Coalescing diffusion particles on the real line

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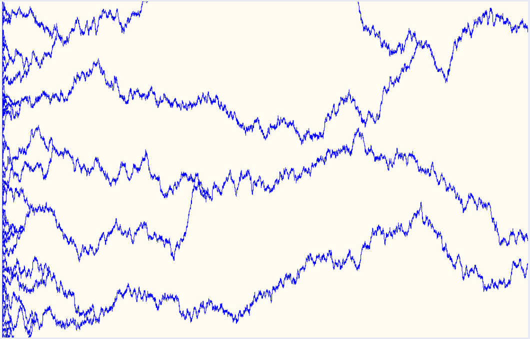
Arratia flow

The system of Brownian particles on the real line that (**Arratia R. A. '79**)

1. start from all points of $\mathbb{R}$;
2. move independently up to the moment of meeting;
3. coalesce;
4. the diffusion is unchangeable and equals 1.



Arratia flow

The Arratia flow, the mathematical description

$\{x(u,t),\ t \geq 0, u \in \mathbb{R}\}$ such that

1. $x(u,\cdot)$ is a Brownian motion;
2. $x(u,0) = u, \quad u \in \mathbb{R}$;
3. $x(u,t) \leq x(v,t), \quad u < v, \ \ t \geq 0$;
4. $\langle x(u,\cdot), x(v,\cdot)\rangle_t = 0, \quad t < \tau_{u,v}$,
   where $\tau_{u,v} = \inf\{t : x(u,t) = x(v,t)\}$.

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Main object

The aim of the talk is to build and investigate the system of diffusion particles on the real line such that particles

1. start from some set of points;
2. move independently up to the moment of the meeting and then coalesce;
3. masses add after coalescing;
4. the diffusion is inversely proportional to the mass;

History of the question

Systems of heavy interacting particles

- Coalescing Brownian particles which have some masses and these masses vary by the some law. The mass does not influence motion of the particles **Dawson D. A. '01, '04**.
- Stochastic differential equation with interaction
$$\begin{cases} dx(u,t) = a(x(u,t),\mu_t,t)dt + \int_{\mathbb{R}} b(x(u,t),\mu_t,t,q)W(dt,dq) \\ x(u,0) = u, \mu_t = \mu_0 \circ x(\cdot,t)^{-1}. \end{cases}$$
Dorogovtsev A. A. '07.
- The cases of finite and infinite numbers of particles with sticking that have mass and speed and their motion obey the conservation law **Sinai Ya. G. '96**.

Main problems of the construction of our system

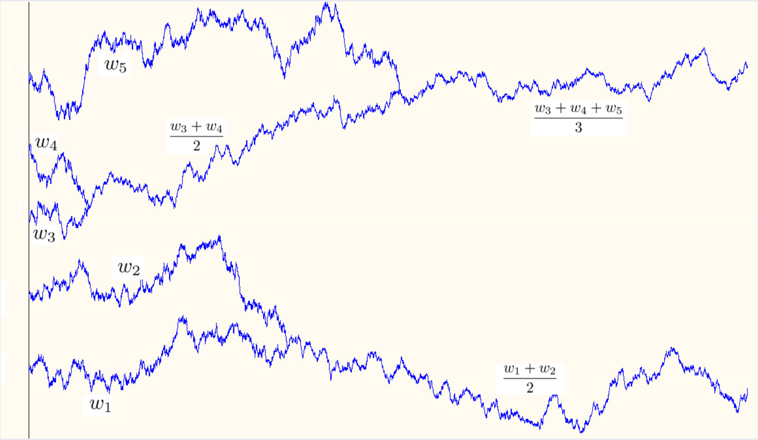
- a priori we cannot start with a finite system
  (the motion of one particle depends on the whole system)
- in case of continuum starting points
  the mass initially is zero $\Leftrightarrow$ infinite diffusion

Steps of construction

1. a finite number of starting points
   (an algebraic construction from independent Wiener processes)
2. a countable number of starting points
   (gaps between independent Wiener processes)
3. the general situation
   (weak compactness in a Skorohod space)

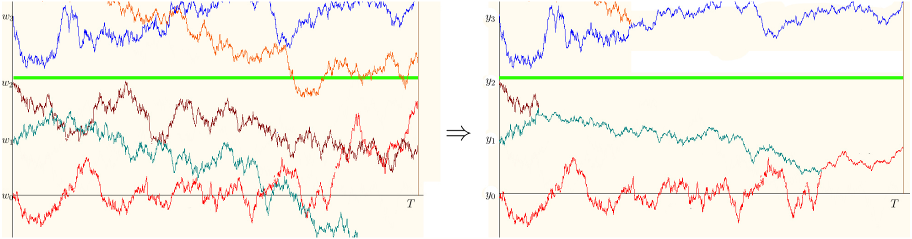
Finite system

Let $w_k(t)$, $t \geq 0$, $k = 1, \ldots, n$, be a set of independent Wiener processes.



Countable system

$\{w_k(t),\ t \geq 0,\ k = 0, 1, 2, \ldots\}$ is a fixed system of independent Wiener processes, $w_k(0) = k, \ \ k = 0, 1, 2, \ldots$



Countable system

Theorem 1 (K. '11)

There exists a sequence of processes $\{y_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$ such that

1°) $y_k(\cdot)$ is a continuous square integrable martingale with respect to

$$\mathcal{F}_t = \sigma(y_i(s),\ s \leq t,\ i \in \mathbb{Z});$$

2°) $y_k(0) = k,\ k \in \mathbb{Z};$

3°) $y_k(t) \leq y_l(t),\ k < l;$

4°) $\langle y_k(\cdot)\rangle_t = \int\limits_0^t \frac{1}{m_k(s)} ds,$
where $m_k(t) = |\{i :\ \exists s \leq t\ y_i(s) = y_k(s)\}|;$

5°) $\langle y_k(\cdot), y_l(\cdot)\rangle_t = 0, \quad t < \tau_{k,l},$
where $\tau_{k,l} = \inf\{t :\ y_k(t) = y_l(t)\}.$

Conditions 1°)–5°) uniquely determine the distribution of $\{y_k(t),\ t \geq 0,\ k \in \mathbb{Z}\}$.

Asymptotic properties of countable system

The estimation of the asymptotic growth of the mass (K. '11)

- $\mathbb{P}\left\{\overline{\lim_{t\to+\infty}} \frac{m_k(t)}{4\sqrt{t\ \ln\ln t}} \leq 1\right\} = 1.$

The asymptotic behaviour of particles

- $\mathbb{P}\left\{\lim_{t\to+\infty} \frac{|y_k(t)|}{\sqrt{2t\ \ln\ln t}} = 0\right\} = 1,$
- $\mathbb{P}\left\{\overline{\lim_{t\to+\infty}} \frac{|y_k(t)|}{\sqrt[4]{t^{1-\varepsilon}}} = \infty\right\} = 1,$ for all $\varepsilon \in (0,1)$.

The asymptotics of the probability of collision time (K. '13)

- $\lim_{t\to 0} t \ln \mathbb{P}\{\tau_{1,n} \leq t\} = -\frac{n^3-n}{24}.$

Continuum system

Theorem 2 (K. '14)

There exists a random element $\{y(u,t),\ u \in [0,1],\ t \in [0,T]\}$ in the Skorohod space $D([0,1], C[0,T])$ such that

1°) $y(u,\cdot)$ is a continuous square integrable martingale with respect to

$$\mathcal{F}_t = \sigma(y(u,s),\ u \in [0,1],\ s \le t), \quad t \in [0,T];$$

2°) $y(u,0) = u,\ u \in [0,1]$;

3°) $y(u,t) \le y(v,t),\ u < v$ and $t \in [0,T]$;

4°) $\langle y(u,\cdot)\rangle_t = \int\limits_0^t \frac{ds}{m(u,s)}$,
where $m(u,t) = \lambda\{v :\ \exists s \le t\ y(v,s) = y(u,s)\}$;

5°) $\langle y(u,\cdot), y(v,\cdot)\rangle_t = 0, \quad t \le \tau_{u,v}$,
where $\tau_{u,v} = \inf\{t :\ y(u,t) = y(v,t)\} \wedge T$.

Properties of particles system

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The mass growth (K. '14)

- For $\beta \in \left(0, \frac{3}{2}\right)$

$$\mathbb{E}\frac{1}{m^\beta(u,t)} \leq \frac{C}{\sqrt[3]{t^\beta}}.$$

The number of clasters (K. '14)

- $\mathbb{E}N(t) \leq \frac{C}{\sqrt[3]{t}}$, where $N(t) = \#\{y(u,t),\ u \in [0,1]\}$.

  compare with the Arratia flow (**Vovchanskii M. B. '13**):

  $\mathbb{E}N'(t) \leq \frac{C}{\sqrt{t}}$, where $N'(t) = \#\{x(u,t),\ u \in [0,1]\}$.

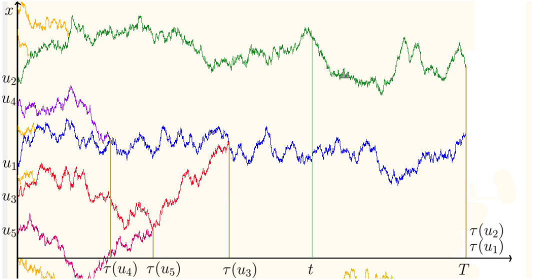
The square mean deviation

- $\mathbb{E}(y(u,t) - u)^2 \leq C\sqrt[3]{t^2}$.

Total free time run of particles from the Arratia flow

Let a set of distinct points $\{u_k, k \in \mathbb{N}\}$ be dense in [0,1].

$$\tau(u_k) = \inf\left\{ t : \prod_{l=1}^{k-1} (x(u_k,t) - x(u_l,t)) = 0 \right\} \wedge T, \quad k \geq 1.$$



Theorem 3 (Dorogovtsev A. A. '04).

The sum $\sum_{k=1}^{\infty} \tau(u_k)$ is finite a.s. and does not depend on the set $\{u_k, k \in \mathbb{N}\}$.

Special stochastic integral with respect to Arratia flow

Theorem 4 (Dorogovtsev A. A. '04).

Let $a : \mathbb{R} \to \mathbb{R}$ be a measurable bounded function. Then the series

$$\sum_{n=1}^{\infty} \int_0^{\tau(u_n)} a(x(u_n, s))dx(u_n, s)$$

is convergent in L_2 and its sum independent of the set $\{u_k, k \in \mathbb{N}\}$.

Denote

$$\int_0^1 \int_0^{\tau(u)} a(x(u, s))dx(u, s) = \sum_{n=1}^{\infty} \int_0^{\tau(u_n)} a(x(u_n, s))dx(u_n, s).$$

Girsanov theorem for diffusion processes with coalescence

Let $a : \mathbb{R} \to \mathbb{R}$ be a bounded Lipschitz continuous function

The Arratia flow with drift

$\{z(u,t),\ u \in [0,1],\ t \in [0,T]\}$ such that

1. $\mathcal{M}(u,\cdot) = z(u,\cdot) - \int\limits_0^{\cdot} a(z(u,s))ds$ is a Brownian motion;
2. $z(u,0) = u, \quad u \in [0,1]$;
3. $z(u,t) \le z(v,t), \quad u < v$;
4. $\langle \mathcal{M}(u,\cdot), \mathcal{M}(v,\cdot)\rangle_t = 0, \quad t < \sigma_{u,v} = \inf\{t : z(u,t) = z(v,t)\} \wedge T$.

Theorem 5 (Dorogovtsev A. A. '07)

The distribution of z is absolutely continuous with respect to the distribution of x in the space $D([0,1], C([0,T]))$ with the density

$$p(x) = \exp\left\{ \int\limits_0^1 \int\limits_0^{\tau(u)} a(x(u,s))dx(u,s) - \frac{1}{2} \int\limits_0^1 \int\limits_0^{\tau(u)} a(x(u,s))ds \right\}.$$

Integral with respect to the stochastic flow of heavy particles

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Proposition 1

For each bounded piecewise continuous function φ there exists the limit

$$\int_0^1 \int_0^{\cdot} \varphi(y(u,s))dy(u,s)du := \lim_{\lambda \to 0} \sum_{k=1}^{n} \int_0^{\cdot} \varphi(y(u_k,s))dy(u_k,s)\Delta u_k$$

in a space of continuous square integrable martingale, and the quadratic characteristic of the limit is

$$\int_0^{\cdot} \int_0^1 \varphi^2(y(u,s))duds.$$

Analog of Ito formula

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Theorem 6 (K. '14)

For each twice continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ having bounded derivatives we have

$$\int_0^1 \varphi(y(u,t))du = \int_0^1 \varphi(u)du +$$

$$+ \int_0^1 \int_0^t \dot{\varphi}(y(u,s))dy(u,s)du + \frac{1}{2}\int_0^t \int_0^1 \frac{\ddot{\varphi}(y(u,s))}{m(u,s)}duds, \quad t \in [0,T].$$

Comparison with integral introduced by Dorogovtsev A. A.

Proposition 2

Let φ be a bounded continuous function. Then a.s.

$$\int_0^1 \int_0^t \frac{\varphi(y(u,s))}{m(u,s)} ds du = \int_0^1 \int_0^{\tau(u)\wedge t} \varphi(y(u,s)) ds.$$

Local time for Arraria flow

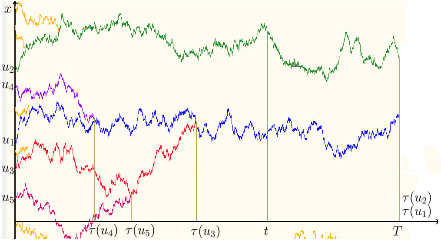
Let $\{x(u,t),\ u\in\mathbb{R},\ t\in[0,T]\}$ be the Arratia flow and $\{u_k\}$ be a dense set of $\mathbb{R}$

Definition (Chernega P. P. '12)

The integral

$$\int\limits_{\mathbb{R}}\int\limits_0^{\tau(u)\wedge t}\delta_0(x(u,s))ds=\sum_{k=1}^{\infty}\int\limits_0^{\tau(u_k)\wedge t}\delta_0(x(u_k,s))ds$$

is called the total local time



Local time for particles with masses

Definition

A random process $\{L(a,t),\ a\in\mathbb{R},\ t\in[0,T]\}$ is said to be the local time for the process $\{y(u,t),\ u\in[0,1],\ t\in[0,T]\}$ if

(a) $(a,t)\to L(a,t)$ is a continuous map a.s.;

(b) for every continuous compactly supported function f

$$\int_0^1\int_0^{\tau(u)\wedge t} f(y(u,s))ds = 2\int_{-\infty}^{+\infty} f(a)L(a,t)da.$$

$L(\cdot,t)$ is a density of

$$\mu(A)=\int_0^1\int_0^{\tau(u)\wedge t}\mathbb{I}_A(y(u,s))ds,\quad A\in\mathcal{B}(\mathbb{R}).$$

Existence of local time

Theorem 7 (K. '14)

The local time for the flow $\{y(u,t),\ u\in[0,1],\ t\in[0,T]\}$ exists. Moreover,

$$L(a,t)=\int_0^1 (y(u,t)-a)^+du-\int_0^1 (u-a)^+du-$$
$$-\int_0^1\int_0^t \mathbb{I}_{(a,+\infty)}(y(u,s))dy(u,s)du.$$

Open problems

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1. Uniqueness of distribution;
2. Asymptotic growth of a particle's mass for small t;
3. Asymptotic behaviour of the particle for small t;
4. Large deviation principle.

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Thank you!