Potts model and spanning forests

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1. Potts model and cluster representation
2. A $q \to 0$ limit: spanning trees, determinants and fermions
3. Another $q \to 0$ limit: spanning forests
   (alias "arboreal gas" alias "tree percolation")
   1. High-T expansion
   2. $1/d$ expansion
   3. Results for $\mathbb{Z}^d$, d=3,4,. . .

References

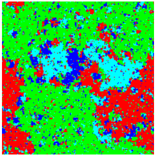
Recent work on Spanning forests/Tree percolation/Arboreal gas

- Transfer matrix, $d = 2$: *Jacobsen, Salas, Sokal,* J. Stat. Phys. 119, 1153 (2005) [cond-mat/0401026]
- Fermionic/Susy field theory: *Jacobsen, Saleur,* Nucl. Phys. B 716, 439 (2005) [cond-mat/0502052]
- MC, $d = 3, 4, 5$: *Deng, Garoni, Sokal,* PRL 98, 030602 (2007) [cond-mat/0610193]
- HT series, all d: MH, WJ, in preparation

Potts model

- *Potts 1952*
- Graph $G = (V, B)$: vertices and bonds
- discrete local degrees of freedom (spins) $s_i \in \{1, \dots, q\}$ on vertices

$$Z = \sum_{\{s_i\}} e^{-\beta H}, \quad H = -J \sum_{b \in B} \delta(s_{b_1}, s_{b_2})$$



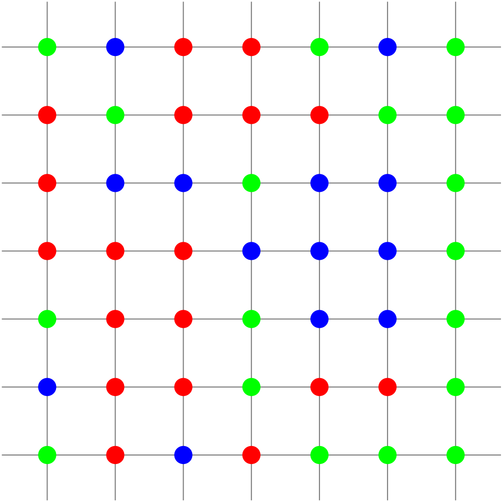
$q = 4$ Potts configuration

Potts model

Infinite volume limit $G \to \mathbb{Z}^d$: Phase transition for some critical value β_c

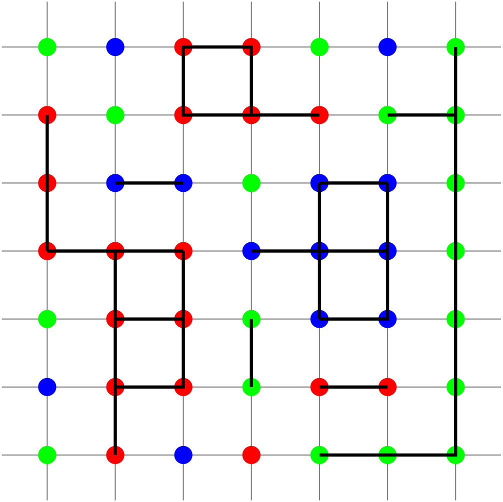
- First order PT for large q
- Second order PT for, e.g., $q \leq 4$ in $d = 2$ and $q \leq 2$ in $d > 2$
  - diverging correlation length, universal critical exponents
  - $\xi \sim |\beta - \beta_c|^{-\nu}$
  - $\chi \sim |\beta - \beta_c|^{-\gamma}$
  - continuum limit can be described by an Euclidean field theory

Clusters



- HT cluster
- *Fortuin, Kasteleyn 1972:* bond active only with probability $1 - e^{-\beta J}$

Clusters



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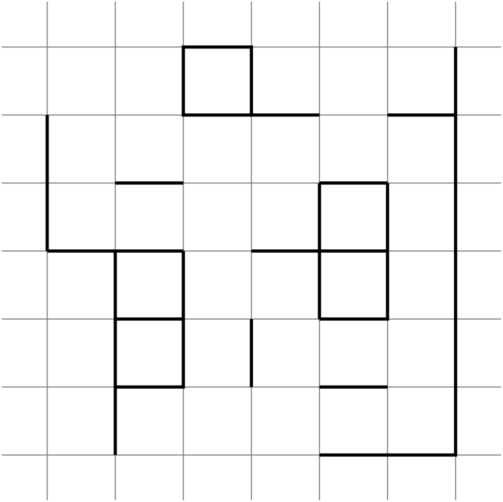

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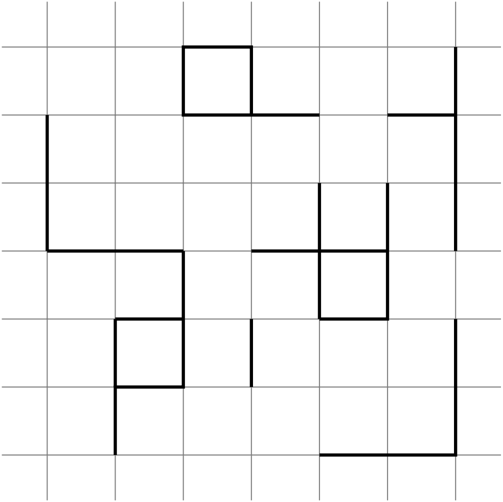

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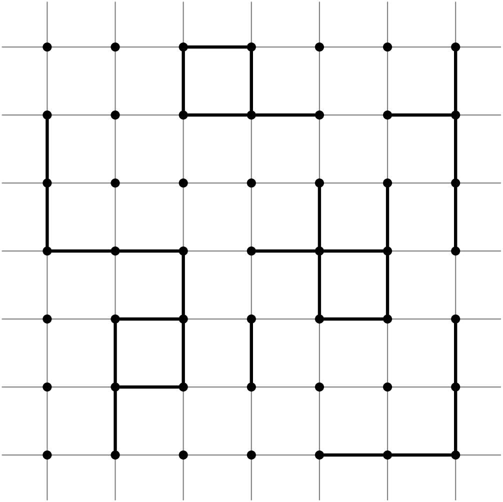

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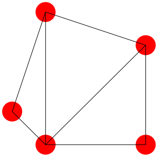
Graphs, clusters, trees and forests

- Graph $G = (V, B)$: vertices and bonds
- Cyclomatic number = number of indept. loops

$$c(G) = |B| - |V| + |G|$$

(bonds - vertices + conn. components)



$|V| = 5$
$|B| = 7$
$|G| = 1$
$c(G) = 3$

(Euler relation is true for all graphs. Special case of planar, connected graphs:
$f = c + 1, \ |G| = 1 \quad \Longrightarrow |V| - |B| + f = 2$)

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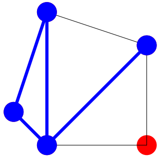

$|V'| = 4$
$|B'| = 4$
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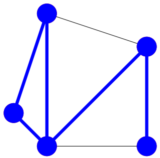

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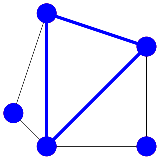

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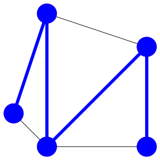

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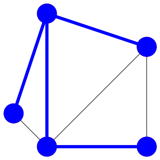

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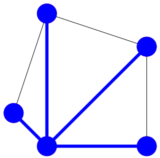

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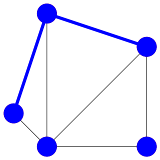

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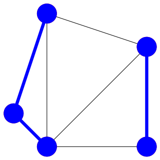

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FK cluster representation

- Potts partition function as cluster sum

$$
\begin{aligned}
Z_G(q,w) &= \sum_{\{s_i\}} \prod_{b \in B_G} (1 + w\delta(s_{b_1}, s_{b_2})) \\
&= \sum_{C \subseteq G} q^{|C|} w^{|B|} \quad \text{where } w = e^{\beta J} - 1
\end{aligned}
$$

- Correlation function and susceptibility:

$$
\begin{aligned}
G(i,j) &= \frac{1}{Z} \sum_{\substack{C_{ij} \subseteq G \\ i \text{ and } j \text{ in same component}}} q^{|C_{ij}|} w^{|B|} \\
\chi_G(q,w) &= \frac{1}{|V|} \sum_{i,j \in V} G(i,j)
\end{aligned}
$$

mean cluster size, magnetic susceptibility as long as $\langle s_i \rangle = 0$ (high-T phase)

Spanning forests

The limit $q \to 0, w/q$ finite describes an ensemble of spanning forests

$$\begin{aligned} Z_G(q,w) &= \sum_{C \subseteq G} q^{|C|} w^{|B|} \\ &= q^{|V|} \sum_{C} q^{c(C)} \left(\frac{w}{q}\right)^{|B|} \\ \lim_{q\to 0} q^{-|V|} Z_G(q, q\alpha) &= F_G(\alpha) = \sum_{c(C)=0} \alpha^{|B|} \quad \text{where } \alpha = w/q \end{aligned}$$

Spanning trees

The limit $q \to 0, w/q^\sigma$ finite, $0 < \sigma < 1$ describes an ensemble of spanning trees

$$\begin{aligned} Z_G(q,w) &= \sum_{C \subseteq G} q^{|C|} w^{|B|} \\ &= q^{\sigma|V|} \sum_C q^{\sigma c(C) + (1-\sigma)|C|} \left(\frac{w}{q^\sigma}\right)^{|B|} \\ \lim_{q\to 0} q^{-\sigma|V|-(1-\sigma)} Z_G(q, q^\sigma \alpha) &= \sum_{c(C)=0, |C|=1} \alpha^{|B|} \quad \text{where } \alpha = w/q^\sigma \\ &= T_G\, \alpha^{|V|-1} \end{aligned}$$

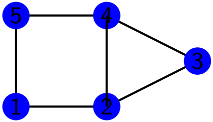
where $T_G = \#\{\text{spanning trees of } G\}$

- trivial, no phase transition

Spanning trees, determinants and fermions

- Adjacency matrix: $|V| \times |V|$ matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$



- Laplacian $\Delta = \text{diag}(\deg(v_1), \dots, \deg(v_{|V|})) - A$

$$\Delta = \begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

- *Kirchhoff 1847:* $T_G = \det \Delta'$

  Proof: $f(G) = f(G \setminus b) + f(G/b)$

  f () $=$ f () $+$ f ()

- free massless symplectic fermions: $T_G = \int \mathcal{D}(\psi, \bar{\psi})\, e^{\bar{\psi}\Delta\psi}$

Spanning forests

- Equivalent to bond percolation with local bond probability $p = \frac{\alpha}{1+\alpha}$ and the nonlocal constraint that clusters are free of loops: tree percolation
- $d > 2$: phase transition at some α_c:
  - $\alpha < \alpha_c$: forests consist of small trees
  - at α_c: one component of the forest percolates
  - $\alpha > \alpha_c$: ensemble is dominated by configurations where a single infinite tree covers a finite fraction of the lattice
  - $\alpha \to \infty$: this fraction approaches 1: spanning trees
- $d = 2$: phase transition only in the antiferromagnetic regime $\alpha_c < 0$.
- Fermionic field theory with $OSp(1|2)$ supersymmetry:

$$\int \mathcal{D}(\psi, \bar{\psi}) \exp\left[\bar{\psi}\Delta\psi + t\sum_i \bar{\psi}_i\psi_i - t\sum_{\langle i,j\rangle} \bar{\psi}_i\psi_i\bar{\psi}_j\psi_j\right] = t^{|V|}F_G(1/t)$$

Series generation techniques - Star graph expansion

Potts model: $\log Z$ and $1/\chi$ have star-graph expansions, i.e. expansions including only biconnected graphs (*no* articulation points)

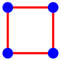
- Construct all star graphs embeddable in $\mathbb{Z}^d$ up to a given order (number of edges E):

order E	8	9	10	11	12	13	14	15	16	17	18	19	20	21
#graphs	2	3	8	9	29	51	142	330	951	2561	7688	23078	55302	165730

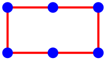
- Count the (weak) embedding numbers $E(G; \mathbb{Z}^d)$
- Calculate Z and correlations $G_{ij} = \langle \delta_{s_i, s_j} \rangle$ for every graph with symbolic parameter q and coupling v (using a cluster representation).
- Calculate $\log Z$, $C_{ij} = G_{ij}/Z$ up to $O(v^N)$
- Inversion of correlation matrix and subgraph subtraction
  $W_\chi(G) = \sum_{i,j} (C^{-1})_{ij} - \sum_{g \subset G} W_\chi(g)$
- Collect the results from all graphs
  $1/\chi = \sum_G E(G; \mathbb{Z}^d)\, W_\chi(G)$

Examples for weak embedding numbers in $\mathbb{Z}^d$

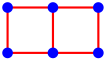
$$d$$



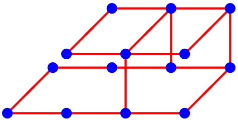
$$\binom{d}{2}$$



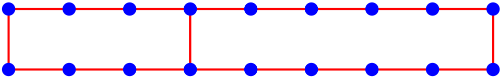
$$2\binom{d}{2} + 16\binom{d}{3}$$



$$2\binom{d}{2} + 12\binom{d}{3}$$



$$12048\binom{d}{3} + 396672\binom{d}{4} + 2127360\binom{d}{5} + 2488320\binom{d}{6}$$



$$8\binom{d}{2} + 275184\binom{d}{3} + 18763392\binom{d}{4} + 208611840\binom{d}{5} + 645442560\binom{d}{6} + 559964160\binom{d}{7}$$

Result: susceptibility series

```
 - 396*q^9*V^20*d^3 - 71664*q^8*V^20*d^3 - 7920*q^8*V^19*d^3 - 35783268*q^7*V^20*d^3 - 4004*q^7*V^20*d^2 - 922320*q^7*V^19*d^3 -
99288*q^7*V^18*d^3 - 2640*q^7*V^17*d^3 + 510996630*q^6*V^20*d^3 + 437960*q^6*V^20*d^2 - 99295644*q^6*V^19*d^3 +
12072*q^6*V^19*d^2 - 3177328*q^6*V^18*d^3 - 4676*q^6*V^18*d^2 - 1035600*q^6*V^17*d^3 - 264*q^6*V^17*d^2 - 16896*q^6*V^16*d^3 -
2160*q^6*V^15*d^3 - 23291841468*q^5*V^20*d^3 - 52837608*q^5*V^20*d^2 + 2790612816*q^5*V^19*d^3 + 7837920*q^5*V^19*d^2 -
284468212*q^5*V^18*d^3 - 1011024*q^5*V^18*d^2 + 20133864*q^5*V^17*d^3 + 97460*q^5*V^17*d^2 - 3599412*q^5*V^16*d^3 -
1524*q^5*V^16*d^2 - 138504*q^5*V^15*d^3 - 880*q^5*V^15*d^2 - 33336*q^5*V^14*d^3 + 360*q^5*V^13*d^3 - 56*q^5*V^12*d^3 +
381920091594*q^4*V^20*d^3 + 882904312*q^4*V^20*d^2 - 53105970234*q^4*V^19*d^3 - 176144660*q^4*V^19*d^2 + 7219713352*q^4*V^18*d^3
+ 33581524*q^4*V^18*d^2 - 884226162*q^4*V^17*d^3 - 6026888*q^4*V^17*d^2 + 112403526*q^4*V^16*d^3 + 996896*q^4*V^16*d^2 -
12004566*q^4*V^15*d^3 - 150264*q^4*V^15*d^2 + 1014426*q^4*V^14*d^3 + 19192*q^4*V^14*d^2 - 164070*q^4*V^13*d^3 -
1212*q^4*V^13*d^2 - 6240*q^4*V^12*d^3 - 72*q^4*V^12*d^2 - 672*q^4*V^11*d^3 - 2985257047506*q^3*V^20*d^3 -
5811546800*q^3*V^20*d^2 + 475828906620*q^3*V^19*d^3 + 1347121220*q^3*V^19*d^2 - 74406392514*q^3*V^18*d^3 -
305887016*q^3*V^18*d^2 + 11347178160*q^3*V^17*d^3 + 67763284*q^3*V^17*d^2 - 1685070330*q^3*V^16*d^3 - 14593908*q^3*V^16*d^2 +
246754864*q^3*V^15*d^3 + 3048028*q^3*V^15*d^2 - 33128280*q^3*V^14*d^3 - 612404*q^3*V^14*d^2 + 4650456*q^3*V^13*d^3 +
116016*q^3*V^13*d^2 - 512634*q^3*V^12*d^3 - 20756*q^3*V^12*d^2 + 45720*q^3*V^11*d^3 + 3384*q^3*V^11*d^2 - 7200*q^3*V^10*d^3 -
336*q^3*V^10*d^2 - 336*q^3*V^9*d^3 + 12030371402052*q^2*V^20*d^3 + 18358300112*q^2*V^20*d^2 - 2100969671688*q^2*V^19*d^3 -
4690241864*q^2*V^19*d^2 + 363320333260*q^2*V^18*d^3 + 1186104664*q^2*V^18*d^2 - 62205280752*q^2*V^17*d^3 -
296386828*q^2*V^17*d^2 + 10536240300*q^2*V^16*d^3 + 73141232*q^2*V^16*d^2 - 1768902744*q^2*V^15*d^3 - 17842272*q^2*V^15*d^2 +
291713952*q^2*V^14*d^3 + 4307276*q^2*V^14*d^2 - 47163528*q^2*V^13*d^3 - 1027340*q^2*V^13*d^2 + 7376632*q^2*V^12*d^3 +
240976*q^2*V^12*d^2 - 1039056*q^2*V^11*d^3 - 54760*q^2*V^11*d^2 + 157656*q^2*V^10*d^3 + 11652*q^2*V^10*d^2 - 17032*q^2*V^9*d^3 -
2372*q^2*V^9*d^2 + 1560*q^2*V^8*d^3 + 476*q^2*V^8*d^2 - 360*q^2*V^7*d^3 - 60*q^2*V^7*d^2 - 23867573497488*q*V^20*d^3 -
27918229076*q*V^20*d^2 + 4446689058192*q*V^19*d^3 + 7649974704*q*V^19*d^2 - 825214527528*q*V^18*d^3 - 2085619352*q*V^18*d^2 +
152588563584*q*V^17*d^3 + 565096960*q*V^17*d^2 - 28142760960*q*V^16*d^3 - 152239804*q*V^16*d^2 + 5176071360*q*V^15*d^3 +
40889376*q*V^15*d^2 - 948210168*q*V^14*d^3 - 10991164*q*V^14*d^2 + 172033392*q*V^13*d^3 + 2961448*q*V^13*d^2 -
30725832*q*V^12*d^3 - 796880*q*V^12*d^2 + 5318208*q*V^11*d^3 + 212544*q*V^11*d^2 - 912336*q*V^10*d^3 - 55824*q*V^10*d^2 +
149664*q*V^9*d^3 + 14448*q*V^9*d^2 - 22080*q*V^8*d^3 - 3628*q*V^8*d^2 + 4320*q*V^7*d^3 + 816*q*V^7*d^2 - 480*q*V^6*d^3 -
180*q*V^6*d^2 + 56*q*V^5*d^2 - 12*q*V^4*d^2 + 18153055172544*V^20*d^3 + 16434101440*V^20*d^2 + 2*V^20*d - 3538929660864*V^19*d^3
- 4749969504*V^19*d^2 - 2*V^19*d + 689190414432*V^18*d^3 + 1369608320*V^18*d^2 + 2*V^18*d - 134132531520*V^17*d^3 -
393581088*V^17*d^2 - 2*V^17*d + 26118927936*V^16*d^3 + 112837280*V^16*d^2 + 2*V^16*d - 5088226944*V^15*d^3 - 32394816*V^15*d^2 -
2*V^15*d + 990596448*V^14*d^3 + 9361040*V^14*d^2 + 2*V^14*d - 192127104*V^13*d^3 - 2729472*V^13*d^2 - 2*V^13*d +
36865536*V^12*d^3 + 800496*V^12*d^2 + 2*V^12*d - 6970368*V^11*d^3 - 234720*V^11*d^2 - 2*V^11*d + 1299264*V^10*d^3 +
68512*V^10*d^2 + 2*V^10*d - 237120*V^9*d^3 - 19776*V^9*d^2 - 2*V^9*d + 41088*V^8*d^3 + 5536*V^8*d^2 + 2*V^8*d - 7680*V^7*d^3 -
1472*V^7*d^2 - 2*V^7*d + 1152*V^6*d^3 + 400*V^6*d^2 + 2*V^6*d - 128*V^5*d^2 - 2*V^5*d + 32*V^4*d^2 + 2*V^4*d - 2*V^3*d + 2*V^2*d
- 2*V*d + 1
```

Large dimensionality expansion

Critical point equation $1/\chi(d, w_c) = 0$ can be iteratively solved:
Large-d expansion for w_c in terms of $\sigma = 2d - 1$

$(v = \frac{w}{w+q})$

$$v_c(q,\sigma) = \frac{1}{\sigma}\left[1 + \frac{8-3q}{2\sigma^2} + \frac{3(8-3q)}{2\sigma^3} + \frac{3\left(68-31q+q^2\right)}{2\sigma^4} + \frac{8664-3798q-11q^2}{12\sigma^5}\right.$$

$$+ \frac{78768-36714q+405q^2-50q^3}{12\sigma^6} + \frac{1476192-685680q-2760q^2-551q^3}{24\sigma^7}$$

$$\left. + \frac{7446864-3524352q-11204q^2-6588q^3-9q^4}{12\sigma^8} + \cdots \right]$$

Critical properties of spanning forests

Table: Critical points for hypercubic lattices $\mathbb{Z}^D$ for dimensions $D \geq 3$.

	MC		HT series	
D	α_c	γ	α_c	γ
3	0.433 65(2)	2.77(10)	0.433 33(5)	2.785(5)
4	0.210 302(10)	1.73(3)	0.209 97(3)	1.71(1)
5	0.140 36(2)	1.22(6)	0.140 31(3)	1.31(1)
6			0.106 68(3)	1.0(1)
7			0.086 74(1)	1.00(2)

- Upper critical dimension is $d = 6$ with logarithmic corrections $\chi \sim (\alpha_c - \alpha)^{-1}(\log(\alpha_c - \alpha))^{\delta}$, $\delta = 0.65(5)$

Conclusions

- Tree percolation is an interesting system with a geometric phase transition
- New universality class with upper critical dimension 6
- Geometric formulation is non-local (MC difficult) but local supersymmetric field theory exists
- Series expansion works