distribution and the law of ξ^n is absolutely continuous with respect to the law of $\mathrm{T}(\mathcal{X})$, which is the law of a sequence of independent standard Brownian motions.

Theorem 1.4. *The value of the conditional distribution of $\mathcal{X} = (\mathcal{Y}, \mathcal{W})$ to the event $\{\mathrm{T}(\mathcal{X}) = 0\}$ along $\{\xi^n\}$ is the law of $(\mathcal{Y}, \mathcal{Y})$.*

Our initial hope was to prove that result for any sequence $\{\xi^n\}$ satisfying (B1)-(B2), but unfortunately this seems to be not achievable and possibly even not true. Nevertheless, a sequence of Ornstein-Uhlenbeck processes is already a reasonable choice of $\{\xi^n\}$ satisfying (B1)-(B2). We refer to Theorem 3.12 for a more precise statement after having carefully defined T and $\{\xi^n\}_{n\geqslant 1}$ among others.

In brief, starting from a modified massive Arratia flow $\mathcal{Y}$, we are able to construct a cylindrical Wiener process $\mathcal{W}$ driving the evolution of $\mathcal{Y}$, and the conditional distribution of $\mathcal{W}$ to the event of coalescing paths along some direction is the law of $\mathcal{Y}$. Of course, this is only a partial answer to our initial question, since we are not able to start from a cylindrical Wiener process $\mathcal{W}$ and to recover the law of a MMAF. As we will see, this is possible in finite dimension. In infinite dimension, the additional difficulty comes from the fact that it is unknown - and seemingly a difficult problem - whether given $\mathcal{W}$, equation (3) admits a unique strong solution.

However, the characterization of MMAF as a conditional distribution of a cylindrical Wiener process to the event of coalescence, given by Theorem 1.4, is interesting. It explains e.g. the form of the rate function in the large deviation principle for the MMAF which is the restriction of the rate function of cylindrical Wiener process to the set of coalescing paths (see [Kon14, KvR19]).

1.4 Law of the coupling $(\mathcal{Y}, \mathcal{W})$

The following statement ensures that the law of a pair $(\mathcal{Y}, \mathcal{W})$ coupled by equation (3) is uniquely determined by the law of $\mathcal{Y}$.

Theorem 1.5. *Let $\mathcal{Y}_t$, $t \geqslant 0$, be a MMAF starting at g. Let $\mathcal{W}$ and $\widetilde{\mathcal{W}}$ be cylindrical Wiener processes in L_2 starting at g and such that $(\mathcal{Y}, \mathcal{W})$ and $(\mathcal{Y}, \widetilde{\mathcal{W}})$ satisfy equation* (3)*. Then* $\mathrm{Law}(\mathcal{Y}, \mathcal{W}) = \mathrm{Law}(\mathcal{Y}, \widetilde{\mathcal{W}})$.

Theorem 1.5 has an interest which is independent of the conditional distribution problem, but it is proved using the same techniques as for Theorem 1.4. Moreover, as a corollary, one can see that steps (S1) and (S2) in the statement of the main result can be replaced by starting from any pair $(\mathcal{Y}, \mathcal{W})$ coupled by (3), which is a stronger result.

1.5 Finite dimensional case

As a particular case of the above introduced method, we come up with a complete answer to our initial question in the finite dimensional case. Let $[n] := \{1, \dots, n\}$. Let $W_k(t),\ t \geqslant 0,\ k \in [n]$, be a family of independent Brownian motions starting at x_k^0, $k \in [n]$, with diffusion rates $\sigma_k^2 = \frac{1}{m_k}$, $k \in [n]$, where $x_1^0 \leqslant \ldots \leqslant x_n^0$ and $m_1 + \dots + m_n = 1$. Define

$$g := \sum_{k=1}^{n} x_k^0 \mathbb{1}_{\pi_k^0}, \tag{4}$$

where $\pi_k^0 = [a_{k-1}, a_k)$, $a_0 = 0$, and $a_k = a_{k-1} + m_k$, $k \in [n]$. Let $\mathcal{Y}$ be a MMAF starting at g. Then by the coalescing property of the MMAF, it is easily seen that there exists a unique family of processes $y_k(t)$, $t \geqslant 0$, $k \in [n]$, such that almost surely

$$\mathcal{Y}_t = \sum_{k=1}^{n} y_k(t) \mathbb{1}_{\pi_k^0}, \quad t \geqslant 0.$$

Moreover, y_k, $k \in [n]$, describe the evolution of the diffusion particles in the MMAF and satisfy properties similar to (E1)–(E4) of Definition 1.3 (see also properties (F1)-(F4) in [Kon17a]).

Theorem 1.6. *Let $X := (W_k(t))_{k=1}^n$, $t \geqslant 0$. Then the conditional distribution of X to the event $\{X \text{ coalesces}\}$[1] is the law of a MMAF $(y_k(t))_{k=1}^n$, $t \geqslant 0$, starting at $(x_k^0)_{k=1}^n$.*

We obtain that stronger result for several reasons. Mainly, we know that the law of a MMAF starting at a step function g is uniquely determined. That is, for any given g as in (4) and any X as in Theorem 1.6, there is a unique strong solution to equation (3). Moreover, the map T can now be more easily defined. E.g, in the case of our initial problem of two Brownian motions with the same variance, $\mathrm{T} : \mathcal{C}[0,\infty)^2 \to \mathcal{C}_0[0,\infty)$ is defined by

$$\mathrm{T}(x)(t) = \begin{cases} \frac{x_1(\tau+t) - x_2(\tau+t)}{2}, & \text{if } \tau < \infty, \\ 0, & \text{if } \tau = \infty, \end{cases} \quad t \geqslant 0,$$

where $\tau = \inf\{t \geqslant 0 : \ x_1(t) = x_2(t)\}$. Furthermore, the regular conditional probability p of X given $\mathrm{T}(X)$ is now continuous at 0, which allows us to

[1] see Section 5 for the precise definition of this set.

define a conditional distribution of X to $\{\mathrm{T}(X)=0\}$ independently of the direction.

Content of the paper. In Section 2, we propose a method to effectively construct a conditional distribution according to Definition 1.1. In Section 3, we recall needed properties of the MMAF and we define the non-coalescing map T, using a construction of an orthonormal basis in $L_2[0,1]$ which is tailored for the MMAF. Finally in that section, we state the main result in Theorem 3.12. Sections 4, 5 and 6 are devoted to the proofs of Theorem 3.12, Theorem 1.6 and Theorem 1.5, respectively.

2 On conditional distributions

2.1 On the definition of conditional distribution

Definition 1.1 is consistent with the continuous case. Indeed, if $z \mapsto p(\cdot, z)$ is continuous at z_0, then by the continuous mapping theorem $p(\cdot, z_0) = \mathrm{Law}_{\{\xi^n\}}(X|\mathrm{T}(X)=z_0)$ for any sequence $\{\xi^n\}_{n\geqslant 1}$ satisfying $(B1)$ and $(B2)$. Actually, it is an equivalence, as the following lemma shows.

Lemma 2.1. *Let z_0 belong to the support of $\mathbb{P}^{\mathrm{T}(X)}$. There exists a probability measure ν such that $\nu = \mathrm{Law}_{\{\xi^n\}}(X|\mathrm{T}(X)=z_0)$ along any sequence $\{\xi^n\}_{n\geqslant 1}$ satisfying (B1) and (B2) if and only if there exists a version of p which is continuous at $z_0 \in \mathbf{F}$. In this case, ν is equal to the value of the continuous version of p at z_0.*

We postpone the proof of the lemma to Section A.2 in the appendix.

Remark 2.2. Definition 1.1 extends the intuitive definition of the conditional distribution of X given $\{X \in C\}$ as the weak limit

$$\mathbb{P}\left[X \in \cdot\ | X \in C\right] = \lim_{\varepsilon \to 0} \mathbb{P}\left[X \in \cdot\ | X \in C_\varepsilon\right],$$

where C denotes a closed subset of $\mathbf{E}$ and C_ε is its ε-extension, *i.e.* $C_\varepsilon = \{x \in \mathbf{E} :\ d_{\mathbf{E}}(C,x) < \varepsilon\}$. We assume $\mathbb{P}\left[X \in C_\varepsilon\right] > 0$ for any $\varepsilon > 0$. Then T can be defined by $\mathrm{T}(x) := d_{\mathbf{E}}(C,x)$. We note that $\{X \in C\} = \{\mathrm{T}(X)=0\}$ and $\{X \in C_\varepsilon\} = \{\mathrm{T}(X) < \varepsilon\}$ for all $\varepsilon > 0$. The sequence $\{\xi^n\}$ could then be defined by

$$\mathbb{P}\left[\xi^n \in A\right] = \frac{1}{\mathbb{P}\left[\mathrm{T}(X) < \frac{1}{n}\right]} \int_A \mathbb{1}_{\{x<\frac{1}{n}\}} \mathbb{P}^{\mathrm{T}(X)}(\mathrm{d}x), \quad A \in \mathcal{B}(\mathbf{E}).$$

One can easily check that $\{\xi^n\}$ satisfies conditions (B1) and (B2) with $z_0 = 0$, and that

$$\mathbb{E}\left[\int_{\mathbf{E}} f(x) p(\mathrm{d}x, \xi^n)\right] = \int_{\mathbf{E}} f(x) \mathbb{P}\left[X \in \mathrm{d}x | X \in C_{1/n}\right].$$

Therefore, the weak limit of $\mathbb{P}\left[X \in \cdot \,|X \in C_{1/n}\right]$ coincides with the measure $\mathrm{Law}_{\{\xi^n\}}(X|\mathrm{T}(X) = 0)$.

2.2 Method of construction of conditional distribution

We introduce here an idea to build a conditional distribution of X given $\{\mathrm{T}(X) = z_0\}$ along a sequence $\{\xi^n\}$. The idea is to split the random element X into two independent parts, Y and Z, so that Z has the same law as $\mathrm{T}(X)$.

More precisely, we assume that there exists a quadruple $(\mathbf{G}, \Psi, Y, Z)$ satisfying the following conditions

(P1) $\mathbf{G}$ is a measurable space;

(P2) Y and Z are independent random elements in $\mathbf{G}$ and $\mathbf{F}$, respectively;

(P3) $\Psi : \mathbf{G} \times \mathbf{F} \to \mathbf{E}$ is a measurable map such that $\mathrm{T}(\Psi(Y, Z)) = Z$ a.s.;

(P4) X and $\Psi(Y, Z)$ have the same distribution.

Proposition 2.3. *Let $(\mathbf{G}, \Psi, Y, Z)$ be a quadruple satisfying (P1)-(P4). The map p defined by*

$$p(A, z) := \mathbb{P}\left[\Psi(Y, z) \in A\right], \quad A \in \mathcal{B}(\mathbf{E}),\ z \in \mathbf{F} \tag{5}$$

is a regular conditional probability of X given $\mathrm{T}(X)$.

Moreover, if $\{\xi^n\}_{n\geqslant 1}$ is a sequence of random elements in $\mathbf{F}$ independent of Y and satisfying (B1) and (B2) of Definition 1.1, then $\Psi(Y, \xi^n)$ converges in distribution to the measure $\mathrm{Law}_{\{\xi^n\}}(X|\mathrm{T}(X) = z_0)$.

Proof. Since Ψ is measurable, p defined by (5) satisfies properties (R1) and (R2) of Definition A.1. Moreover, for every $A \in \mathcal{B}(\mathbf{E})$ and $B \in \mathcal{B}(\mathbf{F})$

$$\begin{aligned}
\mathbb{P}\left[X \in A,\ \mathrm{T}(X) \in B\right] &\overset{(P4)}{=} \mathbb{P}\left[\Psi(Y, Z) \in A,\ \mathrm{T}(\Psi(Y, Z)) \in B\right] \\
&\overset{(P3)}{=} \mathbb{P}\left[\Psi(Y, Z) \in A,\ Z \in B\right] \\
&\overset{(P2)}{=} \int_B p(A, z)\mathbb{P}^Z(\mathrm{d}z).
\end{aligned}$$

Furthermore, since X and $\Psi(Y, Z)$ have the same law, $\mathrm{T}(X)$ and $Z = \mathrm{T}(\Psi(Y, Z))$ have the same law too, so $\mathbb{P}^Z = \mathbb{P}^{\mathrm{T}(X)}$. This concludes the proof of (R3).

Let $f \in \mathcal{C}_b(\mathbf{E})$. By (5) and Proposition A.2, we know that for any regular conditional probability p of X given $\mathrm{T}(X)$, the equality $\int_{\mathbf{E}} f(x)p(\mathrm{d}x, z) = \mathbb{E}\left[f(\Psi(Y, z))\right]$ holds for $\mathbb{P}^{\mathrm{T}(X)}$-almost all $z \in \mathbf{F}$. It also holds $\mathbb{P}^{\xi^n}$-almost everywhere by Property (B1). By independence of ξ^n and Y and Fubini's theorem,

$$\mathbb{E}\left[f(\Psi(Y,\xi^n))\right] = \int_{\mathbf{F}} \mathbb{E}\left[f(\Psi(Y,z))\right] \mathbb{P}^{\xi^n}(\mathrm{d}z) = \int_{\mathbf{F}} \int_{\mathbf{E}} f(x) p(\mathrm{d}x, z) \mathbb{P}^{\xi^n}(\mathrm{d}z).$$

By (1), the last term tends to $\int_{\mathbf{E}} f(x)\nu(\mathrm{d}x)$, where $\nu = \mathrm{Law}_{\{\xi^n\}}(X|\mathrm{T}(X) = z_0)$. This concludes the proof of the convergence in distribution. $\square$

We show in appendix, see Section A.3, how to apply this method to the well-known Brownian bridge.

3 Statement of the main result

In Section 1.3, we announced the construction of several objects, including a modified massive Arratia flow (MMAF) and a non-coalescing remainder map T. The main part of this construction will be the definition of an orthonormal basis of $L_2[0,1]$ which is tailored for the MMAF. In this section, we will follow the steps (S1)-(S4) of Section 1.3 and finally, we will state again Theorem 1.4 in a more precise form, see Theorem 3.12.

3.1 MMAF and set of coalescing paths

In this section, we define the set **Coal** of coalescing trajectories in an infinite-dimensional space and we recall important properties of the MMAF introduced in Definition 1.3 to show that it takes values almost surely in **Coal**. Since they are not the central issue of this paper, the proofs of this section will be succinct, but we will refer to previous works or to the appendix for the detailled versions.

Fix g belonging to the set $L_{2+}^{\uparrow}$ that consists of all non-decreasing càdlàg functions $g:(0,1)\to\mathbb{R}$ satisfying $\int_0^1 |g(u)|^{2+\varepsilon}\mathrm{d}u < \infty$ for some $\varepsilon > 0$.

Let St denote the set of non-decreasing step functions $f:[0,1)\to\mathbb{R}$ of the form

$$f = \sum_{j=1}^{n} f_j \mathbb{1}_{\pi_j}, \tag{6}$$

where $n \geqslant 1$, $f_1 < \dots < f_n$ and $\{\pi_1, \dots \pi_n\}$ is an ordered partition of $[0,1)$ into half-open intervals of the form $\pi_j = [a_j, b_j)$. The natural number n is denoted by $N(f)$ and is by definition finite for every $f \in \mathrm{St}$. Recall that $L_2 := L_2[0,1]$ and that $L_2^{\uparrow}$ is the subset of L_2 consisting of all functions which have non-decreasing versions.

Definition 3.1. We define **Coal** as the set of functions y from $\mathcal{C}([0,\infty), L_2^{\uparrow})$ such that

(G1) y has a version in $D((0,1),\mathcal{C}[0,\infty))$, the space of càdlàg functions from $(0,1)$ to $\mathcal{C}([0,\infty),\mathbb{R})$;

(G2) $y_0 = g$;

(G3) for each $t > 0$, $y_t \in \mathrm{St}$;

(G4) for each $u, v \in (0,1)$ and $s \geqslant 0$, $y_s(u) = y_s(v)$ implies $y_t(u) = y_t(v)$ for every $t \geqslant s$;

(G5) $t \mapsto N(y_t)$, $t \geqslant 0$, is a càdlàg non-increasing integer-valued function with jumps of height one and which is constant equal to 1 for sufficiently large time.

We can interpret y as a deterministic particle system, where $y_t(u)$, $t \geqslant 0$, describes the trajectory of a particle labeled by u. Condition (G3) means that there is only a finite number of particles at each positive time. By Condition (G4), two particles coalesce when they meet. Moreover, by Condition (G5), there can be at most one coalescence at each time, and the number of particles is equal to one for large time.

Note that, according to Lemma B.2 in appendix, the set **Coal** is measurable in $\mathcal{C}([0,\infty), L_2^{\uparrow})$. We will also consider **Coal** as a metric subspace of $\mathcal{C}([0,\infty), L_2^{\uparrow})$.

Recall the following existence property of modified massive Arratia flow.

Proposition 3.2. *Let $g \in L_{2+}^{\uparrow}$. There exists a MMAF starting at g.*

Proof. See [Kon17a, Theorem 1.1]. □

Remark 3.3. However it is not known if properties (E1)-(E4) uniquely determine the distribution of a MMAF starting at g, except the case where $g \in \mathrm{St}$ (see e.g [Kon17a, Proposition 3.3]).

Equivalently, we may also define a MMAF as an $L_2^{\uparrow}$-valued process, in the following sense. For every $f \in L_2^{\uparrow}$, pr_f denotes the orthogonal projection operator in L_2 onto the subspace of $\sigma(f)$-measurable functions.

Lemma 3.4. *Let $g \in L_{2+}^{\uparrow}$ and $\{\mathcal{Y}(u,t),\ u \in (0,1),\ t \in [0,\infty)\}$ be a MMAF starting at g. Then the process $\mathcal{Y}_t$, $t \geqslant 0$, defined by $\mathcal{Y}_t := \mathcal{Y}(\cdot,t)$, $t \geqslant 0$, satisfies*

(M1) $\mathcal{Y}_t$, $t \geqslant 0$, is a continuous $L_2^{\uparrow}$-valued process with $\mathbb{E}\left[\|\mathcal{Y}_t\|_{L_2}^2\right] < \infty$, $t \geqslant 0$;

(M2) for every $h \in L_2$ the L_2-inner product $(\mathcal{Y}_t, h)_{L_2}$, $t \geqslant 0$, is a continuous square integrable martingale with respect to the filtration generated by $\mathcal{Y}_t$, $t \geqslant 0$, that trivially coincides with $(\mathcal{F}_t^{\mathcal{Y}})_{t\geqslant 0}$;

(M3) the joint quadratic variation of $(\mathcal{Y}_t, h_1)_{L_2}$*,* $t \geqslant 0$*, and* $(\mathcal{Y}_t, h_2)_{L_2}$*,* $t \geqslant 0$*, equals* $\langle(\mathcal{Y}_\cdot, h_1)_{L_2}, (\mathcal{Y}_\cdot, h_2)_{L_2}\rangle_t = \int_0^t (\mathrm{pr}_{\mathcal{Y}_s} h_1, h_2)_{L_2} \mathrm{d}s$*,* $t \geqslant 0$*.*

Furthermore, if a process $\mathcal{Y}_t$*,* $t \geqslant 0$*, starting at* g *satisfies (M1)-(M3), then there exists a MMAF* $\{\mathcal{Y}(u,t),\ u \in (0,1),\ t \in [0,\infty)\}$ *such that* $\mathcal{Y}_t = \mathcal{Y}(\cdot, t)$ *in* L_2 *a.s. for all* $t \geqslant 0$*.*

Proof. The first part of the statement follows directly from Lemma B.3 in appendix, for Property (M1), and from [KvR19, Lemma 3.1], for properties (M1) and (M2). As regards the second part of the lemma, it is proved in [Kon17a, Theorem 6.4]. □

According to Lemma 3.4, we may identify the modified massive Arratia flow $\{\mathcal{Y}(u,t),\ u \in (0,1),\ t \in [0,\infty)\}$ and the $L_2^{\uparrow}$-valued martingale $\mathcal{Y}_t$, $t \geqslant 0$, using both notations for the same object.

Lemma 3.5. *The process* $\mathcal{Y}_t$*,* $t \geqslant 0$*, belongs almost surely to* **Coal***.*

Proof. By construction, the process satisfies properties (G1) and (G2). Properties (G3) and (G4) were proved in [Kon17a], propositions 6.2 and 2.3 ibid, respectively. Property (G5) is stated in Lemma B.4 in appendix. □

3.2 MMAF and cylindrical Wiener process

The goal of this section is to explain how to construct, given a MMAF $\mathcal{Y}$, a cylindrical Wiener process $\mathcal{W}$ starting at the same point which satisfies equation (3), in order to complete step (S2) of Section 1.3.

For any $f \in L_2^{\uparrow}$, let $L_2(f)$ denote the subspace of L_2 consisting of $\sigma(f)$-measurable functions. In particular if f is of the form (6), then $L_2(f)$ consists of all step functions which are constant on each π_j. For any $f \in L_2^{\uparrow}$, let pr_f (resp. $\mathrm{pr}_f^{\perp}$) denote the orthogonal projection in L_2 onto $L_2(f)$ (resp. onto $L_2(f)^{\perp}$). Moreover, for any progressively measurable process κ_t, $t \geqslant 0$, in L_2 and for any cylindrical Wiener process B in L_2, we denote

$$\int_0^t \kappa_s \cdot \mathrm{d}B_s := \int_0^t K_s \mathrm{d}B_s.$$

where $K_t = (\kappa_t, \cdot)_{L_2}$, $t \geqslant 0$.

Proposition 3.6. *Let* $g \in L_{2+}^{\uparrow}$ *and* $\mathcal{Y}_t$*,* $t \geqslant 0$*, be a MMAF starting at* g*. Let* B_t*,* $t \geqslant 0$*, be a cylindrical Wiener process in* L_2 *starting at* 0 *defined on the same probability space and independent of* $\mathcal{Y}$*. Then the process* $\mathcal{W}_t$*,* $t \geqslant 0$*, defined by*

$$\mathcal{W}_t := \mathcal{Y}_t + \int_0^t \mathrm{pr}_{\mathcal{Y}_s}^{\perp} \mathrm{d}B_s, \quad t \geqslant 0, \tag{7}$$

is a cylindrical Wiener process in L_2 starting at g, where equality (7) *should be understood*[2] *as follows:*

$$\mathcal{W}_t(h) := (\mathcal{Y}_t, h)_{L_2} + \int_0^t \mathrm{pr}^{\perp}_{\mathcal{Y}_s} h \cdot \mathrm{d}B_s, \quad t \geqslant 0, \ h \in L_2.$$

Moreover, $(\mathcal{Y}, \mathcal{W})$ satisfies equation (3).

Proof. It follows from Property (M3) and from [GM11, Corollary 2.2] that there exists a cylindrical Wiener process $\tilde{B}$ in L_2 starting at 0 (possibly on an extended probability space also denoted by $(\Omega, \mathcal{F}, \mathbb{P})$) such that

$$\mathcal{Y}_t = g + \int_0^t \mathrm{pr}_{\mathcal{Y}_s} \mathrm{d}\tilde{B}_s, \quad t \geqslant 0.$$

Moreover, we may assume that $\tilde{B}$ is independent of B. It is trivial that the map $\mathcal{W}_t : L_2 \to L_2(\Omega, \mathcal{F}, \mathbb{P})$ defined by (7) is linear. Let $(\mathcal{F}_t)_{t \geqslant 0}$ be the natural filtration generated by $\tilde{B}$ and B. Let us check that $\mathcal{W}_t(h)$, $t \geqslant 0$, is an $(\mathcal{F}_t)$-Brownian motion starting at $(g, h)_{L_2}$ with diffusion rate $\|h\|^2_{L_2}$ for any $h \in L_2$. Using the independence of $\tilde{B}$ and B, we have that $\mathcal{W}_t(h)$, $t \geqslant 0$, is a continuous $(\mathcal{F}_t)$-martingale with quadratic variation

$$\langle \mathcal{W}(h) \rangle_t = \int_0^t \| \mathrm{pr}_{\mathcal{Y}_s} h\|^2_{L_2} \mathrm{d}s + \int_0^t \| \mathrm{pr}^{\perp}_{\mathcal{Y}_s} h\|^2_{L_2} \mathrm{d}s = \int_0^t \|h\|^2_{L_2} \mathrm{d}s = t\|h\|^2_{L_2}.$$

This implies ii) and iii) of Definition 1.2 by Lévy's characterization and by the polarization equality, respectively.

Moreover, for every $h \in L_2$ and $t \geqslant 0$,

$$\begin{aligned} \int_0^t \mathrm{pr}_{\mathcal{Y}_s} h \cdot \mathrm{d}\mathcal{W}_s &= \int_0^t (\mathcal{Y}_s, \mathrm{pr}_{\mathcal{Y}_s} h)_{L_2} \mathrm{d}s + \int_0^t \mathrm{pr}^{\perp}_{\mathcal{Y}_s} \circ \mathrm{pr}_{\mathcal{Y}_s} h \cdot \mathrm{d}\mathcal{B}_s \\ &= \int_0^t (\mathcal{Y}_s, h)_{L_2} \mathrm{d}s = (\mathcal{Y}_t, h)_{L_2} - (g, h)_{L_2}. \end{aligned}$$

Therefore $\mathcal{Y}_t = g + \int_0^t \mathrm{pr}_{\mathcal{Y}_s} \mathrm{d}\mathcal{W}_s$, which is equality (3). □

Note that it is not obvious whether each cylindrical Wiener process $\mathcal{W}$ in L_2 starting at g and satisfying (3) is necessary of the form (7). Actually, this is the result of Theorem 1.5 and will be proved in Section 6.

[2] The process $\mathrm{pr}^{\perp}_{\mathcal{Y}_t}$, $t \geqslant 0$, does not take values in the space of Hilbert-Schmidt operators in L_2. Therefore, the integral $\int_0^t \mathrm{pr}^{\perp}_{\mathcal{Y}_s} \mathrm{d}B_s$ is not well-defined but $h \mapsto \int_0^t \mathrm{pr}^{\perp}_{\mathcal{Y}_s} h \cdot \mathrm{d}B_s$ is.

3.3 Construction of non-coalescing remainder map

Up to now and until the end of Section 4, we fix a strictly increasing function g in $L_{2+}^{\uparrow}$ and $\mathcal{X} := (\mathcal{Y}, \mathcal{W})$, where $\mathcal{Y}_t$, $t \geqslant 0$, is a modified massive Arratia flow starting at g and $\mathcal{W}_t$, $t \geqslant 0$, is defined by (7). In particular, the assumption on g implies that $L_2(g) = L_2$. In this section, we consider step (S3) of Section 1.3.

Let us introduce for every $y \in \mathbf{Coal}$ the corresponding coalescence times:

$$\tau_k^y := \inf\{t \geqslant 0 : \ N(y_t) \leqslant k\}, \quad k \geqslant 0. \tag{8}$$

Since g is a strictly increasing function, one has that $N(g) = +\infty$, and therefore, the family $\{\tau_k^y,\ k \geqslant 0\}$ is strictly decreasing for all $y \in \mathbf{Coal}$, i.e.

$$0 < \cdots < \tau_2^y < \tau_1^y < \tau_0^y = +\infty,$$

by Condition (G5).

Now we are going to define an orthonormal basis $\{e_k^y,\ k \geqslant 0\}$ in L_2 which depends on $y \in \mathbf{Coal}$. Since y_t, $t \geqslant 0$, is an L_2-valued continuous function and $L_2(g) = L_2$ due to the strong increase of g, it is easily seen that the closure of $\bigcup_{k=1}^{\infty} L_2(y_{\tau_k^y})$ coincides with L_2. Let H_k^y be the orthogonal complement of $L_2(y_{\tau_k^y})$ in L_2, $k \geqslant 1$.

Lemma 3.7. *For every $y \in$* **Coal** *there exists a unique orthonormal basis $\{e_l^y,\ l \geqslant 0\}$ of L_2 such that*

1) the family $\{e_l^y,\ 0 \leqslant l < k\}$ is a basis of $L_2(y_{\tau_k^y})$ for each $k \geqslant 1$;

2) $(e_l^y, \mathbb{1}_{[0,u]})_{L_2} \geqslant 0$ for every $u \in (0,1)$.

Moreover, the family $\left\{e_l^y,\ l \geqslant k\right\}$ is a basis of H_k^y for each $k \geqslant 1$.

In other words, the map $t \mapsto \mathrm{pr}_{y_t}$ is a projection map onto a subspace which decreases from exactly one dimension whenever a coalescence of y occurs, and the basis $\{e_l^y,\ l \geqslant 0\}$ is adapted to that decreasing sequence of subspaces.

Proof. Let us construct the family $\{e_k^y,\ k \geqslant 0\}$ explicitly. Since $y_{\tau_1^y}$ is constant on $[0,1]$, the only choice is $e_0^y = \mathbb{1}_{[0,1]}$.

We say that an interval I is a step of a map f if f is constant on I but not constant on any interval strictly larger than I. At time τ_k^y a coalescence occurs. So there exist $a < b < c$ such that $[a,b)$ and $[b,c)$ are steps of $y_{\tau_{k+1}^y}$, and $[a,c)$ is a step of $y_{\tau_k^y}$. We call b *the coalescence point* of $y_{\tau_k^y}$. The only possible choice for e_k^y so that it has norm 1, it belongs to $L_2(y_{\tau^y_{k+1}})$, it is orthogonal to every element of $L_2(y_{\tau^y_k})$ and it satisfies Condition 2) is:

$$e_k^y = \frac{1}{\sqrt{c-a}}\left(\sqrt{\frac{c-b}{b-a}}\mathbb{1}_{[a,b)} - \sqrt{\frac{b-a}{c-b}}\mathbb{1}_{[b,c)}\right). \tag{9}$$

Since $\overline{\bigcup_{k=1}^{\infty} L_2(y_{\tau^y_k})} = L_2$, we get that $\{e_k^y,\ k \geqslant 0\}$ form a basis of L_2.

The last part of the statement follows from the fact that for each $k \geqslant 1$, $H_k^y = L_2(y_{\tau^y_k})^{\perp}$. □

Remark 3.8. The construction of the basis $\left\{e_k^y,\ k \geqslant 0\right\}$ in the above proof easily implies that the map $\mathbf{Coal} \ni y \mapsto e_k^y \in L_2$ is measurable for any $k \geqslant 0$, where **Coal** is endowed with the induced topology of $\mathcal{C}([0,\infty), L_2^{\uparrow})$. Moreover, by (9), for every $k \geqslant 1$, e_k^y is uniquely determined by $y_{\cdot \wedge \tau_k^y}$.

According to step (S3), given $\mathcal{X} = (\mathcal{Y}, \mathcal{W})$, we will define now the non-coalescing part $\mathrm{T}(\mathcal{X})$ of $\mathcal{W}$. Note that $\tau_k^{\mathcal{Y}}$ are $(\mathcal{F}_t^{\mathcal{Y}})$-stopping times for all $k \geqslant 0$, where $(\mathcal{F}_t^{\mathcal{Y}})_{t\geqslant 0}$ is the complete right-continuous filtration generated by the MMAF $\mathcal{Y}$. Furthermore, Remark 3.8 yields that $e_k^{\mathcal{Y}}$ is an $\mathcal{F}^{\mathcal{Y}}_{\tau_k^{\mathcal{Y}}}$-measurable random element in L_2. *To simplify the notation, we will write e_k and τ_k instead of $e_k^{\mathcal{Y}}$ and $\tau_k^{\mathcal{Y}}$, respectively.*

Recall that $\mathcal{W}$ is defined by equality (7). In particular, the real-valued process $\mathcal{W}_t(e_k)$, $t \geqslant 0$, satisfies:

$$\mathcal{W}_t(e_k) = (\mathcal{Y}_t, e_k)_{L_2} + \int_0^t \mathbb{1}_{\{s \geqslant \tau_k\}} e_k \cdot \mathrm{d}B_s,$$

because $\mathrm{pr}^{\perp}_{\mathcal{Y}_s} e_k = \mathbb{1}_{\{s\geqslant\tau_k\}} e_k$. By construction of e_k in Lemma 3.7, $(\mathcal{Y}_t, e_k)_{L_2}$ vanishes for all $t \geqslant \tau_k$. Thus we note that for $t \in [0, \tau_k]$, $\mathcal{W}_t(e_k) = (\mathcal{Y}_t, e_k)_{L_2}$ and that $\mathcal{W}_{\tau_k}(e_k) = 0$, whereas for $t \geqslant \tau_k$, $\mathcal{W}_t(e_k) = B_t(e_k) - B_{\tau_k}(e_k)$. Since B is independent of $\mathcal{Y}$ and thus of e_k, $B_t(e_k)$ is well-defined by $B_t(e_k) = \int_0^t e_k \cdot \mathrm{d}B_s$, $t \geqslant 0$. To recap, in space direction e_k, the projection of $\mathcal{W}$ is equal to the projection of its coalescing part $\mathcal{Y}$ before stopping time τ_k, and is equal to the projection of a noise B which is independent of $\mathcal{Y}$ after τ_k. Therefore, we define formally $\xi = \mathrm{T}(\mathcal{X}) = \mathrm{T}(\mathcal{Y}, \mathcal{W})$ as follows

$$\xi_t = \sum_{k=1}^{\infty} e_k \mathcal{W}_{t+\tau_k}(e_k), \quad t \geqslant 0.$$

More rigorously[3], we define ξ_t as a map from the Hilbert space $L_2^0 := L_2 \ominus$

[3] Similarly as for the cylindrical Wiener process $\mathcal{W}$, ξ can not be defined as a random process taking values in L_2.

$\operatorname{span}\{\mathbb{1}_{[0,1]}\}$ to $L_2(\Omega)$. We set

$$\xi_t(h) := \sum_{k=1}^{\infty} (e_k, h)_{L_2} \mathcal{W}_{t+\tau_k}(e_k), \quad t \geqslant 0, \quad h \in L_2^0. \tag{10}$$

Proposition 3.9. *For every $h \in L_2^0$ the sum* (10) *converges almost surely in $\mathcal{C}[0,\infty)$. Moreover, ξ_t, $t \geqslant 0$, is a cylindrical Wiener process in L_2^0 starting at* 0 *that is independent of the MMAF $\mathcal{Y}$.*

In order to prove the above statement, we start with the following lemma.

Lemma 3.10. *The processes $\mathcal{W}_{\cdot+\tau_k}(e_k)$, $k \geqslant 1$, are independent standard Brownian motions that do not depend on the MMAF $\mathcal{Y}$.*

Proof. Let us denote

$$\eta_k(t) := \mathcal{W}_{t+\tau_k}(e_k) = B_{t+\tau_k}(e_k) - B_{\tau_k}(e_k), \quad t \geqslant 0, \quad k \geqslant 1. \tag{11}$$

We fix $n \geqslant 1$ and show that the processes $\mathcal{Y}$, η_k, $k \in [n]$, are independent and that η_k, $k \in [n]$, are standard Brownian motions. Let

$$F_0 : \mathcal{C}([0,\infty), L_2^{\uparrow}) \to \mathbb{R}, \quad F_k : \mathcal{C}[0,\infty) \to \mathbb{R}, \quad k \in [n],$$

be bounded measurable functions. By strong Markov property of B and the independence of B and $\mathcal{Y}$, $B_{\cdot+\tau_k} - B_{\tau_k}$ is also independent of $\mathcal{Y}$. Moreover for every $y \in \mathbf{Coal}$,

$$\eta_k^y(t) := B_{t+\tau_k^y}(e_k^y) - B_{\tau_k^y}(e_k^y), \quad t \geqslant 0, \quad k \in [n],$$

are independent standard Brownian motions. Therefore, we can compute

$$\begin{aligned}
\mathbb{E}\left[F_0(\mathcal{Y}) \prod_{k=1}^{n} F_k(\eta_k)\right] &= \mathbb{E}\left[\mathbb{E}\left[F_0(\mathcal{Y}) \prod_{k=1}^{n} F_k(\eta_k) \Big| \mathcal{Y}\right]\right] \\
&= \mathbb{E}\left[F_0(\mathcal{Y}) \,\mathbb{E}\left[\prod_{k=1}^{n} F_k(\eta_k^y)\right]\Bigg|_{y=\mathcal{Y}}\right] \\
&= \mathbb{E}\left[F_0(\mathcal{Y}) \,\mathbb{E}\left[\prod_{k=1}^{n} F_k(w_k)\right]\Bigg|_{y=\mathcal{Y}}\right] = \mathbb{E}\left[F_0(\mathcal{Y})\right] \prod_{k=1}^{n} \mathbb{E}\left[F_k(w_k)\right],
\end{aligned}$$

where w_k, $k \in [n]$, are independent standard Brownian motions that do not depend on $\mathcal{Y}$. This completes the proof of the lemma. □

Proof of Proposition 3.9. Let $h \in L_2^0$ and $y \in \mathbf{Coal}$ be fixed. For every $n \in \mathbb{N}$ we define

$$M_t^{y,n}(h) := \sum_{k=1}^{n} (e_k^y, h)_{L_2} \eta_k(t), \quad t \geqslant 0,$$

where η_k, $k \geqslant 1$, are defined by (11). By Lemma 3.10, η_k, $k \geqslant 1$, are independent standard Brownian motions, hence $M_t^{y,n}(h)$, $t \geqslant 0$, is a continuous square-integrable martingale with respect to the filtration $(\mathcal{F}_t^\eta)_{t\geqslant 0}$ generated by η_k, $k \geqslant 1$, with quadratic variation

$$\langle M^{y,n}(h)\rangle_t = \sum_{k=1}^{n}(e_k^y, h)_{L_2}^2 t, \quad t \geqslant 0.$$

Moreover, for each $T > 0$ the sequence of processes $\{M^{y,n}(h)\}_{n\geqslant 1}$ restricted to the interval $[0, T]$ converges in $L_2(\Omega, \mathcal{C}[0, T])$. Indeed, for each $m < n$, by Doob's inequality

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]} |M_t^{y,n}(h) - M_t^{y,m}(h)|^2\right] &= \mathbb{E}\left[\sup_{t\in[0,T]}\left|\sum_{k=m+1}^{n}(e_k^y, h)_{L_2}\eta_k(t)\right|^2\right] \\ &\leqslant 4\sum_{k=m+1}^{n}(e_k^y, h)_{L_2}^2 T,\end{aligned}$$

The sum $\sum_{k=1}^n (e_k^y, h)_{L_2}^2$ converges to $\|h\|_{L_2}^2$ because $\{e_k^y,\ k \geqslant 1\}$ is an orthonormal basis of L_2^0. Thus, $\{M^{y,n}(h)\}_{n\geqslant 1}$ is a Cauchy sequence in $L_2(\Omega, \mathcal{C}[0, T])$, and hence, it converges to a limit denoted by $M^y(h) = \sum_{k=1}^\infty (e_k^y, h)_{L_2}\eta_k$. Trivially, $M_t^y(h)$ can be well-defined for all $t \geqslant 0$, and, by [CE05, Lemma B.11], $M_t^y(h)$, $t \geqslant 0$, is a continuous square-integrable $(\mathcal{F}_t^\eta)$-martingale with quadratic variation $\langle M^y(h)\rangle_t = \lim_{n\to\infty}\langle M^{y,n}(h)\rangle_t = \|h\|_{L_2}^2 t$, $t \geqslant 0$.

Remark that $\sum_{k=1}^\infty (e_k^y, h)_{L_2}\eta_k$ is a sum of independent random elements in $\mathcal{C}[0, T]$. Hence, by Itô-Nisio's Theorem [IN68, Theorem 3.1], the sequence $\{M^{y,n}(h)\}_{n\geqslant 1}$ converges almost surely to $M^y(h)$ in $\mathcal{C}[0, T]$ for every $T > 0$, and therefore, in $\mathcal{C}[0, \infty)$. Recall that by Lemma 3.10, the sequence $\{\eta_k\}_{k\geqslant 1}$ is independent of $\mathcal{Y}$, and by Lemma 3.5, $\mathcal{Y}$ belongs to **Coal** almost surely. Then $\sum_{k=1}^\infty (e_k, h)_{L_2}\eta_k$ also converges almost surely in $\mathcal{C}[0, \infty)$ to a limit that we have called $\xi(h)$.

Moreover, similarly as the proof of Lemma 3.10, we show that the processes $\mathcal{Y}$ and $\{\xi(h_i),\ i \in [n]\}$ for every $h_i \in L_2^0$, $i \in [n]$, $n \geqslant 1$, are independent. We conclude that ξ is independent of $\mathcal{Y}$.

Let us show that ξ is a cylindrical Wiener process. Obviously, $h \mapsto \xi(h)$ is a linear map. We denote $\tilde{\mathcal{F}}_t^{\eta,\mathcal{Y}} = \mathcal{F}_t^\eta \vee \sigma(\mathcal{Y})$, $t \geqslant 0$. We need to check that for every $h \in L_2^0$, $\xi(h)$ is an $(\tilde{\mathcal{F}}_t^{\eta,\mathcal{Y}})$-Brownian motion. According to Lévy's characterization of Brownian motion [IW89, Theorem II.6.1], it is enough to show that $\xi(h)$ is a continuous square-integrable $(\tilde{\mathcal{F}}_t^{\eta,\mathcal{Y}})$-martingale with quadratic variation $\|h\|_{L_2}^2 t$. So, we take $n \geqslant 1$ and a bounded measurable function

$$F : \mathcal{C}[0, \infty)^n \times \mathcal{C}([0, \infty), L_2) \to \mathbb{R}.$$

Then using Lemma 3.10 and the fact that $M^y(h)$ is an $(\mathcal{F}_t^\eta)$-martingale, we have for every $s < t$

$$\begin{aligned}
\mathbb{E}[\xi_t(h)F\left((\eta_k(\cdot \wedge s))_{k=1}^n, \mathcal{Y}\right)] &= \mathbb{E}\left[\mathbb{E}\left[\xi_t(h)F\left((\eta_k(\cdot \wedge s))_{k=1}^n, \mathcal{Y}\right) \middle| \mathcal{Y}\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[M_t^y(h)F\left((\eta_k(\cdot \wedge s))_{k=1}^n, y\right)\right]\Big|_{y=\mathcal{Y}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[M_s^y(h)F\left((\eta_k(\cdot \wedge s))_{k=1}^n, y\right)\right]\Big|_{y=\mathcal{Y}}\right] \\
&= \mathbb{E}\left[\xi_s(h)F\left((\eta_k(\cdot \wedge s))_{k=1}^n, \mathcal{Y}\right)\right].
\end{aligned}$$

Hence, $\xi(h)$ is an $(\tilde{\mathcal{F}}_t^{\eta,\mathcal{Y}})$-martingale. Similarly, one can prove that $\xi_t(h)^2 - \|h\|_{L_2}^2 t$, $t \geqslant 0$, is also an $(\tilde{\mathcal{F}}_t^{\eta,\mathcal{Y}})$-martingale. This proves that $\xi(h)$ is a continuous square-integrable $(\tilde{\mathcal{F}}_t^{\eta,\mathcal{Y}})$-martingale with quadratic variation $\|h\|_{L_2}^2 t$, $t \geqslant 0$. The equality $\mathbb{E}\left[\xi_t(h_1)\xi_t(h_2)\right] = t(h_1, h_2)_{L_2}$, $t \geqslant 0$, trivially follows from the polarization equality and the fact that $\xi(h_1)$ and $\xi(h_2)$ are martingales with respect to the same filtration $(\tilde{\mathcal{F}}_t^{\eta,\mathcal{Y}})_{t\geqslant 0}$. Thus, ξ is an $(\tilde{\mathcal{F}}_t^{\eta,\mathcal{Y}})$-cylindrical Wiener process in L_2^0 starting at 0. This finishes the proof of the proposition. □

We conclude this section by defining properly the space $\mathbf{E}$ on which the random element $\mathcal{X}$ take values and the non-coalescing remainder map $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ needed to achieve step (S3) of Section 1.3. However, as we already noted, the cylindrical Wiener process $\mathcal{W}$ is not a random element in $\mathcal{C}([0,\infty), L_2)$. So we define $\mathbf{E} := \mathcal{C}([0,\infty), L_2^{\uparrow}) \times \mathcal{C}[0,\infty)^{\mathbb{N}_0}$ and $\mathbf{F} := \mathcal{C}_0[0,\infty)^{\mathbb{N}}$. Here, $\mathcal{C}[0,\infty)$ is the space of continuous functions from $[0,\infty)$ to $\mathbb{R}$ equipped with its usual Fréchet distance, $\mathcal{C}_0[0,\infty)$ denotes the subspace of all functions vanishing at 0 and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Equipped with the metric induced by the product topology, $\mathbf{E}$ is a Polish space.

Now, we fix an orthonormal basis $\{h_j,\ j \geqslant 0\}$ of L_2 such that $h_0 = \mathbb{1}_{[0,1]}$. In particular, $\{h_j,\ j \geqslant 1\}$ is an orthonormal basis of L_2^0. We identify the cylindrical Wiener process $\mathcal{W}$ with the following random element in $\mathcal{C}[0,\infty)^{\mathbb{N}_0}$:

$$\widehat{\mathcal{W}}_t = \left(\widehat{\mathcal{W}}_j(t)\right)_{j\geqslant 0} := \left(\mathcal{W}_t(h_j)\right)_{j\geqslant 0}, \quad t \geqslant 0.$$

Indeed $\mathcal{W}$ and $\widehat{\mathcal{W}}$ are related by $\mathcal{W}_t(h) = \sum_{j=0}^{\infty} \widehat{\mathcal{W}}_j(t)(h, h_j)_{L_2}$, for all $t \geqslant 0$ and $h \in L_2$, where the series converges in $\mathcal{C}[0,\infty)$ almost surely for every $h \in L_2$.

Similarly, we identify ξ with $\widehat{\xi}_t = \left(\widehat{\xi}_j(t)\right)_{j\geqslant 1} := \left(\xi_t(h_j)\right)_{j\geqslant 1}$, $t \geqslant 0$, and $\mathcal{Y}$ with $\widehat{\mathcal{Y}}_t = \left(\widehat{\mathcal{Y}}_j(t)\right)_{j\geqslant 0} := \left((\mathcal{Y}_t, h_j)_{L_2}\right)_{j\geqslant 0}$, $t \geqslant 0$. By equality (10), $\widehat{\xi}$ and $\widehat{\mathcal{W}}$

are related by

$$\widehat{\xi}_j(t) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} (e_k, h_j)_{L_2} (e_k, h_i)_{L_2} \widehat{\mathcal{W}}_i(t + \tau_k), \quad t \geqslant 0, \quad j \geqslant 1. \tag{12}$$

We define $\widehat{\mathcal{X}} = \left(\mathcal{Y}, \widehat{\mathcal{W}}\right)$, which is a random element on in $\mathbf{E}$. By (12), there exists a measurable map $\widehat{\mathrm{T}} : \mathbf{E} \to \mathbf{F}$ such that

$$\widehat{\xi} = \widehat{\mathrm{T}}(\widehat{\mathcal{X}}) \tag{13}$$

almost surely.

3.4 Statement of the main result

Let us clarify step (S4) of Section 1.3. According to Definition 1.1, we need to define a random sequence $\{\xi^n\}_{n \geqslant 1}$ in $\mathbf{F} = \mathcal{C}_0[0, \infty)^{\mathbb{N}}$ converging to 0 in distribution and such that $\mathbb{P}^{\xi^n}$ is absolutely continuous with respect to the law of $\widehat{\mathrm{T}}(\widehat{\mathcal{X}})$. By (13) and Proposition 3.9, $\mathbb{P}^{\widehat{\mathrm{T}}(\widehat{\mathcal{X}})}$ is the law of a sequence of independent Brownian motions.

Let for each $n \geqslant 1$, $\xi^n := (\xi_j^n)_{j \geqslant 1}$ be the sequence of Ornstein-Uhlenbeck processes, independent of $\mathcal{Y}$, that are strong solutions to the equations

$$\begin{cases} \mathrm{d}\xi_j^n(t) = -\alpha_j^n \mathbb{1}_{\{t \leqslant n\}} \xi_j^n(t) \mathrm{d}t + \mathrm{d}\widehat{\xi}_j(t), \\ \xi_j^n(0) = 0, \end{cases} \tag{14}$$

where $\{\alpha_j^n, \ n, j \geqslant 1\}$ is a family of non-negative real numbers such that

(O1) for every $n \geqslant 1$ the series $\sum_{j=1}^{\infty} (\alpha_j^n)^2 < +\infty$;

(O2) for every $j \geqslant 1$, $\alpha_j^n \to +\infty$ as $n \to \infty$.

Remark 3.11. (i) Using Kakutani's theorem [Kak48, p. 218] and Jensen's inequality, it is easily seen that Condition (O1) guaranties the absolute continuity of $\mathbb{P}^{\xi^n}$ with respect to $\mathbb{P}^{\widehat{\xi}}$ on $\mathcal{C}[0, \infty)^{\mathbb{N}}$. The indicator function in the drift is important, otherwise the law is singular. Hence, Assumption (B1) of Definition 1.1 is satisfied by the sequence $\{\xi^n\}_{n \geqslant 1}$.

(ii) Condition (O2) yields the convergence in distribution of $\{\xi^n\}_{n \geqslant 1}$ to 0 in $\mathcal{C}[0, \infty)^{\mathbb{N}}$ (see Lemma 4.7 below). Thus Assumption (B2) is also satisfied.

The following theorem is the main result of the paper.

Theorem 3.12. *The value of the conditional distribution of* $\widehat{\mathcal{X}} = (\mathcal{Y}, \widehat{\mathcal{W}})$ *to the event* $\{\widehat{\mathrm{T}}(\widehat{\mathcal{X}}) = 0\}$ *along* $\{\xi^n\}$ *is the law of* $(\mathcal{Y}, \widehat{\mathcal{Y}})$.

The event $\{\widehat{\mathrm{T}}(\widehat{\mathcal{X}}) = 0\}$, which equals to $\{\widehat{\xi} = 0\}$, is by construction the event where the non-coalescing part of $\widehat{\mathcal{W}}$ vanishes.

Remark 3.13. For simplicity, we assumed in sections 3.3 and 3.4 that the initial condition g is strictly increasing. Actually, everything remains true if g is an arbitrary element of $L_{2+}^{\uparrow}$, up to replacing the space L_2 by the space $L_2(g)$. In particular, if g is a step function, then $L_2(g)$ has finite dimension, equal to $N(g)$, and the orthonormal basis constructed in Lemma 3.7 and the sum in the definition of $\widehat{\xi}$ consists of finitely many summands.

4 Proof of the main theorem

In order to prove Theorem 3.12, we follow the strategy introduced in Section 2.2. We start by the construction of a quadruple $(\mathbf{G}, \Psi, Y, Z)$ satisfying (P1)-(P4). The idea behind the construction of Ψ is inspired by the result of Proposition 3.6, stating that $\mathcal{W}$ can be build from the MMAF $\mathcal{Y}$ and some independent process.

4.1 Construction of quadruple

Define $\mathbf{G} := \mathbf{Coal}$, $Y := \mathcal{Y}$ and $Z := \widehat{\mathcal{Z}}$, where $\mathcal{Z}$ is a cylindrical Wiener process in L_2^0 starting at 0 that is independent of $\mathcal{Y}$. By the same identification as previously, for the same basis $\{h_j,\ j \geqslant 0\}$, $\widehat{\mathcal{Z}}_t = \left(\widehat{\mathcal{Z}}_j(t)\right)_{j\geqslant 1} := (\mathcal{Z}_t(h_j))_{j\geqslant 1}$, $t \geqslant 0$, is a sequence of independent standard Brownian motions and is a random element in $\mathbf{F}$. Therefore, properties (P1) and (P2) are satisfied.

We define

$$\psi(\mathcal{Y}, \mathcal{Z}) := (\mathcal{Y}, \varphi(\mathcal{Y}, \mathcal{Z}))\,,$$

where $\varphi_t(\mathcal{Y}, \mathcal{Z})$ is a map from L_2 to $L_2(\Omega)$ defined by

$$\varphi_t(\mathcal{Y}, \mathcal{Z})(h) = (\mathcal{Y}_t, h)_{L_2} + \sum_{k=1}^{\infty} (e_k, h)_{L_2} \mathbb{1}_{\{t \geqslant \tau_k\}} \mathcal{Z}_{t-\tau_k}(e_k) \tag{15}$$

for all $t \geqslant 0$ and $h \in L_2$. As in the proof of Lemma 3.10, one can show that $\mathcal{Z}(e_k)$, $k \geqslant 1$, are independent standard Brownian motions that do not depend on $\mathcal{Y}$.

Lemma 4.1. *For each $h \in L_2$, the sum in* (15) *converges almost surely in $\mathcal{C}[0, \infty)$. Furthermore, $\varphi(\mathcal{Y}, \mathcal{Z})$ is a cylindrical Wiener process in L_2 starting at g and the law of $\psi(\mathcal{Y}, \mathcal{Z})$ is equal to the law of $\mathcal{X} = (\mathcal{Y}, \mathcal{W})$.*

Remark 4.2. Before giving the proof of the lemma, note that the map φ constructs a cylindrical Wiener process from $\mathcal{Y}$, by adding to $\mathcal{Y}$ some non-coalescing term. Actually, for each $y \in \mathbf{Coal}$, $\varphi(y, z)$ belongs to $\mathbf{Coal}$ if and only if $z = 0$. This statement is proved in Lemma B.9.

Proof of Lemma 4.1. Let us first show that the sum in (15) converges almost surely in $\mathcal{C}[0,\infty)$. Fixing $y \in \mathbf{Coal}$ and $h \in L_2$, we define for every $n \geqslant 1$

$$R_t^{y,n}(h) := \sum_{k=1}^{n} (e_k^y, h)_{L_2} \mathbb{1}_{\{t \geqslant \tau_k^y\}} \mathcal{Z}_{t-\tau_k^y}(e_k), \quad t \geqslant 0.$$

Since $\mathcal{Z}(e_k)$, $k \geqslant 1$, are independent standard Brownian motions, one can easily check that $R_t^{y,n}(h)$, $t \geqslant 0$, is a continuous square-integrable martingale with respect to the filtration generated by $\mathcal{Z}_{t-\tau_k^y}(e_k)$, $k \geqslant 1$. As in the proof of Proposition 3.9, one can show that the sequence of partial sums $\{R^{y,n}(h)\}_{n\geqslant 1}$ converges in $\mathcal{C}[0,\infty)$ almost surely for each $y \in \mathbf{Coal}$. By the independence of $\mathcal{Z}(e_k)$, $k \geqslant 1$, and $\mathcal{Y}$, one can see that the series

$$R_t^{\mathcal{Y}}(h) := \sum_{k=1}^{\infty} (e_k, h)_{L_2} \mathbb{1}_{\{t \geqslant \tau_k\}} \mathcal{Z}_{t-\tau_k}(e_k), \quad t \geqslant 0,$$

also converges almost surely in $\mathcal{C}[0,\infty)$.

Next, we claim that there exists a cylindrical Wiener process θ_t, $t \geqslant 0$, in L_2^0 starting at 0 independent of $\mathcal{Y}$ such that

$$\mathcal{W}_t = \mathcal{Y}_t + \int_0^t \mathrm{pr}_{\mathcal{Y}_s}^{\perp} \, \mathrm{d}\theta_s, \quad t \geqslant 0. \tag{16}$$

Indeed, by Proposition 3.6, there is a cylindrical Wiener process B_t, $t \geqslant 0$, in L_2 starting at 0 independent of $\mathcal{Y}$ and satisfying equation (7). Taking θ equal to the restriction of B to the sub-Hilbert space L_2^0, we easily check that $\int_0^t \mathrm{pr}_{\mathcal{Y}_s}^{\perp} \, \mathrm{d}\theta_s = \int_0^t \mathrm{pr}_{\mathcal{Y}_s}^{\perp} \, \mathrm{d}B_s$, $t \geqslant 0$, since for all $s \geqslant 0$, $\mathrm{pr}_{\mathcal{Y}_s}^{\perp} = \mathrm{pr}_{L_2^0} \circ \mathrm{pr}_{\mathcal{Y}_s}^{\perp}$ almost surely. Furthermore, almost surely

$$\int_0^t \mathrm{pr}_{\mathcal{Y}_s}^{\perp} \, \mathrm{d}\theta_s = \sum_{k=1}^{\infty} e_k \mathbb{1}_{\{t \geqslant \tau_k\}} (\theta_{t\wedge\tau_k}(e_k) - \theta_{\tau_k}(e_k)), \quad t \geqslant 0.$$

For each fixed $y \in \mathbf{Coal}$, the family

$$\left\{ \mathbb{1}_{\{t \geqslant \tau_k^y\}} (\theta_{t\wedge\tau_k^y}(e_k^y) - \theta_{\tau_k^y}(e_k^y)), \ t \geqslant 0, \ k \geqslant 1 \right\},$$

has the same distribution as

$$\left\{ \mathbb{1}_{\{t \geqslant \tau_k^y\}} \mathcal{Z}_{t-\tau_k^y}(e_k^y), \ t \geqslant 0, \ k \geqslant 1 \right\}.$$

Therefore, using the independence of $\mathcal{Y}$ and θ on the one hand and the independence of $\mathcal{Y}$ and $\mathcal{Z}$ on the other hand, we get the equality

$$\begin{aligned} &\mathrm{Law}\left\{ \left(\mathcal{Y}_t, \int_0^t \mathrm{pr}_{\mathcal{Y}_s}^{\perp} \, \mathrm{d}\theta_s \right), t \geqslant 0 \right\} \\ &\qquad\qquad = \mathrm{Law}\left\{ \left(\mathcal{Y}_t, \sum_{k=1}^{\infty} e_k \mathbb{1}_{\{t \geqslant \tau_k\}} \mathcal{Z}_{t-\tau_k}(e_k) \right), t \geqslant 0 \right\}. \end{aligned}$$

This relation and equalities (15) and (16) yield that the law of $\mathcal{X} = (\mathcal{Y}, \mathcal{W})$ is equal to the law of $\psi(\mathcal{Y}, \mathcal{Z}) = (\mathcal{Y}, \varphi(\mathcal{Y}, \mathcal{Z}))$. In particular, $\varphi(\mathcal{Y}, \mathcal{Z})$ is a cylindrical Wiener process in L_2 starting at g. □

Moreover, there exists a measurable map $\widehat{\varphi} : \mathbf{E} \to \mathcal{C}[0, \infty)^{\mathbb{N}_0}$ such that

$$\widehat{\varphi}(\mathcal{Y}, \widehat{\mathcal{Z}}) = \widehat{\varphi(\mathcal{Y}, \mathcal{Z})}.$$

almost surely. Let us define $\Psi : \mathbf{G} \times \mathbf{F} \to \mathbf{E}$ by

$$\Psi(y, z) := (y, \widehat{\varphi}(y, z)) \,. \tag{17}$$

It follows from the last two equalities and from Lemma 4.1 that

Corollary 4.3. *The laws of $\Psi(\mathcal{Y}, \widehat{\mathcal{Z}})$ and of $\widehat{\mathcal{X}} = (\mathcal{Y}, \widehat{\mathcal{W}})$ are the same.*

Hence Property (P4) is satisfied. It remains to check (P3). By equalities (12) and (13), we compute $\widehat{\mathrm{T}}(\Psi(\mathcal{Y}, \widehat{\mathcal{Z}}))$:

$$\widehat{\mathrm{T}}(\Psi(\mathcal{Y}, \widehat{\mathcal{Z}}))_j(t) = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}(e_k, h_j)_{L_2}(e_k, h_i)_{L_2}\widehat{\varphi(\mathcal{Y}, \mathcal{Z})}_i(t+\tau_k), \quad t \geqslant 0, \;\; j \geqslant 1.$$

Proposition 4.4. *Almost surely $\widehat{\mathrm{T}}(\Psi(\mathcal{Y}, \widehat{\mathcal{Z}})) = \widehat{\mathcal{Z}}$.*

Proof. By continuity in t of $\widehat{\mathrm{T}}(\Psi(\mathcal{Y}, \widehat{\mathcal{Z}}))_j(t)$ and $\widehat{\mathcal{Z}}_j(t)$, it is enough to show that for each $t \geqslant 0$ and $j \geqslant 1$ almost surely $\widehat{\mathrm{T}}(\Psi(\mathcal{Y}, \widehat{\mathcal{Z}}))_j(t) = \widehat{\mathcal{Z}}_j(t)$. Since $\{h_i,\ i \geqslant 1\}$ is an orthonormal basis of L_2^0, we have

$$\begin{aligned}\widehat{\mathrm{T}}(\Psi(\mathcal{Y}, \widehat{\mathcal{Z}}))_j(t) &= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}(e_k, h_j)_{L_2}(e_k, h_i)_{L_2}\varphi_{t+\tau_k}(\mathcal{Y}, \mathcal{Z})(h_i) \\ &= \sum_{k=1}^{\infty}(e_k, h_j)_{L_2}\varphi_{t+\tau_k}(\mathcal{Y}, \mathcal{Z})(e_k).\end{aligned}$$

By (15) and Lemma 3.7, we have

$$\begin{aligned}\varphi_{t+\tau_k}(\mathcal{Y}, \mathcal{Z})(e_k) &= (\mathcal{Y}_{t+\tau_k}, e_k)_{L_2} + \sum_{l=1}^{\infty}(e_l, e_k)_{L_2}\mathbb{1}_{\{t+\tau_k \geqslant \tau_l\}}\mathcal{Z}_{t+\tau_k-\tau_l}(e_l) \\ &= \mathbb{1}_{\{t+\tau_k \geqslant \tau_k\}}\mathcal{Z}_{t+\tau_k-\tau_k}(e_k) = \mathcal{Z}_t(e_k).\end{aligned}$$

Hence, almost surely

$$\widehat{\mathrm{T}}(\Psi(\mathcal{Y}, \widehat{\mathcal{Z}}))_j(t) = \sum_{k=1}^{\infty}(e_k, h_j)_{L_2}\mathcal{Z}_t\,(e_k) = \mathcal{Z}_t(h_j) = \widehat{\mathcal{Z}}_j(t),$$

because $\{e_k,\ k \geqslant 1\}$ is an orthonormal basis of L_2^0. □

Thus, Property (P3) holds. Hence, by Proposition 2.3, the probability kernel p defined by

$$p(A,z) := \mathbb{P}\left[\Psi(\mathcal{Y},z) \in A\right] = \mathbb{P}\left[(\mathcal{Y},\widehat{\varphi}\,(\mathcal{Y},z)) \in A\right] \tag{18}$$

for all $A \in \mathcal{B}(\mathbf{E})$ and $z \in \mathbf{F}$, is a regular conditional probability of $\widehat{\mathcal{X}}$ given $\widehat{\mathrm{T}}(\widehat{\mathcal{X}})$.

Remark 4.5. Informally, we understand the event $\{\widehat{\mathrm{T}}(\widehat{\mathcal{X}}) = 0\}$ as an equivalent to the event $\{\mathcal{W} \in \mathbf{Coal}\}$. Nevertheless, we should not expect to prove that both events are equal, since the map $\widehat{\mathrm{T}}$ was defined up to a set of measure zero with respect to the law of $\widehat{\mathcal{X}}$. But in view of Remark 4.2, this equivalence seems reasonable.

4.2 Value of p along a sequence of Ornstein-Uhlenbeck processes

According to Proposition 2.3, it remains to show the following to complete the proof of Theorem 3.12. Let $\{\xi^n\}_{n\geqslant 1}$ be the sequence defined by (14) and independent of $\mathcal{Y}$. Let Ψ be defined by (17). Then $\Psi(\mathcal{Y},\xi^n)$ converges in distribution to $(\mathcal{Y},\widehat{\mathcal{Y}})$.

For $y \in \mathbf{Coal}$ we consider

$$\Psi(y,\xi^n) = (y,\widehat{\varphi}(y,\xi^n)),$$

where the map $\widehat{\varphi} : \mathbf{E} \to \mathcal{C}[0,\infty)^{\mathbb{N}_0}$ was defined in Section 4.1. Since for every $n \geqslant 1$ the law of ξ^n is absolutely continuous with respect to $\mathbb{P}^{\widehat{\xi}}$ (which is equal to $\mathbb{P}^{\widehat{\mathcal{Z}}}$), we have that for almost all $y \in \mathbf{Coal}$ with respect to $\mathbb{P}^{\mathcal{Y}}$

$$\widehat{\varphi}_j\,(y,\xi^n) = (y_\cdot,h_j) + \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(e_k^y,h_j)_{L_2}(h_l,e_k^y)_{L_2}\mathbb{1}_{\left\{\cdot\geqslant\tau_k^y\right\}}\xi_l^n(\cdot-\tau_k^y) \tag{19}$$

for each $j \geqslant 0$, where the series converges in $\mathcal{C}[0,\infty)$ almost surely. Without loss of generality, we may assume that equality (19) holds for all $y \in \mathbf{Coal}$. Otherwise, we can work with a measurable subset of $\mathbf{Coal}$ of $\mathbb{P}^{\mathcal{Y}}$-measure one for which equality (19) holds.

Proposition 4.6. *Let $\varepsilon \in (0,1)$ and $y \in$ **Coal** be such that the series $\sum_{k=1}^{\infty}(\tau_k^y)^{1-\varepsilon}$ converges. Then the sequence of processes $\Psi(y,\xi^n)$, $n \geqslant 1$, converges in distribution to $(y,\widehat{y})$ in $\mathbf{E} = \mathcal{C}([0,\infty),L_2^{\uparrow}) \times \mathcal{C}[0,\infty)^{\mathbb{N}_0}$, where $\widehat{y} = ((y_\cdot,h_j)_{L_2})_{j\geqslant 0}$.*

Let us fix $y \in \mathbf{Coal}$ satisfying the assumption of Proposition 4.6. Before starting the proof, we define for all $j \geqslant 0$

$$R_j^n(t) := \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(e_k^y,h_j)_{L_2}(h_l,e_k^y)_{L_2}\mathbb{1}_{\left\{t\geqslant\tau_k^y\right\}}\xi_l^n(t-\tau_k^y), \quad t \geqslant 0,$$

and $R_t^n := (R_j^n(t))_{j\geqslant 0}$, $t \geqslant 0$. Remark that $R_0^n = 0$. Note that it is sufficient to prove that

$$R^n \xrightarrow{d} 0 \quad \text{in} \quad \mathcal{C}[0,\infty)^{\mathbb{N}_0}, \quad n \to \infty. \tag{20}$$

Indeed, this will imply that

$$\Psi(y, \xi^n) = (y, \widehat{\varphi}(y, \xi^n)) = (y, \widehat{y} + R^n) \xrightarrow{d} (y, \widehat{y}) \quad \text{in} \quad \mathbf{E}.$$

Let us first prove some auxiliary lemmas.

Lemma 4.7. *The sequence of random elements* $\{\xi^n\}_{n\geqslant 1}$ *converges in distribution to* 0 *in* $\mathcal{C}[0,\infty)^{\mathbb{N}}$.

Proof. In order to prove the lemma, we first show that the sequence $\{\xi^n\}_{n\geqslant 1}$ is tight in $\mathcal{C}[0,\infty)^{\mathbb{N}}$. This will imply that the sequence $\{\xi^n\}_{n\geqslant 1}$ is relatively compact, by Prohorov's theorem. Then we will show that every (weakly) convergent subsequence of $\{\xi^n\}_{n\geqslant 1}$ converges to 0. This will immediately yield that $\xi^n \xrightarrow{d} 0$ in $\mathcal{C}[0,\infty)^{\mathbb{N}}$.

According to [EK86, Proposition 3.2.4], the tightness of $\{\xi^n\}_{n\geqslant 1}$ will follow from the tightness of $\{\xi_j^n\}_{n\geqslant 1}$ in $\mathcal{C}[0,\infty)$ for every $j \geqslant 1$. So, let $j \geqslant 1$ and $T > 0$ be fixed. Since the covariance of Ornstein-Uhlenbeck processes is well-known, one can easily check that for every $n \geqslant 1$ and every $0 \leqslant s \leqslant t \leqslant n$,

$$\mathbb{E}\left[\left(\xi_j^n(t) - \xi_j^n(s)\right)^2\right] \leqslant \frac{1}{\alpha_j^n} \wedge (t-s), \tag{21}$$

where $\frac{1}{0} := +\infty$. Since ξ_j^n is a Gaussian process, it follows that for every $0 \leqslant s \leqslant t \leqslant T$ and every $n \geqslant T$,

$$\mathbb{E}\left[\left(\xi_j^n(t) - \xi_j^n(s)\right)^4\right] \leqslant 3\mathbb{E}\left[\left(\xi_j^n(t) - \xi_j^n(s)\right)^2\right]^2 \leqslant 3(t-s)^2.$$

Moreover, $\xi_j^n(0) = 0$. Hence, by Kolmogorov-Chentsov tightness criterion (see e.g. [Kal02, Corollary 16.9]), the sequence of processes $\{\xi_j^n\}_{n\geqslant 1}$ restricted to $[0,T]$ is tight in $\mathcal{C}[0,T]$. Since $T > 0$ was arbitrary, we get that $\{\xi_j^n\}_{n\geqslant 1}$ is tight in $\mathcal{C}[0,\infty)$. Hence, $\{\xi^n\}_{n\geqslant 1}$ is tight in $\mathcal{C}[0,\infty)^{\mathbb{N}}$.

Next, let $\{\xi^n\}_{n\geqslant 1}$ converges in distribution to ξ^∞ in $\mathcal{C}[0,\infty)^{\mathbb{N}}$ along a subsequence $N \subseteq \mathbb{N}$. Then for every $t \geqslant 0$ and $j \geqslant 1$ $\{\xi_j^n(t)\}_{n\geqslant 1}$ converges in distribution to $\xi_j^\infty(t)$ in $\mathbb{R}$ along N. But on the other hand, for each $n \geqslant t$,

$$\mathbb{E}\left[(\xi_j^n(t))^2\right] \leqslant \frac{t}{\alpha_j^n} \to 0, \quad n \to \infty,$$

by (21) and Assumption (O2) in Section 3.4. Hence, $\xi_j^\infty(t) = 0$ almost surely for all $t \geqslant 0$ and $j \geqslant 1$. Thus, we have obtained that $\xi^\infty = 0$, and therefore, $\xi^n \xrightarrow{d} 0$ in $\mathcal{C}[0,\infty)^{\mathbb{N}}$ as $n \to \infty$. □

To prove that $\{R^n\}_{n\geqslant 1}$ converges to 0, we will use the same argument as in the proof of Lemma 4.7. So, we start from the tightness of $\{R^n\}$.

Lemma 4.8. *Under the assumption of Proposition 4.6, the sequence* $\{R^n\}_{n\geqslant 0}$ *is tight in* $\mathcal{C}[0,\infty)^{\mathbb{N}_0}$.

Proof. Again, according to [EK86, Proposition 3.2.4], it is enough to check that the sequence $\{R_j^n\}_{n\geqslant 1}$ is tight in $\mathcal{C}[0,\infty)$ for every $j \geqslant 0$. For $j=0$, $R_0^n = 0$ so the result is obvious. So, let $j \geqslant 1$ be fixed. We set

$$R_j^{n,1}(t) := \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(e_k^y, h_j)_{L_2}(h_l, e_k^y)_{L_2}\xi_l^n(t), \quad t \geqslant 0,$$

and

$$R_j^{n,2}(t) := \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(e_k^y, h_j)_{L_2}(h_l, e_k^y)_{L_2}\left(\mathbb{1}_{\{t\geqslant\tau_k^y\}}\xi_l^n(t-\tau_k^y) - \xi_l^n(t)\right), \quad t \geqslant 0.$$

Then $R_j^n = R_j^{n,1} + R_j^{n,2}$. We will prove the tightness separately for $\{R_j^{n,1}\}_{n\geqslant 1}$ and $\{R_j^{n,2}\}_{n\geqslant 1}$.

Tightness of $\{R_j^{n,1}\}_{n\geqslant 1}$. Using the fact that $\{e_k^y,\ k\geqslant 1\}$ and $\{h_l,\ l\geqslant 1\}$ are bases of L_2^0, a simple computation shows that almost surely

$$\Gamma_j(\widehat{\xi}) := \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(e_k^y, h_j)_{L_2}(h_l, e_k^y)_{L_2}\widehat{\xi}_l = \widehat{\xi}_j.$$

Due to the absolute continuity of the law of ξ^n with respect to the law of $\widehat{\xi}$ and the equality $\Gamma_j(\xi^n) = R_j^{n,1}$, we get that $R_j^{n,1} = \xi_j^n$. Hence it follows from Lemma 4.7 that $R_j^{n,1}$ converges in distribution to 0 in $\mathcal{C}[0,\infty)$. In particular, $\{R_j^{n,1}\}_{n\geqslant 1}$ is tight in $\mathcal{C}[0,\infty)$, according to Prohorov's theorem.

Tightness of $\{R_j^{n,2}\}_{n\geqslant 1}$.

Step I. For any $t\in[0,n]$ the vector

$$V_t^n := \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}e_k^y(e_k^y, h_l)_{L_2}\left(\mathbb{1}_{\{t\geqslant\tau_k^y\}}\xi_l^n(t-\tau_k^y) - \xi_l^n(t)\right)$$

belongs almost surely to L_2^0 and $\mathbb{E}\left[\|V_t^n\|_{L_2}^2\right] \leqslant \sum_{k=1}^{\infty}(t\wedge\tau_k^y) < \infty$.

Indeed, by Parseval's equality (with respect to the orthonormal family $\{e_k^y,\ k\geqslant 1\}$) and by the independence of $\{\xi_l^n\}_{l\geqslant 1}$,

$$\mathbb{E}\left[\|V_t^n\|_{L_2}^2\right] = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(e_k^y, h_l)_{L_2}^2 E_{k,l}^n(t), \tag{22}$$

where $E^n_{k,l}(t) := \mathbb{E}\left[\left(\mathbb{1}_{\{t\geqslant\tau^y_k\}}\xi^n_l(t-\tau^y_k)-\xi^n_l(t)\right)^2\right]$. Since $\xi^n_l(0)=0$, we have

$$E^n_{k,l}(t)=\mathbb{1}_{\{t\geqslant\tau^y_k\}}\mathbb{E}\left[\left(\xi^n_l(t-\tau^y_k)-\xi^n_l(t)\right)^2\right]+\mathbb{1}_{\{t<\tau^y_k\}}\mathbb{E}\left[\left(\xi^n_l(0)-\xi^n_l(t)\right)^2\right]. \tag{23}$$

By inequality (21), we can deduce that

$$E^n_{k,l}(t)\leqslant \mathbb{1}_{\{t\geqslant\tau^y_k\}}\tau^y_k+\mathbb{1}_{\{t<\tau^y_k\}}t=t\wedge\tau^y_k. \tag{24}$$

Therefore,

$$\mathbb{E}\left[\|V^n_t\|^2_{L_2}\right]\leqslant\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(e^y_k,h_l)^2_{L_2}(t\wedge\tau^y_k)=\sum_{k=1}^{\infty}(t\wedge\tau^y_k), \tag{25}$$

by Parseval's identity (with respect to the orthonormal family $\{h_l,\ l\geqslant 1\}$). Moreover, $\sum_{k=1}^{\infty}(t\wedge\tau^y_k)\leqslant t^\varepsilon\sum_{k=1}^{\infty}(\tau^y_k)^{1-\varepsilon}<\infty$. Therefore, for any $t\in[0,n]$, V^n_t belongs to L^0_2 almost surely. In particular, for every $t\in[0,n]$ the inner product $(V^n_t,h_j)_{L_2}$ is well-defined, and almost surely $R^{n,2}_j(t)=(V^n_t,h_j)_{L_2}$.

Step II. Let $T>0$. There exists $C_{y,\varepsilon}$ depending on y and ε such that for all $0\leqslant s\leqslant t\leqslant T$ and $n\geqslant T$,

$$\mathbb{E}\left[\left(R^{n,2}_j(t)-R^{n,2}_j(s)\right)^2\right]\leqslant C_{y,\varepsilon}(t-s)^\varepsilon.$$

Indeed, proceeding as in Step I, we get

$$\begin{aligned}
&\mathbb{E}\left[\left(R^{n,2}_j(t)-R^{n,2}_j(s)\right)^2\right]\leqslant\mathbb{E}\left[\|V^n_t-V^n_s\|^2_{L_2}\right]\\
&\leqslant\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(e^y_k,h_l)^2_{L_2}\mathbb{E}\left[\left(\mathbb{1}_{\{t\geqslant\tau^y_k\}}\xi^n_l(t-\tau^y_k)-\xi^n_l(t)-\mathbb{1}_{\{s\geqslant\tau^y_k\}}\xi^n_l(s-\tau^y_k)+\xi^n_l(s)\right)^2\right]\\
&\leqslant\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(e^y_k,h_l)^2_{L_2}4\left((t-s)\wedge\tau^y_k\right)=4\sum_{k=1}^{\infty}\left((t-s)\wedge\tau^y_k\right)\leqslant 4(t-s)^\varepsilon\sum_{k=1}^{\infty}(\tau^y_k)^{1-\varepsilon},
\end{aligned}$$

where we use as previously inequality (21). By assumption on y, the series $\sum_{k=1}^{\infty}(\tau^y_k)^{1-\varepsilon}$ converges, so the proof of Step II is achieved.

Step III. There exists $\alpha>0$, $\beta>0$ and $C_{y,\varepsilon}$ depending on y and ε such that for all $0\leqslant s\leqslant t\leqslant T$ and $n\geqslant T$,

$$\mathbb{E}\left[\left|R^{n,2}_j(t)-R^{n,2}_j(s)\right|^\alpha\right]\leqslant C_{y,\varepsilon}(t-s)^{1+\beta}.$$

Indeed, for any $s\leqslant t$ from $[0,T]$, $R^{n,2}_j(t)-R^{n,2}_j(s)$ is a random variable with normal distribution $\mathcal{N}(0,\sigma^2)$. By Step II, $\sigma^2\leqslant C_{y,\varepsilon}(t-s)^\varepsilon$. Therefore, for any $p \geqslant 1$,

$$\mathbb{E}\left[\left|R_j^{n,2}(t) - R_j^{n,2}(s)\right|^{2p}\right] \leqslant (2p-1)!!\ (\sigma^2)^p \leqslant C_{p,y,\varepsilon}(t-s)^{\varepsilon p}.$$

The statement of Step III follows by choosing p larger than $\frac{1}{\varepsilon}$.

Step IV. By Kolmogorov-Chentsov tightness criterion (see e.g. [Kal02, Corollary 16.9]), it follows from Step III and the equality $R_j^{n,2}(0) = 0$, $n \geqslant 1$, that the sequence of processes $\{R_j^{n,2}\}_{n\geqslant 1}$ restricted to $[0,T]$ is tight in $\mathcal{C}[0,T]$ for every $T > 0$. Hence, $\{R_j^{n,2}\}_{n\geqslant 1}$ is tight in $\mathcal{C}[0,\infty)$.

Conclusion of the proof. As the sum of two tight sequences, the sequence $\{R_j^n\}_{n\geqslant 1}$ is tight in $\mathcal{C}[0,\infty)$ for any $j \geqslant 1$. Since $\mathcal{C}[0,\infty)^{\mathbb{N}}$ is equipped with the product topology, it follows from [EK86, Proposition 3.2.4] that the sequence $\{R^n\}_{n\geqslant 1}$ is tight in $\mathcal{C}[0,\infty)^{\mathbb{N}}$. □

Lemma 4.9. *For every $j \geqslant 1$ and $t \geqslant 0$, $\mathbb{E}\left[(R_j^n(t))^2\right] \to 0$ as $n \to \infty$.*

Proof. Let $j \geqslant 1$ and $t \geqslant 0$ be fixed. We recall that $R_j^n = R_j^{n,1} + R_j^{n,2}$. Remark that $R_j^{n,1} = \xi_j^n$ almost surely. Thus, $\mathbb{E}\left[\left(R_j^{n,1}(t)\right)^2\right] \to 0$ follows immediately from inequality (21).

Due to the equality $R_j^{n,2}(t) = (V_t^n, h_j)_{L_2}$, we can estimate for $n \geqslant t$

$$\mathbb{E}\left[\left(R_j^{n,2}(t)\right)^2\right] \leqslant \mathbb{E}\left[\|V_t^n\|_{L_2}^2\right] = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty}(e_k^y, h_l)_{L_2}^2 E_{k,l}^n(t).$$

By (23) and (21), we have for every $k, l \geqslant 1$

$$0 \leqslant E_{k,l}^n(t) \leqslant \frac{1}{\alpha_l^n} \to 0, \quad n \to \infty.$$

Therefore, inequalities (24) and (25) and the dominated convergence theorem imply that $\mathbb{E}\left[\|V_t^n\|_{L_2}^2\right] \to 0$. This concludes the proof. □

Proof of Proposition 4.6. Lemma 4.8 and Prohorov's theorem yield that the sequence $\{R^n\}_{n\geqslant 1}$ is relatively compact in $\mathcal{C}[0,\infty)^{\mathbb{N}_0}$. Moreover, by Lemma 4.9, we deduce that each weakly convergent subsequence of $\{R^n\}_{n\geqslant 1}$ converges in distribution to 0. It implies convergence (20), which achives the proof of the proposition. □

Proof of Theorem 3.12. By lemmas 3.5 and B.6, $\mathcal{Y}$ belongs almost surely to **Coal** and the series $\sum_{k=1}^{\infty}(\tau_k^{\mathcal{Y}})^{1-\varepsilon}$ converges almost surely for each $\varepsilon \in (0, \frac{1}{2})$. Therefore, Proposition 4.6 and the independence of $\mathcal{Y}$ and $\{\xi^n\}_{n\geqslant 1}$ imply that $\Psi(\mathcal{Y},\xi^n)$, $n \geqslant 1$, converges in distribution to $(\mathcal{Y},\widehat{\mathcal{Y}})$ in $\mathbf{E}$. By Proposition 2.3, the same sequence converges in distribution to the conditional law $\mathrm{Law}_{\{\xi^n\}}(\mathcal{X}|\mathrm{T}(\mathcal{X})=0)$. Thus $\mathrm{Law}_{\{\xi^n\}}(\mathcal{X}|\mathrm{T}(\mathcal{X})=0)=\mathrm{Law}(\mathcal{Y},\widehat{\mathcal{Y}})$. $\square$

5 Finite dimensional case

Let $W_k(t)$, $t \geqslant 0$, $k \in [n]$, be a family of independent Brownian motions starting at x_k^0, $k \in [n]$, with diffusion rates $\sigma_k^2 = \frac{1}{m_k}$, $k \in [n]$, where $\sum_{k=1}^n m_k = 1$. Without restriction of generality, we may assume that $x_1^0 < \dots < x_n^0$. Let $g = \sum_{k=1}^n x_k^0 \mathbb{1}_{\pi_k^0}$ be the initial condition given by (4). Then

$$\mathcal{W}_t^g := \sum_{k=1}^n W_k(t)\mathbb{1}_{\pi_k^0}, \quad t \geqslant 0,$$

is a continuous process taking values in $L_2(g)$ and it can be considered as a cylindrical Wiener process in $L_2(g)$ starting at g, taking $\mathcal{W}_t^g(h) = (\mathcal{W}_t^g, h)_{L_2}$.

Let $\mathcal{Y}$ be a MMAF obtained as the unique solution to the equation

$$\mathcal{Y}_t = g + \int_0^t \mathrm{pr}_{\mathcal{Y}_s} \,\mathrm{d}\mathcal{W}_s^g, \quad \mathcal{Y}_t \in L_2^{\uparrow}, \quad t \geqslant 0,$$

which exists according to Corollary B.8. We remark that $\mathcal{X} = (\mathcal{Y}, \mathcal{W}^g)$ is now a random element in $\mathbf{E} := \mathcal{C}([0,\infty), L_2^{\uparrow}(g)) \times \mathcal{C}([0,\infty), L_2(g))$, where $L_2^{\uparrow}(g) := L_2^{\uparrow} \cap L_2(g)$. Since the space $L_2(g)$ is finite dimensional,

$$\xi = \mathrm{T}(\mathcal{X}) = \sum_{k=1}^{n-1} e_k (\mathcal{W}_{\cdot+\tau_k}, e_k)_{L_2}$$

is a random element in $\mathbf{F} := \mathcal{C}([0,\infty), L_2^0(g))$ with $L_2^0(g) = L_2^0 \cap L_2(g)$, where the stopping times $\tau_k = \tau_k^{\mathcal{Y}}$, $k \in [n-1]$, and the basis $e_k = e_k^{\mathcal{Y}}$, $k \in [n-1] \cup \{0\}$, of $L_2(g)$ are defined as in Section 3.3. By equality (18) and Remark 3.13, we can conclude that the regular conditional probability of $\mathcal{X}$ given $\mathrm{T}(\mathcal{X})$ is defined as

$$p(A,z) = \mathbb{P}\left[\mathcal{X} \in A | \mathrm{T}(\mathcal{X}) = z\right] = \mathbb{P}\left[(\mathcal{Y}, \varphi(\mathcal{Y},z)) \in A\right], \quad A \in \mathcal{B}(\mathbf{E}), \quad z \in \mathbf{F},$$

where

$$\varphi(\mathcal{Y},z) = \mathcal{Y} + \sum_{k=1}^{n-1} e_k \mathbb{1}_{\{\cdot \geqslant \tau_k\}} (z_{\cdot - \tau_k}, e_k)_{L_2}.$$

Obviously, the map $z \mapsto p(\cdot, z)$ is continuous. Thus $p(\cdot, 0)$ is the value of the conditional distribution of $\mathcal{X}$ to the event $\{\mathrm{T}(\mathcal{X}) = 0\}$ along every direction. In particular, for every $A \in \mathcal{B}(\mathcal{C}([0,\infty), L_2(g)))$

$$\mathbb{P}\left[\mathcal{W}^g \in A | \mathrm{T}(\mathcal{Y}, \mathcal{W}^g) = 0\right] = \mathbb{P}\left[\varphi(\mathcal{Y},0) \in A\right] = \mathbb{P}\left[\mathcal{Y} \in A\right]. \qquad (26)$$

Denote $\mathbf{E}^g := \{x \in \mathcal{C}([0,\infty), L_2(g)) : \; x_0 = g\}$ and

$$\mathbf{Coal}^{\mathrm{ex}} := \left\{ x \in \mathbf{E}^g : \begin{array}{c} \forall u, v \in (0,1) \; \forall s \geqslant 0, \; x_s(u) = x_s(v) \\ \text{implies } x_t(u) = x_t(v), \; \forall t \geqslant s \end{array} \right\}. \tag{27}$$

Remark 5.1. Trivially, every function from $\mathbf{Coal}^{\mathrm{ex}}$ satisfies properties (G1)–(G4) of the definition of the set $\mathbf{Coal}$ (see Definition 3.1). So, the only difference between $\mathbf{Coal}$ and $\mathbf{Coal}^{\mathrm{ex}}$ is that in elements of $\mathbf{Coal}^{\mathrm{ex}}$ there could appear more that one coalescence at the same time and the number of particles does not need to be one for large time.

Lemma 5.2. *There exists a map* $\bar{\mathrm{T}} : \mathbf{E}^g \to \mathbf{F}$ *such that* $\bar{\mathrm{T}}(\mathcal{W}^g) = \mathrm{T}(\mathcal{Y}, \mathcal{W}^g)$ *almost surely and* $\bar{\mathrm{T}}^{-1}(\{0\}) = \mathbf{Coal}^{\mathrm{ex}}$.

Before the proof of the lemma we introduce a map $S : \mathbf{E}_0 \to \mathbf{Coal}^{\mathrm{ex}}$ setting $S(x) = y$ for the unique solution from $\mathbf{Coal}^{\mathrm{ex}}$ to the deterministic equation

$$y_t = g + \int_0^t \mathrm{pr}_{y_s} \, \mathrm{d}x_s, \quad t \geqslant 0,$$

which exists, by Lemma B.7. By the construction of y it is easily seen that the map S is measurable, where $\mathbf{Coal}^{\mathrm{ex}}$ is considered as a metric space with the induced topology of $\mathbf{E}_0$.

Proof of Lemma 5.2. We are going to define the map $\bar{\mathrm{T}}$ as follows

$$\bar{\mathrm{T}}(x) := \sum_{k=1}^{n-1} \mathbb{1}_{\{\tau_k^y < \infty\}} e_k^y \left(x_{\cdot + \tau_k^y}, e_k^y \right)_{L_2}, \quad y = S(x), \quad x \in \mathbf{E}_0. \tag{28}$$

However, the problem is that the stopping times τ_k^y, $k \in [n-1]$, and the basis e_k^y, $k \in [n-1] \cup \{0\}$ in $L_2(g)$ were defined in Section 3.3 only for $y \in \mathbf{Coal}$. The stopping times τ_k^y, $k \in [n-1]$, can be defined as in (8) also for $y \in \mathbf{Coal}^{\mathrm{ex}}$. If $\tau_k^y = +\infty$ for some $k \geqslant 1$, the definition of the corresponding e_k^y does not matter since it does not appear in (28).

Let $y \in \mathbf{Coal}^{\mathrm{ex}}$ such that there exists $k \in [n-2]$ and $p \in [n-k-1]$ such that $\tau_{k+p+1}^y < \tau_{k+p}^y = \tau_k^y < \tau_{k-1}^y$, where $\tau_0^y = +\infty$ and $\tau_n^y = 0$. This means that at time τ_k^y there are $p+1$ coalescence points, according to the terminology of Lemma 3.7. Then we define $e_k^y, \ldots, e_{k+p}^y$ as we would do in the proof of Lemma 3.7, if those $p+1$ coalescences happened one after another, in the increasing order of coalescence points. Moreover, the equality

$$(e_k^y, y_t)_{L_2} = 0, \quad t \geqslant \tau_k^y, \tag{29}$$

remains true for $y \in \mathbf{Coal}^{\mathrm{ex}}$.[4] Hence the map $\bar{\mathrm{T}}$ is well-defined for every $x \in \mathbf{E}_0$. Since $\mathbb{P}[\mathcal{Y} \in \mathbf{Coal}] = 1$ and $S(\mathcal{W}^g) = \mathcal{Y}$, we have $\bar{\mathrm{T}}(\mathcal{W}^g) = \mathrm{T}(\mathcal{Y}, \mathcal{W}^g)$ a.s. This completes the proof of the first part of the lemma.

[4] This equality holds for $y \in \mathbf{Coal}$ by Lemma 3.7.

Next, we are going to show that $\bar{\mathrm{T}}^{-1}(\{0\}) = \mathbf{Coal}^{\mathrm{ex}}$. Let $x \in \mathbf{Coal}^{\mathrm{ex}}$. Then, trivially, $y = S(x) = x$. Hence, if $\tau_k^y < \infty$, one can conclude that $(x_{t+\tau_k^y}, e_k^y)_{L_2} = (y_{t+\tau_k^y}, e_k^y)_{L_2} = 0$ for all $k \in [n-1]$, by (29). Consequently, $\bar{\mathrm{T}}(x) = 0$.

We next assume that $\bar{\mathrm{T}}(x) = 0$. Let $y = S(x)$. We fix $t \geqslant \tau_k^y$, for some $k \in [n-1]$ and deduce from the equality $\bar{\mathrm{T}}(x) = 0$ that $(x_t, e_l^y)_{L_2} = 0$ for all $l \in \{k, \dots, n-1\}$. Consequently, $x_t \in L_2(y_{\tau_k^y})$ that implies that

$$\mathrm{pr}_{y_{\tau_k^y}} x_t = x_t \text{ for all } t \geqslant \tau_k^y.$$

For $k = n$, the above equality follows from the fact that $x_t \in L_2(g)$, $t \geqslant 0$. Therefore, for all $t \geqslant 0$

$$y_t = g + \int_0^t \mathrm{pr}_{y_s} \, \mathrm{d}x_s = g + \sum_{l=1}^{n} \int_{\tau_l^y \wedge t}^{\tau_{l-1}^y \wedge t} \mathrm{pr}_{y_{\tau_l^y}} \, \mathrm{d}x_s = x_t.$$

This implies that $x \in \mathbf{Coal}^{\mathrm{ex}}$. The lemma is proved. □

Corollary 5.3. *Let $\mathcal{W}^g$ and $\mathcal{Y}$ be defined as above and $\bar{\mathrm{T}}$ be given by* (28)*. Then for every $A \in \mathcal{B}(\mathbf{E}_0)$*

$$\mathbb{P}\left[\mathcal{W}^g \in A | \mathcal{W}^g \in \mathbf{Coal}^{\mathrm{ex}}\right] = \mathbb{P}\left[\mathcal{W}^g \in A | \bar{T}(\mathcal{W}^g) = 0\right] = \mathbb{P}\left[\mathcal{Y} \in A\right].$$

Proof. The corollary directly follows from equality (26) and Lemma 5.2. □

Proof of Theorem 1.6. Remark that there exists an isomorphism Ξ_m between $L_2(g)$ and the Hilbert space $\mathbb{R}^n$ furnished with the inner product $(a, b)_m = \sum_{k=1}^n a_k b_k m_k$, $a, b \in \mathbb{R}^n$, defined as

$$\Xi_m(f) = (f_k)_{k=1}^n,$$

where $f = \sum_{k=1}^n f_k \mathbb{1}_{\pi_k^0} \in L_2(g)$. We also determine $\tilde{\Xi}_m : \mathcal{C}\left([0, \infty), L_2(g)\right) \to \mathcal{C}[0, \infty)^n$ as

$$\tilde{\Xi}_m(y)(t) = \Xi_m(y_t), \quad t \geqslant 0.$$

Set

$$\begin{aligned}\mathbf{Coal}_m &:= \tilde{\Xi}_m^{-1}(\mathbf{Coal}^{\mathrm{ex}}) \\ &= \left\{ x = (x_k)_{k=1}^n \in \mathcal{C}[0, \infty)^n : \begin{array}{l} \forall k, l \in [n],\ \forall s \geqslant 0,\ x_k(s) = x_l(s) \\ \text{implies } x_k(t) = x_l(t),\ \forall t \geqslant s \end{array} \right\}.\end{aligned}$$

We define for $y \in \mathbf{Coal}_m$ the times $\tau_k^y := \tau_k^{\Xi^{-1}(y)}$ and vectors $e_k^y = \Xi(e_k^{\Xi^{-1}(y)})$, $k \geqslant 1$. Let

$$\bar{\mathrm{T}}^m(x) = \left(\left(x(\cdot + \tau_k^y), e_k^y\right)_{L_2}\right)_{k=1}^{n-1}, \quad y = (\tilde{\Xi} \circ S \circ \tilde{\Xi}^{-1})(x),$$

be a map from $\mathcal{C}[0,\infty)^n$ to $\mathcal{C}[0,\infty)^{n-1}$.

By the construction of $\mathcal{W}^g$, we have that $X = (W_k)_{k=1}^n = \tilde{\Xi}(\mathcal{W}^g)$. We remark that $\bar{\mathrm{T}}(X)$ is a standard Brownian motion in $\mathbb{R}^{n-1}$, according to Lemma 3.10 and Remark 3.13. Define $y = (y_k)_{k=1}^n := \tilde{\Xi}(\mathcal{Y})$. Then using the fact that $L_2(g)$ and $\mathbb{R}^n$ are isomorphic Hilbert spaces and Corollary 5.3, we can conclude that for every $A \in \mathcal{B}\left(\mathcal{C}[0,\infty)^n\right)$

$$\mathbb{P}\left[X \in A | X \in \mathbf{Coal}_m\right] = \mathbb{P}\left[X \in A | \bar{T}_m(X) = 0\right] = \mathbb{P}\left[y \in A\right].$$

This completes the proof of the theorem. □

6 Coupling of MMAF and cylindrical Wiener process

We have already seen, in Proposition 3.6 and its proof, that for every MMAF $\mathcal{Y}$ starting at g there exists a cylindrical Wiener process $\mathcal{W}$ in L_2 starting at g such that equation (3) holds. However, it is unknown whether equation (3) has a strong solution. At least, we know that strong well-posedness holds in finite dimension, see Section 5.

In Proposition 3.6, we considered a process $\mathcal{W}$ defined by (7) and we proved that the pair $(\mathcal{Y}, \mathcal{W})$ satisfies (3). The reverse statement holds true, in the following sense.

Proposition 6.1. *Let $\mathcal{Y}_t$, $t \geqslant 0$, be a MMAF and $\mathcal{W}_t$, $t \geqslant 0$ be a cylindrical Wiener process in L_2 both starting at g and such that $(\mathcal{Y}, \mathcal{W})$ satisfies* (3)*. Then there exists a cylindrical Wiener process B_t, $t \geqslant 0$, in L_2 starting at* 0 *independent of $(\mathcal{Y}, \mathcal{W})$ such that for every $h \in L_2$ almost surely*

$$\mathcal{W}_t(h) = (\mathcal{Y}_t, h)_{L_2} + \int_0^t \mathrm{pr}_{\mathcal{Y}_s}^{\perp} h \cdot \mathrm{d}B_s, \quad t \geqslant 0. \tag{30}$$

As a direct consequence, we prove Theorem 1.5.

Proof of Theorem 1.5. By Proposition 6.1, there are cylindrical Wiener processes B and $\widetilde{B}$ in L_2 starting at 0, independent of $(\mathcal{Y}, \mathcal{W})$ and of $(\mathcal{Y}, \widetilde{\mathcal{W}})$, respectively, such that equation (30) holds for $(\mathcal{Y}, \mathcal{W}, B)$ and for $(\mathcal{Y}, \widetilde{\mathcal{W}}, \widetilde{B})$, respectively. Thus $(\mathcal{Y}, B)$ and for $(\mathcal{Y}, \widetilde{B})$ have the same distribution, and it follows from (30) that $(\mathcal{Y}, \mathcal{W})$ and for $(\mathcal{Y}, \widetilde{\mathcal{W}})$ have the same distribution too. □

In Section 6.1, our goal is to prove Lemma 6.2 and we will show several auxiliary statements in order to achieve this. Then we will apply Lemma 6.2 to show Proposition 6.1 in Section 6.2. Interestingly, the proofs rely on the basis $\{e_k^y,\ k \geqslant 0\}$ defined by Lemma 3.7.

6.1 Some auxiliary lemmas

Recall that we denote $e_k := e_k^{\mathcal{Y}}$ and $\tau_k := \tau_k^{\mathcal{Y}}$, and that for every $k \geqslant 1$, the random element e_k is $\mathcal{F}_{\tau_k}^{\mathcal{Y}}$-measurable. Let $(\mathcal{F}_t^{\mathcal{X}})_{t \geqslant 0}$ be the complete right-continuous filtration generated by $\mathcal{X} := (\mathcal{Y}, \mathcal{W})$.

For every $k \geqslant 1$ we remark that $\mathcal{W}_t^k := \mathcal{W}_{t+\tau_k} - \mathcal{W}_{\tau_k}$, $t \geqslant 0$, is a cylindrical Wiener process starting at 0 independent of $\mathcal{F}_{\tau_k}^{\mathcal{X}}$. Moreover, if $l \geqslant k$, then $\tau_l \leqslant \tau_k$ almost surely and the random element e_l is $\mathcal{F}_{\tau_l}^{\mathcal{X}}$-measurable, hence also $\mathcal{F}_{\tau_k}^{\mathcal{X}}$-measurable. Therefore, the process

$$\mathcal{W}_t^k(e_l) := \int_0^t e_l \cdot \mathrm{d}\mathcal{W}_s^k = \int_{\tau_k}^{t+\tau_k} e_l \cdot \mathrm{d}\mathcal{W}_s, \quad t \geqslant 0, \tag{31}$$

is well-defined.

Lemma 6.2. *The processes $\mathcal{Y}$, $\mathcal{W}^k(e_k)$, $k \geqslant 1$, are independent.*

In order to prove that lemma, we start by some auxiliary definitions and results. The process

$$\zeta_t^k := \int_0^t \mathrm{pr}_{\mathcal{Y}_{\tau_k}} \mathrm{d}\mathcal{W}_s^k, \quad t \geqslant 0,$$

is a well-defined continuous L_2-valued $(\mathcal{F}_{t+\tau_k}^{\mathcal{X}})$-martingale, because $\mathrm{pr}_{\mathcal{Y}_{\tau_k}}$ is $\mathcal{F}_{\tau_k}^{\mathcal{X}}$-measurable and $\mathcal{W}_t^k$, $t \geqslant 0$, is independent of $\mathcal{F}_{\tau_k}^{\mathcal{X}}$. Let $\mathcal{G}_k$ be the complete σ-algebra generated by $\mathcal{X}(t \wedge \tau_k) = (\mathcal{Y}_{t\wedge\tau_k}, \mathcal{W}_{t\wedge\tau_k})$, $t \geqslant 0$, and by ζ_t^k, $t \geqslant 0$.

Lemma 6.3. *For every $k \geqslant 1$ the MMAF $\mathcal{Y}$ is $\mathcal{G}_k$-measurable as a map from Ω to $\mathcal{C}([0,\infty), L_2^{\uparrow})$.*

Proof. In order to show the measurability of $\mathcal{Y}$ with respect to $\mathcal{G}_k$, it is enough to show the measurability of $\mathcal{Y}_{\tau_k+t}$, $t \geqslant 0$.

By Corollary B.8, we know that for every $g \in \mathrm{St}$ and cylindrical Wiener process W, there exists a unique continuous $L_2^{\uparrow}$-valued process Y such that almost surely

$$Y_t = g + \int_0^t \mathrm{pr}_{Y_s} \mathrm{d}W_s^g, \quad t \geqslant 0,$$

where $W_t^g = \int_0^t \mathrm{pr}_g \mathrm{d}W_s$, $t \geqslant 0$.

Let us consider the equation

$$Z_t = \mathcal{Y}_{\tau_k} + \int_0^t \mathrm{pr}_{Z_s} \mathrm{d}\zeta_s^k, \quad t \geqslant 0, \tag{32}$$

where $\zeta_t^k = \int_0^t \mathrm{pr}_{\mathcal{Y}_{\tau_k}} \mathrm{d}\mathcal{W}_s^k$. We note that $\mathcal{Y}_{\tau_k}$ belongs to St almost surely and is independent of $\mathcal{W}^k$. Furthermore, the process $\mathcal{Y}_{\tau_k+t}$, $t \geqslant 0$, is a strong solution to (32). Therefore, it is uniquely determined by ζ^k and $\mathcal{Y}_{\tau_k}$, thus it is $\mathcal{G}_k$-measurable. □

Lemma 6.4. *Let $y \in$ **Coal** and $k \geqslant 1$. Then the processes*

$$\mathcal{W}_t^k(e_l^y) = \int_0^t e_l^y \cdot \mathrm{d}\mathcal{W}_s^k, \quad t \geqslant 0, \ l \geqslant k$$

are independent standard Brownian motions that do not depend on

$$\zeta_t^{y,k} := \int_0^t \mathrm{pr}_{y_{\tau_k^y}} \mathrm{d}\mathcal{W}_s^k, \quad t \geqslant 0.$$

Proof. By Lemma 3.7, the family $\{e_l^y, l \geqslant 0\}$ is orthonormal. Consequently, $\mathcal{W}^k(e_l^y)$, $l \geqslant 0$, are independent Brownian motions. Moreover, again by Lemma 3.7, $\zeta_t^{y,k} = \sum_{j=0}^{k-1} e_j^k \mathcal{W}_t^k(e_j^y)$, $t \geqslant 0$, thus it is independent to $\mathcal{W}^k(e_l^y)$, $l \geqslant k$. □

Lemma 6.5. *For every $k \geqslant 1$ the processes $\mathcal{W}^k(e_l)$, $l \geqslant k$, are independent Brownian motions and do not depend on $\mathcal{G}_k$. Furthermore, for each $l > k$, $\mathcal{W}^l_{\cdot\wedge\tau_{k,l}}(e_l)$ is $\mathcal{G}_k$-measurable, where $\tau_{k,l} := \tau_k - \tau_l$.*

Proof. Let $n \geqslant k$ and $m \geqslant 1$ be fixed. Let h_j, $j \geqslant 0$, be an arbitrary orthonormal basis of L_2. We consider bounded measurable functions

$$\begin{aligned} G_0 &: \mathcal{C}([0,\infty), L_2^{\uparrow}) \times \mathcal{C}[0,\infty)^m \to \mathbb{R} \\ G_1 &: \mathcal{C}([0,\infty), L_2) \to \mathbb{R} \\ F_l &: \mathcal{C}[0,\infty) \to \mathbb{R}, \quad l = k, \ldots, n. \end{aligned}$$

We then use the independence of $\mathcal{W}^k$ from $\mathcal{F}^{\mathcal{X}}_{\tau_k}$.

$$\begin{aligned} E &:= \mathbb{E}\left[G_0\left(\mathcal{Y}_{\cdot\wedge\tau_k}, (\mathcal{W}_{\cdot\wedge\tau_k}(h_j))_{j=1}^m\right) G_1(\zeta^k) \prod_{l=k}^n F_l\left(\mathcal{W}^k(e_l)\right)\right] \\ &= \mathbb{E}\left[G_0\left(\mathcal{Y}_{\cdot\wedge\tau_k}, (\mathcal{W}_{\cdot\wedge\tau_k}(h_j))_{j=1}^m\right) \mathbb{E}\left[G_1(\zeta^k) \prod_{l=k}^n F_l\left(\mathcal{W}^k(e_l)\right) \middle| \mathcal{F}^{\mathcal{X}}_{\tau_k}\right]\right] \\ &= \mathbb{E}\left[G_0\left(\mathcal{Y}_{\cdot\wedge\tau_k}, (\mathcal{W}_{\cdot\wedge\tau_k}(h_j))_{j=1}^m\right) \mathbb{E}\left[G_1(\zeta^{y,k}) \prod_{l=k}^n F_l\left(\mathcal{W}^k(e_l^y)\right)\right]\Bigg|_{y=\mathcal{Y}_{\cdot\wedge\tau_k}}\right]. \end{aligned}$$

Then we apply Lemma 6.4 and we denote by w_l, $l = k, \ldots, n$, a family of standard independent Brownian motions that do not depend on $\mathcal{Y}$ and $\mathcal{W}$.

$$E = \mathbb{E}\left[G_0\left(\mathcal{Y}_{\cdot\wedge\tau_k}, (\mathcal{W}_{\cdot\wedge\tau_k}(h_j))_{j=1}^m\right) G_1(\zeta^k)\right] \prod_{l=k}^n \mathbb{E}\left[F_l\left(w_l\right)\right],$$

which achieves the proof of the first part of the statement.

Furthermore, for every $l > k$, we remark that e_l and τ_l are $\mathcal{G}_k$-measurable because they are $\mathcal{F}^{\mathcal{Y}}_{\tau_l}$-measurable and $\mathcal{F}^{\mathcal{Y}}_{\tau_l} \subseteq \mathcal{F}^{\mathcal{Y}}_{\tau_k} \subseteq \mathcal{G}_k$. Then the process $\mathcal{W}^l_{t\wedge\tau_{k,l}} = \mathcal{W}_{(t\wedge\tau_{k,l})+\tau_l} - \mathcal{W}_{\tau_l}$, $t \geqslant 0$, is $\mathcal{G}_k$-measurable, and consequently, $\mathcal{W}^l_{\cdot\wedge\tau_{k,l}}(e_l)$ is also $\mathcal{G}_k$-measurable. This finishes the proof of the second part of the lemma. □

Next, we define the gluing map $\mathrm{Gl} : \mathcal{C}_0[0,\infty)^2 \times [0,\infty) \to \mathcal{C}_0[0,\infty)$ as follows

$$\mathrm{Gl}(x_1, x_2, r)(t) = x_1(t \wedge r) + x_2\left((t-r)^+\right), \quad t \geqslant 0, \tag{33}$$

where $a^+ := a \vee 0$. It is easily seen that the map Gl is continuous and therefore measurable.

Since almost surely, $\mathcal{W}^k_t(e_l) = \mathcal{W}^l_{t+\tau_k-\tau_l}(e_l) - \mathcal{W}^l_{\tau_k-\tau_l}(e_l)$, $t \geqslant 0$, for every $l > k \geqslant 1$, a simple computation shows that for every $l > k \geqslant 1$ almost surely

$$\mathcal{W}^l(e_l) = \mathrm{Gl}\left(\mathcal{W}^l_{\cdot\wedge\tau_{k,l}}(e_l), \mathcal{W}^k(e_l), \tau_{k,l}\right), \tag{34}$$

where $\tau_{k,l} := \tau_k - \tau_l$.

Proof of Lemma 6.2. In order to prove this lemma, it is enough to show that for each $k \geqslant 1$, $\mathcal{W}^k(e_k)$ is independent of $\mathcal{Y}$, $\mathcal{W}^l(e_l)$, $l > k$.

Let us denote by $\mathcal{H}_k$ be the complete σ-algebra generated by $\mathcal{G}_k$ and $\mathcal{W}^k(e_l)$, $l > k$. By Lemma 6.5, the process $\mathcal{W}^k(e_k)$ is independent of $\mathcal{H}_k$.

Moreover for every $l > k$, using Lemma 6.3, $\mathcal{Y}$ and $\tau_{k,l}$ are $\mathcal{G}_k$-measurable, hence they are $\mathcal{H}_k$-measurable. By Lemma 6.5 and by the definition of $\mathcal{H}_k$, we also see that $\mathcal{W}^l_{\cdot\wedge\tau_{k,l}}(e_l)$ and $\mathcal{W}^k(e_l)$ are $\mathcal{H}_k$-measurable. By (34), it follows that $\mathcal{W}^l(e_l)$ is $\mathcal{H}_k$-measurable for every $l > k$. Therefore $\mathcal{Y}$, $\mathcal{W}^l(e_l)$, $l > k$, are independent of $\mathcal{W}^k(e_k)$. □

6.2 Proof of Proposition 6.1

Let β_k, $k \geqslant 0$, be independent standard Brownian motions, *independent* of $\mathcal{X} = (\mathcal{Y}, \mathcal{W})$. Recall that $\mathrm{pr}^{\perp}_{\mathcal{Y}_t} e_k = \mathbb{1}_{\{t\geqslant\tau_k\}} e_k$, $t \geqslant 0$, is a right-continuous $(\mathcal{F}^{\mathcal{X}}_t)$-adapted process in L_2. Thus we can define for every $k \geqslant 0$

$$B_k(t) := \beta_k(t \wedge \tau_k) + \int_0^t \mathbb{1}_{\{s\geqslant\tau_k\}} e_k \cdot \mathrm{d}\mathcal{W}_s, \quad t \geqslant 0. \tag{35}$$

Since $\tau_0 = +\infty$, we have in particular $B_0(t) = \beta_0(t)$, $t \geqslant 0$.

Lemma 6.6. *The processes B_k, $k \geqslant 0$, defined by* (35)*, are independent standard Brownian motions.*

Proof. $\mathcal{W}$ is an $(\mathcal{F}_t^{\mathcal{X}})$-cylindrical Wiener process. Since β_k, $k \geqslant 0$, are independent of $\mathcal{X}$, the filtration $(\mathcal{F}_t^{\mathcal{X}})$ can be extended to the complete right-continuous filtration $(\mathcal{F}_t^{\mathcal{X},\beta})_{t\geqslant 0}$ generated by $\mathcal{X}$ and β_k, $k \geqslant 0$. Hence, $\mathcal{W}$ is an $(\mathcal{F}_t^{\mathcal{X},\beta})$-cylindrical Wiener process and β_k, $k \geqslant 0$, are independent standard $(\mathcal{F}_t^{\mathcal{X},\beta})$-Brownian motions. Since the L_2-valued right-continuous processes $\mathbb{1}_{\{t\geqslant\tau_k\}}e_k$, $t \geqslant 0$, are $(\mathcal{F}_t^{\mathcal{X},\beta})$-adapted, they are $(\mathcal{F}_t^{\mathcal{X},\beta})$-progressively measurable, and hence, the processes $B_k(t)$, $t \geqslant 0$, $k \geqslant 0$, are $(\mathcal{F}_t^{\mathcal{X},\beta})$-continuous martingales. Moreover, the quadratic variations satisfy $\langle B_k, B_l\rangle_t = \mathbb{1}_{\{k=l\}}t$ for every $k, l \geqslant 0$ and $t \geqslant 0$. By Lévy's characterization of Brownian motion [IW89, Theorem II.6.1], B_k, $k \geqslant 0$, are independent $(\mathcal{F}_t^{\mathcal{X},\beta})$-Brownian motions. □

We will now use the result of Lemma 6.2 to prove the following lemma.

Lemma 6.7. *The processes $\mathcal{Y}$, B_k, $k \geqslant 0$, are independent.*

Proof. Since $B_0 = \beta_0$ is independent of $\mathcal{Y}$ by definition and of B_k, $k \geqslant 1$, by Lemma 6.6, it is enough to prove that the processes $\mathcal{Y}$, B_k, $k \in [n]$, are independent, for any given n.

Putting together (31), (33) and (35), we have

$$B_k = \mathrm{Gl}\left(\beta_k, \mathcal{W}^k(e_k), \tau_k\right), \quad k \in [n].$$

Since β_k, $k \in [n]$, is independent of $(\mathcal{Y}, \mathcal{W})$ and using Lemma 6.2, we deduce that the processes $\mathcal{Y}$, β_k, $\mathcal{W}^k(e_k)$, $k \in [n]$, are independent. Moreover, τ_k, $k \in [n]$, are measurable with respect to $\mathcal{F}^{\mathcal{Y}} := \sigma(\mathcal{Y})$. Let $G_0 : \mathcal{C}([0,\infty), L_2^{\uparrow}) \to \mathbb{R}$, $F_k : \mathcal{C}[0,\infty) \to \mathbb{R}$, $k \in [n]$, be bounded measurable functions. We have

$$\begin{aligned}\mathbb{E}\left[G_0(\mathcal{Y})\prod_{k=1}^n F_k(B_k)\right] &= \mathbb{E}\left[G_0(\mathcal{Y})\,\mathbb{E}\left[\prod_{k=1}^n F_k\left(\mathrm{Gl}\left(\beta_k, \mathcal{W}^k(e_k), \tau_k\right)\right)\middle|\mathcal{F}^{\mathcal{Y}}\right]\right] \\ &= \mathbb{E}\left[G_0(\mathcal{Y})\,\mathbb{E}\left[\prod_{k=1}^n F_k\left(\mathrm{Gl}\left(\beta_k, \mathcal{W}^k(e_k), \tau_k^y\right)\right)\right]\Bigg|_{y=\mathcal{Y}}\right].\end{aligned}$$

Note that if w_1 and w_2 are independent standard Brownian motions and $r > 0$, then the process $\mathrm{Gl}(w_1, w_2, r)$ is a standard Brownian motion. It follows that for any fixed $y \in \mathbf{Coal}$, $\mathrm{Gl}\left(\beta_k, \mathcal{W}^k(e_k), \tau_k^y\right)$, $k \in [n]$, is a family of independent standard Brownian motions. Thus for every $y \in \mathbf{Coal}$,

$$\mathbb{E}\left[\prod_{k=1}^n F_k\left(\mathrm{Gl}\left(\beta_k, \mathcal{W}^k(e_k), \tau_k^y\right)\right)\right] = \prod_{k=1}^n \mathbb{E}\left[F_k(w_k)\right],$$

where w_k, $k \in [n]$, denotes an arbitrary family of independent standard Brownian motions. Therefore,

$$\mathbb{E}\left[G_0(\mathcal{Y})\prod_{k=1}^{n} F_k(B_k)\right] = \mathbb{E}\left[G_0(\mathcal{Y})\right]\prod_{k=1}^{n}\mathbb{E}\left[F_k(w_k)\right],$$

which achieves the proof of the lemma because B_k, $k \in [n]$, are independent standard Brownian motions by Lemma 6.6. □

Now, we finish the proof of Proposition 6.1.

Proof of Proposition 6.1. Define

$$B_t(h) := \sum_{k=0}^{\infty}(h, e_k)_{L_2} B_k(t), \quad h \in L_2.$$

Since B_k, $k \geqslant 0$, are independent Brownian motions that do not depend on $\mathcal{Y}$ and hence on e_k, $k \geqslant 1$, one can show similarly to the proof of Lemma 3.10 that the series converges in $\mathcal{C}[0,\infty)$ almost surely for every $h \in L_2$, and B_t, $t \geqslant 0$, is a cylindrical Wiener process in L_2 starting at 0.

Moreover, B is independent of $\mathcal{Y}$. Indeed, for any $n \geqslant 1$, for any $h_1, \ldots, h_n$ in L_2, for any bounded and measurable functions $F : \mathcal{C}[0,\infty)^n \to \mathbb{R}$ and $G : \mathcal{C}([0,\infty), L_2^{\uparrow}) \to \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\left[F(B(h_1), \ldots, B(h_n))\, G(\mathcal{Y})\right] &= \mathbb{E}\left[\mathbb{E}\left[F(B(h_1), \ldots, B(h_n)) \,\middle|\, \mathcal{F}^{\mathcal{Y}}\right] G(\mathcal{Y})\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[F(w_1, \ldots, w_n)\right] G(\mathcal{Y})\right] \\
&= \mathbb{E}\left[F(B(h_1), \ldots, B(h_n))\right]\mathbb{E}\left[G(\mathcal{Y})\right],
\end{aligned}$$

where w_k, $k \in [n]$, denotes an arbitrary family of independent standard Brownian motions.

Moreover, since $\mathrm{pr}^{\perp}_{\mathcal{Y}_t} e_k = \mathbb{1}_{\{t \geqslant \tau_k\}} e_k$, we easily check that

$$\begin{aligned}
\int_0^t \mathrm{pr}^{\perp}_{\mathcal{Y}_s} h \cdot \mathrm{d}B_s = \int_0^t \mathrm{pr}^{\perp}_{\mathcal{Y}_s} h \cdot \mathrm{d}\mathcal{W}_s &= \int_0^t h \cdot \mathrm{d}\mathcal{W}_s - \int_0^t \mathrm{pr}_{\mathcal{Y}_s} h \cdot \mathrm{d}\mathcal{W}_s \\
&= \mathcal{W}_t(h) - (g, h)_{L_2} - (\mathcal{Y}_t - g, h)_{L_2}
\end{aligned}$$

for all $t \geqslant 0$, which implies equality (30). □

A Appendix: Regular conditional probability

A.1 Definition

Let $\mathbf{E}$ be a Polish space and $\mathbf{F}$ be a metric space. We consider random elements X and ξ in $\mathbf{E}$ and $\mathbf{F}$, respectively, defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let also $\mathcal{B}(\mathbf{E})$ (resp. $\mathcal{B}(\mathbf{F})$) denote the Borel σ-algebra on $\mathbf{E}$ (resp. $\mathbf{F}$) and $\mathcal{P}(\mathbf{E})$ be the space of probability measures on $(\mathbf{E}, \mathcal{B}(\mathbf{E}))$ endowed with the topology of weak convergence.

Definition A.1. A function $p : \mathcal{B}(\mathbf{E}) \times \mathbf{F} \to [0,1]$ is a *regular conditional probability of X given ξ* if

(R1) for every $z \in \mathbf{F}$, $p(\cdot, z) \in \mathcal{P}(\mathbf{E})$;

(R2) for every $A \in \mathcal{B}(\mathbf{E})$, $z \mapsto p(A, z)$ is measurable;

(R3) for every $A \in \mathcal{B}(\mathbf{E})$ and $B \in \mathcal{B}(\mathbf{F})$,

$$\mathbb{P}\left[X \in A,\ \xi \in B\right] = \int_B p(A, z)\, \mathbb{P}^\xi(\mathrm{d}z),$$

where $\mathbb{P}^\xi := \mathbb{P} \circ \xi^{-1}$ denotes the law of ξ.

Recall the following existence and uniqueness result (see e.g. [Kal02, Theorem 6.3]):

Proposition A.2. *There exists a regular conditional probability of X given ξ. Moreover, it is unique in the following sense: if p and p' are regular conditional probabilities of X given ξ, then*

$$\mathbb{P}^\xi\left[z \in \mathbf{F} :\ p(\cdot, z) = p'(\cdot, z)\right] = 1.$$

A.2 Proof of Lemma 2.1

We first recall that the sufficiency of Lemma 2.1 immediately follows from the continuous mapping theorem.

We next prove the necessity. We first choose a family $\{f_k,\ k \geqslant 1\} \subset \mathcal{C}_b(\mathbf{E})$ which strongly separate points in $\mathbf{E}$. One can show that such a family exists since $\mathbf{E}$ is separable (see also [BK10, Lemma 2]). By [EK86, Theorem 4.5] (or [BK10, Theorem 6] for weaker assumptions on the space $\mathbf{E}$), any sequence $\{\mu_n\}_{n\geqslant 1}$ of probability measures on $\mathbf{E}$ converges weakly to a probability measure μ if and only if

$$\int_{\mathbf{E}} f_k(x)\mu_n(\mathrm{d}x) \to \int_{\mathbf{E}} f_k(x)\mu(\mathrm{d}x), \quad n \to \infty,$$

for all $k \geqslant 1$.

We define the following sets

$$A_m^{k,+} = \left\{ z \in \mathbf{F} : \int_{\mathbf{E}} f_k(x) p(\mathrm{d}x, z) - \int_{\mathbf{E}} f_k(x)\nu(\mathrm{d}x) \geqslant \frac{1}{m} \right\},$$
$$A_m^{k,-} = \left\{ z \in \mathbf{F} : \int_{\mathbf{E}} f_k(x)\nu(\mathrm{d}x) - \int_{\mathbf{E}} f_k(x) p(\mathrm{d}x, z) \geqslant \frac{1}{m} \right\}$$

for all $k \geqslant 1$ and $m \geqslant 1$. Let also $A_m^k = A_m^{k,+} \cup A_m^{k,-}$.

Lemma A.3. *If for every $k \geqslant 1$ and $m \geqslant 1$ there exists $\delta_m^k > 0$ such that*

$$\mathbb{P}^{\mathrm{T}(X)}\left[A_m^k \cap B_m^k\right] = 0, \tag{36}$$

where B_m^k is the ball in $\mathbf{F}$ with center z_0 and radius δ_m^k, then p has a version continuous at z_0. Moreover, it can be taken as

$$p'(\cdot, z) = \begin{cases} p(\cdot, z), & \text{if } z \notin \bigcup_{k,m=1}^{\infty}\left(A_m^k \cap B_m^k\right), \\ \nu, & \text{otherwise.} \end{cases}$$

Proof. We first remark that according to (36), $p' = p$ $\mathbb{P}^{\mathrm{T}(X)}$-a.e. Next, let $z_n \to z_0$ in $\mathbf{F}$ as $n \to \infty$. Without loss of generality, we may assume that $z_n \notin \bigcup_{k,m=1}^{\infty}\left(A_m^k \cap B_m^k\right)$ for all $n \geqslant 1$. Let $m \geqslant 1$ and $k \geqslant 1$ be fixed. Then there exists a number N such that $z_n \in B_m^k$ for all $n \geqslant N$. Consequently, $z_n \notin A_m^k$, $\forall n \geqslant N$, that yields

$$\left|\int_{\mathbf{E}} f_k(x)p(\mathrm{d}x, z_n) - \int_{\mathbf{E}} f_k(x)\nu(\mathrm{d}x)\right| < \frac{1}{m}$$

for all $n \geqslant N$. This finishes the proof of the lemma. □

We come back to the proof of Lemma 2.1. Let us assume that p has no version continuous at z_0. Then, according to Lemma A.3, there exists $k \geqslant 1$ and $m \geqslant 1$ such that for every $\delta > 0$

$$\mathbb{P}^{\mathrm{T}(X)}\left[A_m^k \cap B_\delta\right] > 0,$$

where B_δ denotes the ball with center z_0 and radius δ. Without loss of generality, we may assume that $\mathbb{P}^{\mathrm{T}(X)}\left[A_m^{k,+} \cap B_\delta\right] > 0$ for every $\delta > 0$. For every $n \geqslant 1$, let ξ^n be a random element in $\mathbf{F}$ with distribution

$$\mathbb{P}^{\xi^n}[A] = \int_{\mathbf{F}} q_n(z)\mathbb{P}^{\mathrm{T}(X)}[\mathrm{d}z], \quad A \in \mathcal{B}(\mathbf{F}),$$

where

$$q_n(z) = \frac{1}{\mathbb{P}^{\mathrm{T}(X)}\left[A_m^{k,+} \cap B_{\frac{1}{n}}\right]}\mathbb{1}_{A_m^{k,+} \cap B_{\frac{1}{n}}}(z), \quad z \in \mathbf{F}.$$

By the construction, $\mathbb{P}^{\xi^n} \ll \mathbb{P}^{\mathrm{T}(X)}$, $n \geqslant 1$. Moreover, it is easy to see that $\xi^n \to z_0$ in distribution as $n \to \infty$. But

$$\mathbb{E}\left[\int_{\mathbf{E}} f_k(x)p(\mathrm{d}x, \xi^n)\right] \not\to \int_{\mathbf{E}} f_k(x)\nu(\mathrm{d}x), \quad n \to \infty.$$

Indeed, for every $n \geqslant 1$ the random element ξ^n takes values almost surely in $A_m^{k,+}$, which implies that

$$\mathbb{E}\left[\int_{\mathbf{E}} f_k(x)p(\mathrm{d}x, \xi^n)\right] - \int_{\mathbf{E}} f_k(x)\nu(\mathrm{d}x) \geqslant \frac{1}{m}, \quad n \geqslant 1.$$

We have obtained the contradiction with assumption (1). This finishes the proof of Lemma 2.1.

A.3 One example: Brownian bridge

Let $X = \{W_t,\ t \in [0,1]\}$ be a standard Brownian motion, seen as a random element in $\mathbf{E} = \{x \in \mathcal{C}[0,1] :\ x(0) = 0\}$. Let $\mathbf{F} = \mathbb{R}$ and $\mathrm{T} : \mathbf{E} \to \mathbf{F}$ be defined by $\mathrm{T}(x) = x(1)$. What is the conditional distribution of X to the event $\{\mathrm{T}(X) = z_0\}$?

We construct a quadruple $(\mathbf{G}, \Psi, Y, Z)$ satisfying (P1)-(P4). Set $\mathbf{G} = \{x \in \mathbf{E} :\ x(1) = 0\}$. The process $Y = \{Y_t,\ t \in [0,1]\}$ defined by $Y_t := W_t - tW_1$ and the random variable $Z := W_1$ are independent. Moreover, let $\Psi(y,z)(t) = y(t) + tz$, $t \in [0,1]$. It is easily seen that properties (P3) and (P4) are satisfied.

Moreover, the map $z \mapsto \Psi(y,z)$ is continuous for any fixed $y \in \mathbf{G}$. Therefore, $z \mapsto p(\cdot, z)$ is a continuous map from $\mathbf{F}$ to $\mathcal{P}(\mathbf{E})$. By Lemma 2.1 and Proposition 2.3, we conclude that the law of $\Psi(Y, z_0)$ is the conditional law of the Brownian motion $\{W_t,\ t \in [0,1]\}$ to the event $\{W_1 = z_0\}$, along any sequences $\{\xi^n\}$ satisfying (B1) and (B2). This measure is known as the law of the Brownian bridge from 0 to z_0.

B Some properties of MMAF

B.1 Measurability of coalescing set

We recall that the set $D((0,1), \mathcal{C}[0,\infty))$ denotes the space of càdlàg functions from $(0,1)$ to $\mathcal{C}[0,\infty)$ equipped with the Skorokhod distance, which makes it a Polish space. Set

$$D^{\uparrow} := \{y \in D((0,1), \mathcal{C}[0,\infty)) :\ \forall 0 < u < v < 1,\ y_t(u) \leqslant y_t(v)\ \forall t \geqslant 0\}.$$

It is easily seen that $D^{\uparrow}$ a closed subspace of $D((0,1), \mathcal{C}[0,\infty))$. So, we will consider $D^{\uparrow}$ as a Polish subspace of $D((0,1), \mathcal{C}[0,\infty))$. Let

$$\begin{aligned} D_2^{\uparrow} := &\left\{ y \in D^{\uparrow} :\ \forall T \in \mathbb{N},\ \exists K \in \mathbb{N},\ \exists \delta \in \mathbb{Q}_+,\ \max_{t \in [\frac{1}{T}, T]} \|y_t\|_{L_{2+\delta}} \leqslant K \right\} \\ &\cap \left\{ y \in D^{\uparrow} :\ \|y_t - y_0\|_{L_2} \to 0,\ t \to 0 \right\} =: D^1 \cap D^2. \end{aligned}$$

Lemma B.1. *For every $A \in \mathcal{B}(D^{\uparrow})$ the set $A \cap D_2^{\uparrow}$ is a Borel measurable subset of $\mathcal{C}L_2^{\uparrow} := \mathcal{C}([0,\infty), L_2^{\uparrow})$.*

Proof. First we are going to show that $D_2^{\uparrow}$ is a subset of $\mathcal{C}L_2^{\uparrow}$. So, we take $y \in D_2^{\uparrow}$ and check that y is a continuous L_2-valued function. The continuity of y at 0 follows from the definition of $D_2^{\uparrow}$. Let $t > 0$ and $t_n \to t$ as $n \to \infty$. Without loss of generality, we may assume that $t_n \in [\frac{1}{T}, T]$ for some $T \in \mathbb{N}$ and all $n \geqslant 1$. We are going to show that $y_{t_n} \to y_t$ in L_2, $n \to \infty$. Let us note that the sequence $\{y_{t_n}\}_{n\geqslant 1}$ is relatively compact, according to [Kon17a, Lemma 5.1] and the fact that $y_{t_n} \in L_2^{\uparrow}$, $n \geqslant 1$, are uniformly bounded in $L_{2+\delta}$-norm. This implies that there exists a subsequence $N \subseteq \mathbb{N}$ and $f \in L_2^{\uparrow}$ such that $y_{t_n} \to f$ in L_2 along N. On the other hand, $y_{t_n} \to y_t$ pointwise, that implies the equality $f = y_t$. Moreover, it yields that every convergent subsequence of $\{y_{t_n}\}_{n\geqslant 1}$ converges to y_t in L_2. Using the relatively compactness of $\{y_{t_n}\}_{n\geqslant 1}$, we can conclude that $y_{t_n} \to y_t$ in L_2 as $n \to \infty$. Thus, $y \in \mathcal{C}L_2^{\uparrow}$.

Next, we will check that the set $D_2^{\uparrow}$ is measurable in $D^{\uparrow}$. We fix $t \geqslant 0$ and make the following observation. For every $y \in D^{\uparrow}$ the real-valued function y_t is non-decreasing on $(0,1)$. This implies that it has at most countable number of discontinuous points. Hence, by [EK86, Proposition 3.5.3], the convergence $y^n \to y$ in $D^{\uparrow}$ implies the convergence of $y_t^n \to y_t$ a.e. (with respect to the Lebesgue measure on $[0,1]$). Using Fatou's lemma, we get that the set

$$\Lambda(t, f, K, p) := \left\{ y \in D^{\uparrow} : \ \|y_t - f\|_{L_p} \leqslant K \right\} \quad \text{is closed in } D^{\uparrow} \tag{37}$$

for every $K \geqslant 0$, $p \geqslant 2$ and $f \in L_p$. Hence the set

$$D^1 = \bigcap_{T=1}^{\infty} \bigcup_{K=1}^{\infty} \bigcup_{\delta \in \mathbb{Q}_+} \bigcap_{t \in \left[\frac{1}{T}, T\right]} \Lambda(t, 0, K, 2+\delta)$$

is Borel measurable in $D^{\uparrow}$. Using the standard argument and (37), one can check the measurability of D^2. So, the set $D_2^{\uparrow} = D^1 \cap D^2$ is Borel measurable in $D^{\uparrow}$.

We claim that the identity map $\Phi : D_2^{\uparrow} \to \mathcal{C}L_2^{\uparrow}$ is Borel measurable. Indeed, let

$$B_r^T(y) := \left\{ x \in \mathcal{C}L_2^{\uparrow} : \ \max_{t \in [0,T]} \|x_t - y_t\|_{L_2} \leqslant r \right\}.$$

Then the preimage

$$\Phi^{-1}\left(B_r^T(y)\right) = \bigcap_{t \in [0,T]} \Lambda(t, y_t, r, 2)$$

is a closed set in $D^{\uparrow}$, by (37). Since the Borel σ-algebra on $\mathcal{C}L_2^{\uparrow}$ is generated by the family $\left\{ B_r^T(y),\ T, r > 0,\ y \in \mathcal{C}L_2^{\uparrow} \right\}$, Φ is a Borel measurable function. Moreover, it is an injective map. So, using the Kuratowski theorem (see [Par67, Theorem 3.9]) and the fact that $A \cap D_2^{\uparrow} \in \mathcal{B}(D^{\uparrow})$, we obtain that the image $\Phi(A \cap D_2^{\uparrow}) = A \cap D_2^{\uparrow} \in \mathcal{B}(\mathcal{C}L_2^{\uparrow})$ for every $A \in \mathcal{B}(D^{\uparrow})$. □

Lemma B.2. *Let* **Coal** *be defined in Section 3.1. Then* **Coal** *is a Borel measurable subset of* $\mathcal{C}L_2^{\uparrow}$.

Proof. Let $\mathbf{Coal}_D$ consists of all functions from $D^{\uparrow}$ which satisfies conditions (G2)-(G5) of the definition of **Coal** in Section 3.1. Since every function $f \in \mathrm{St}$ has a finite L_p-norm for every $p \geqslant 2$, it is easily seen that

$$\mathbf{Coal}_D \cap D_2^{\uparrow} = \mathbf{Coal}.$$

Hence, according to Lemma B.1, the statement of the lemma will immediately follow from the measurability of $\mathbf{Coal}_D$ in $D^{\uparrow}$. However, this follows from the fact that the set $\mathbf{Coal}_D$ can be determined via values of $y(u) \in \mathcal{C}[0,\infty)$ for u from a countable set U. We leave a detailed proof for the reader. □

B.2 Properties of MMAF

Let $\{\mathcal{Y}(u,t),\ u \in (0,1),\ t \in [0,\infty)\}$ be a MMAF starting at $g \in L_{2+}^{\uparrow}$, and $\mathcal{Y}_t = \mathcal{Y}(\cdot,t)$, $t \geqslant 0$.

Lemma B.3. *If* $\|g\|_{L_{2+\varepsilon}} < \infty$ *for some* $\varepsilon > 0$, *then for every* $T > 0$ *and* $\delta \in \left(0, \frac{\varepsilon}{2+\varepsilon}\right)$ *there exists* $C_{T,\delta}$ *such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\mathcal{Y}_t - g\|_{L_{2+\delta}}^{2+\delta}\right] \leqslant C_{T,\delta}\left(1 + \|g\|_{L_{2+\varepsilon}}\right).$$

Proof. In order to check the estimate, one needs to repeat the proof of [Kon17a, Proposition 4.4] replacing the summation with the integration. □

Recall that for every $f \in \mathrm{St}$, $N(f)$ denotes the number of steps of f. We write $N(f) = \infty$ for each non-decreasing càdlàg function f which does not belong to St. For any $y \in \mathcal{C}([0,\infty), L_2^{\uparrow})$, define

$$\tau_k^y = \inf\{t \geqslant 0 : N(y_t) \leqslant k\}, \quad k \geqslant 0.$$

The following lemma states that a MMAF satisfies almost surely Property (G5) of Definition 3.1.

Lemma B.4. *Let* $\mathcal{Y}_t$, $t \geqslant 0$, *be a MMAF starting at* g. *Then*

$$\mathbb{P}\left[\forall k < N(g), \tau_{k+1}^{\mathcal{Y}} < \tau_k^{\mathcal{Y}}\right] = 1 \quad \text{and} \quad \mathbb{P}\left[\tau_1^{\mathcal{Y}} < +\infty\right] = 1. \tag{38}$$

Proof. The proof of the statement follows from the fact that with probability one, three or more independent Brownian motions cannot meet at the same time, and from mathematical induction. $\square$

Lemma B.5. *For every $y \in \mathbf{Coal}$, $\beta > 0$ and $n \geqslant 1$ one has*

$$\sum_{k=n}^{\infty} \left(\tau_k^y\right)^{\beta} = \beta \int_0^{\tau_n^y} (N(y_t) - n) t^{\beta-1} \mathrm{d}t.$$

Proof. For simplicity of notation we will omit the superscript y in τ_k^y. We write for $m > n$

$$\begin{aligned}
\sum_{k=n}^{m} \tau_k^{\beta} &= \sum_{k=n+1}^{m} (k-n)\left(\tau_{k-1}^{\beta} - \tau_k^{\beta}\right) + (m+1-n)\tau_m^{\beta} \\
&= \sum_{k=n+1}^{m} (N(y_{\tau_k}) - n)\left(\tau_{k-1}^{\beta} - \tau_k^{\beta}\right) + (N(y_{\tau_{m+1}}) - n)\tau_m^{\beta} \\
&= \int_0^{\tau_n} \left(N(y_{t\vee\tau_{m+1}}) - n\right) \mathrm{d}t^{\beta} = \beta \int_0^{\tau_n} \left(N(y_{t\vee\tau_{m+1}}) - n\right) t^{\beta-1}\mathrm{d}t,
\end{aligned}$$

Hence, the statement of the lemma follows from the monotone convergence theorem. $\square$

Lemma B.6. *Let $\mathcal{Y}$ be a MMAF starting at $g \in L_{2+}^{\uparrow}$. Then for every $\beta > \frac{1}{2}$, $\sum_{k=1}^{\infty}(\tau_k^{\mathcal{Y}})^{\beta} < +\infty$ almost surely.*

Proof. Let $g \in L_{2+\varepsilon}$ for some $\varepsilon > 0$. In order to prove the lemma, we will use the estimate

$$\mathbb{E}\left[N(\mathcal{Y}_t)\right] \leqslant \frac{C_{\varepsilon,T}}{\sqrt{t}} \left(1 + \|g\|_{L_{2+\varepsilon}}\right), \quad t \in (0,T],$$

from [Kon17a, Remark 4.6], where $C_{\varepsilon,T}$ is a constant depending on ε and $T > 0$. Take an arbitrary number $T > 0$ and estimate for $\beta > \frac{1}{2}$

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\tau_1^{\mathcal{Y}}\wedge T} N(\mathcal{Y}_t) t^{\beta-1}\mathrm{d}t\right] &\leqslant \int_0^T \mathbb{E}\left[N(\mathcal{Y}_t)\right] t^{\beta-1}\mathrm{d}t \\
&\leqslant C_{\varepsilon,T}\left(1 + \|g\|_{L_{2+\varepsilon}}\right) \int_0^T t^{\beta-\frac{3}{2}}\mathrm{d}t < +\infty.
\end{aligned}$$

Thus $\int_0^{\tau_1^{\mathcal{Y}}\wedge T} N(\mathcal{Y}_t)t^{\beta-1}\mathrm{d}t < \infty$ almost surely, for all $T > 0$. Since $\tau_1^{\mathcal{Y}} < \infty$ almost surely by (38), $\int_0^{\tau_1^{\mathcal{Y}}} N(\mathcal{Y}_t)t^{\beta-1}\mathrm{d}t < \infty$ almost surely. Thus, the statement of the lemma follows directly from Lemma B.5. $\square$

B.3 Uniqueness of solutions to deterministic equation

Let $g \in \mathrm{St}$ be fixed and $N(g) = n$. For functions $x \in \mathcal{C}([0,\infty), L_2(g))$ and $y \in \mathbf{Coal}^{\mathrm{ex}}$, defined by (27), we introduce the integral

$$\int_0^t \mathrm{pr}_{y_s} \, \mathrm{d}x_s = \sum_{k=1}^{n} \left(\mathrm{pr}_{y_{\tau_k^y \wedge t}} x_{\tau_{k-1}^y \wedge t} - \mathrm{pr}_{y_{\tau_k^y \wedge t}} x_{\tau_k^y \wedge t} \right). \tag{39}$$

Lemma B.7. *For every $x \in \mathcal{C}([0,\infty), L_2(g))$, there exists a unique $y \in$ $\mathbf{Coal}^{\mathrm{ex}}$ such that*

$$y_t = g + \int_0^t \mathrm{pr}_{y_s} \, \mathrm{d}x_s, \quad t \geqslant 0. \tag{40}$$

Proof. Without loss of generality, we assume that $x_0 = g$. The function $y \in \mathbf{Coal}^{\mathrm{ex}}$ can be constructed step by step. First take $\sigma_0 = 0$ and $\tilde{y}_t^0 = g$, $t \geqslant 0$. Then set

$$\tilde{y}_t^k := \tilde{y}_{\sigma_{k-1}}^{k-1} + \mathrm{pr}_{\tilde{y}_{\sigma_{k-1}}^{k-1}} x_t - \mathrm{pr}_{\tilde{y}_{\sigma_{k-1}}^{k-1}} x_{\sigma_{k-1}}, \quad t \geqslant \sigma_{k-1},$$

and

$$\sigma_k := \inf \left\{ t > \sigma_{k-1} : \ \dim L_2(\tilde{y}_t^k) < \dim L_2(\tilde{y}_{\sigma_{k-1}}^{k-1}) \right\}$$

for all $k \in [n-1]$. Remark that $\dim L_2(\tilde{y}_{\sigma_k}^k) \in [n-k]$ for each $0 \leqslant k \leqslant n-1$. We set $\sigma_n = +\infty$. The function y can be defined as

$$y_t = \tilde{y}_t^k \quad \text{for } t \in [\sigma_{k-1}, \sigma_k), \quad k \in [n].$$

By construction, y belongs to $\mathbf{Coal}^{\mathrm{ex}}$, satisfies (40) and is uniquely determined. □

Corollary B.8. *Let W be a cylindrical Wiener process in $L_2(g)$ starting at g. Then there exists a unique $(\mathcal{F}_t^W)$-adapted process Y_t, $t \geqslant 0$, such that*

$$Y_t = g + \int_0^t \mathrm{pr}_{Y_s} \, \mathrm{d}W_s, \quad Y_t \in L_2^{\uparrow}, \quad t \geqslant 0,$$

where $(\mathcal{F}_t^W)_{t \geqslant 0}$ is the filtration generated by W.

Proof. The statement of the lemma directly follows from Lemma B.7 and the fact that $L_2^{\uparrow}$-valued continuous martingales starting from g belongs to **Coal** almost surely (see [Kon17a, Proposition 2.2]). □

B.4 On map φ

In Remark 4.2, we announced the following result.

Lemma B.9. *For every $y \in$ **Coal** and $z = (z_k)_{k\geqslant 1} \in \mathcal{C}_0[0,\infty)^{\mathbb{N}}$ define similarly to* (15)

$$\varphi_t(y,z) = y_t + \sum_{k=1}^{\infty} e_k^y \mathbb{1}_{\left\{t \geqslant \tau_k^y\right\}} z_k(t-\tau_k^y), \quad t \geqslant 0,$$

*if the series converges in $\mathcal{C}([0,\infty),L_2)$. Then for each $y \in$ **Coal**, $\varphi(y,z)$ belongs to **Coal** if and only if $z=0$.*

Proof. It is obvious that $\varphi(y,0) = y \in$ **Coal**.

We assume now that $\varphi(y,z)$ belongs to **Coal** and prove that $z=0$. Set

$$\gamma(y,z) = \sum_{k=1}^{\infty} e_k^y \mathbb{1}_{\left\{t \geqslant \tau_k^y\right\}} z_k(t-\tau_k^y), \quad t \geqslant 0,$$

and show that $\gamma(y,z) = 0$. This will immediately imply $z=0$.

Step I. Let $k \geqslant 1$ be fixed. By (9), there exist $a<b<c$ such that

$$e_k^y = \frac{1}{\sqrt{c-a}}\left(\sqrt{\frac{c-b}{b-a}}\mathbb{1}_{[a,b)} - \sqrt{\frac{b-a}{c-b}}\mathbb{1}_{[b,c)}\right).$$

The goal of this step is to show that $\varphi_{\tau_k^y}(y,z)(u) = y_{\tau_k^y}(u)$ for every $u \in [a,c)$, in other words, that $\gamma_{\tau_k^y}(y,z)$ is equal to zero on the interval $[a,c)$.

By the construction of τ_k^y and e_k^y, $y_{\tau_k^y}$ is constant on the interval $[a,c)$. Furthermore, since $\varphi(y,z) \in$ **Coal**, $\varphi_{\tau_k^y}(y,z)$ belongs to $L_2^{\uparrow}$. Hence, we can deduce that $\gamma_{\tau_k^y}(y,z) = \varphi_{\tau_k^y}(y,z) - y_{\tau_k^y}(y,z)$ is non-decreasing on $[a,c)$, as a difference of a non-decreasing function and a constant function. Furthermore,

$$\gamma_{\tau_k^y}(y,z) = \sum_{l=k}^{\infty} e_l^y z_l(\tau_k^y - \tau_l^y) = \sum_{l=k+1}^{\infty} e_l^y z_l(\tau_k^y - \tau_l^y),$$

since $z_k(0)=0$. Hence, $\gamma_{\tau_k^y}(y,z)$ belongs to $\operatorname{span}\left\{e_l^y,\ l \geqslant k+1\right\}$, whereas $\mathbb{1}_{[a,b)}$ and $\mathbb{1}_{[a,c)}$ both belong to $\operatorname{span}\left\{e_l^y,\ l \leqslant k\right\}$. Indeed, $\mathbb{1}_{[a,c)} \in L_2(y_{\tau_k^y}) = \operatorname{span}\left\{e_l^y,\ l<k\right\}$, by Lemma 3.7, and $\mathbb{1}_{[a,b)} \in \operatorname{span}\left\{\mathbb{1}_{[a,c)}, e_k^y\right\}$. Recall that $\{e_l^y,\ l \geqslant 0\}$ is an orthonormal basis of L_2. Thus,

$$\left(\gamma_{\tau_k^y}(y,z), \mathbb{1}_{[a,b)}\right)_{L_2} = \left(\gamma_{\tau_k^y}(y,z), \mathbb{1}_{[a,c)}\right)_{L_2} = 0.$$

So, we can deduce that $u \mapsto (\gamma_{\tau_k^y}(y,z), \mathbb{1}_{[a,u)})_{L_2}$ is a convex function on $[a,c]$ which vanishes at a, b and c. Thus, it is zero everywhere on $[a,c]$. In particular, $\gamma_{\tau_k^y}(y,z)(u) = 0$ for every $u \in (a,c)$. Consequently, $\varphi_{\tau_k^y}(y,z)(u) = y_{\tau_k^y}(u)$ for every $u \in (a,c)$. The equality also holds for $u = a$, by the right-continuity of $\varphi_{\tau_k^y}(y,z)$ and $y_{\tau_k^y}$.

Step II. Now let $t > 0$ be fixed. By Property (G3) of the definition of **Coal** in Section 3.1, y_t belongs to St, and thus,

$$y_t(u) = \sum_{j=1}^{n} y_j \mathbb{1}_{[a_j,c_j)}(u),$$

for pairwise distinct y_j, $j \in [n]$. Fix $j \in [n]$. By coalescence Property (G4), there exists $k \geqslant 1$ such that $u \mapsto y_s(u)$ is constant on $[a_j,c_j)$ for every $s \geqslant \tau_k^y$ and non-constant on $[a_j,c_j)$ for every $s < \tau_k^y$. By Step I, $y_{\tau_k^y} = \varphi_{\tau_k^y}(y,z)$ on $[a_j,c_j)$. Thus, $\varphi_{\tau_k^y}(y,z)$ is constant on $[a_j,c_j)$. By Property (G4) again, now applied to $\varphi(y,z)$, $\varphi_t(y,z)$ is constant on $[a_j,c_j)$ due to $t \geqslant \tau_k^y$. As the difference of two constant functions, $\gamma_t(y,z)$ is also constant on $[a_j,c_j)$. Moreover, by the construction of γ and Lemma 3.7, $\gamma_t(y,z)$ is orthogonal to $L_2(y_t)$. Hence $\gamma_t(y,z)$ is also orthogonal to $\mathbb{1}_{[a_j,c_j)}$. Therefore, we can conclude that $\gamma_t(y,z) = 0$ on $[a_j,c_j)$. Since $j \in [n]$ and $t > 0$ were arbitrary, we deduce that $\gamma_t(y,z) = 0$ on $[0,1)$ for every $t > 0$. This finishes the proof of the lemma. $\square$

References

[BK10] Douglas Blount and Michael A. Kouritzin. On convergence determining and separating classes of functions. *Stochastic Process. Appl.*, 120(10):1898–1907, 2010.

[CE05] Alexander S. Cherny and Hans-Jürgen Engelbert. *Singular stochastic differential equations*, volume 1858 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005.

[EK86] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986. Characterization and convergence.

[GM11] Leszek Gawarecki and Vidyadhar Mandrekar. *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations*. Probability and its Applications (New York). Springer, Heidelberg, 2011.

[IN68] Kiyosi Itô and Makiko Nisio. On the convergence of sums of independent Banach space valued random variables. *Osaka Math. J.*, 5:35–48, 1968.

[IW89] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam; Kodansha, Ltd., Tokyo, second edition, 1989.

[Kak48] Shizuo Kakutani. On equivalence of infinite product measures. *Ann. of Math. (2)*, 49:214–224, 1948.

[Kal02] Olav Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[Kon10] V. V. Konarovskiĭ. On an infinite system of diffusing particles with coalescing. *Teor. Veroyatn. Primen.*, 55(1):157–167, 2010.

[Kon14] V. V. Konarovskyi. Large deviations principle for finite system of heavy diffusion particles. *Theory Stoch. Process.*, 19(1):37–45, 2014.

[Kon17a] Vitalii Konarovskyi. On asymptotic behavior of the modified Arratia flow. *Electron. J. Probab.*, 22:Paper No. 19, 31, 2017.

[Kon17b] Vitalii Konarovskyi. A system of coalescing heavy diffusion particles on the real line. *Ann. Probab.*, 45(5):3293–3335, 2017.

[KvR19] Vitalii Konarovskyi and Max-K. von Renesse. Modified massive Arratia flow and Wasserstein diffusion. *Comm. Pure Appl. Math.*, 72(4):764–800, 2019.

[Mar18] Victor Marx. A new approach for the construction of a Wasserstein diffusion. *Electron. J. Probab.*, 23:Paper No. 124, 54, 2018.

[Par67] K. R. Parthasarathy. *Probability measures on metric spaces*. Probability and Mathematical Statistics, No. 3. Academic Press, Inc., New York-London, 1967.