2D Critical Systems, Fractals and SLE

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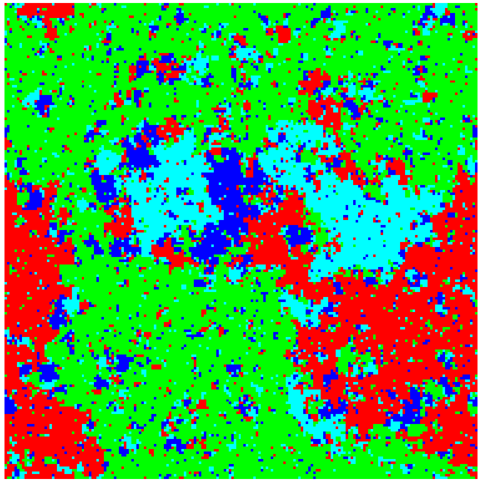
- Statistical models, clusters, loops
- Fractal dimensions
- Stochastic/Schramm Loewner evolution (SLE)
- Outlook

Lattice models: Percolation, Ising, Potts ...

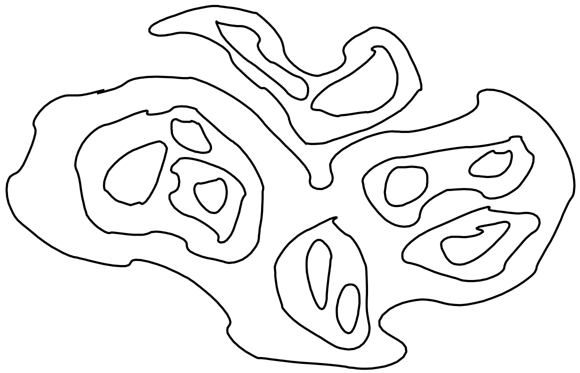
discrete local degrees of freedom (spins), $Z = \sum_{\{s_i\}} e^{-\beta H}$, $H = -\sum_{n.n.} \delta(s_i, s_j)$



$q = 4$ Potts cluster

cluster representation

$\Longrightarrow$



loop representation:
gas of non-intersecting loops,
boundaries of clusters

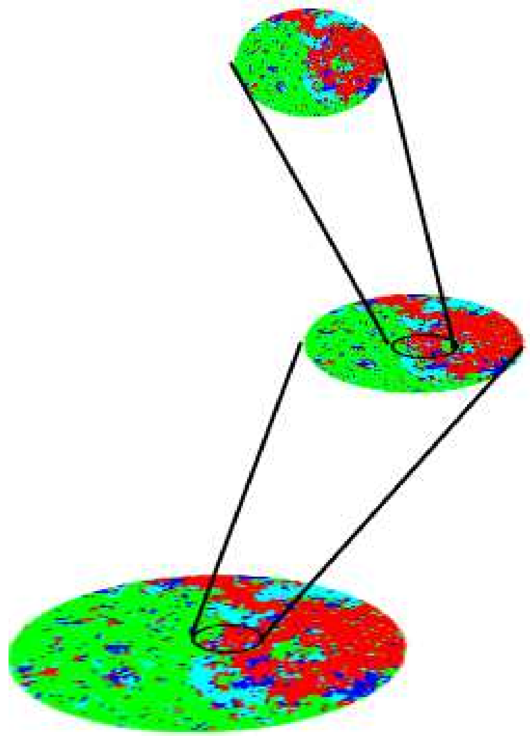
$\Longrightarrow$ level lines of Coulomb gas

$$Z = \sum p^E (1-p)^{N-E} q^C$$

$$p = \frac{e^\beta - 1}{e^\beta - 1 + q}$$

Critical behaviour

- 2. order phase transition at some critical point β_c
- self-similarity of cluster/loop ensemble $\Rightarrow$ fractals
- universal critical exponents: $\xi \sim (\beta - \beta_c)^{-\nu}$, $\chi \sim (\beta - \beta_c)^{-\gamma}$



- continuum limit of critical system can be constructed and gives conformal Euclidean field theory
- critical exponents related to:
  - conformal weights of operators
  - fractal dimensions of geometric objects

Local interpretation of cluster models

$$Z = \sum_{\{\text{cluster}\}} p^E (1-p)^{N-E} q^C$$

$$\stackrel{?}{=} \sum_{\{\text{local d.o.f. } s_i\}} \prod (\text{pos. local terms}) = \sum \exp\left(\sum (\text{local terms})\right)$$

- True for $q = 2, 3, 4, \ldots$ (Potts model) and for Behara numbers:

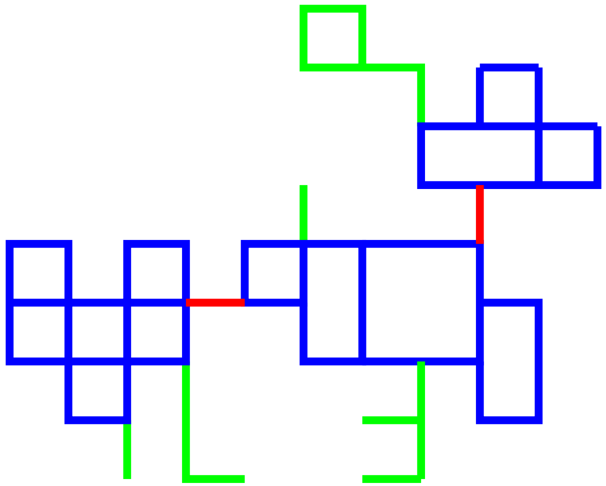
$$q = 4\cos^2(\frac{\pi}{n}), \qquad n = 2, 3, 4, \ldots \infty$$

$$q = 0,\ 1,\ 2,\ 2.61803,\ 3,\ 3.24698,\ 3.41421,\ 3.53209,\ 3.61803, \ldots 4$$

local interpretation as A_{n-1} RSOS model

Structure of clusters

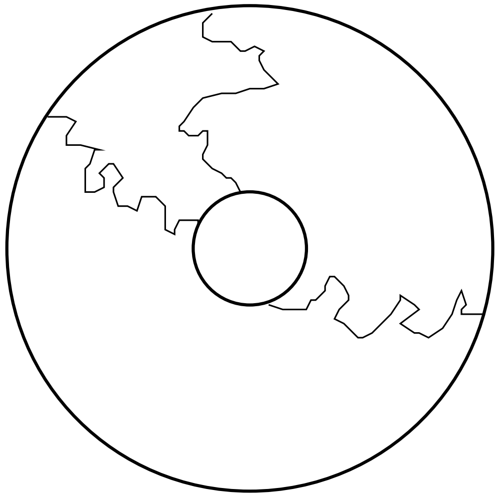
k**-block** = can not be separated into disconnected parts by cutting fewer than k vertices
k**-bone** = set of all points connected to k endpoints by k disjoint paths



- cluster = 1-block
- blue bonds = 2-blocks (blobs)
- red bonds connect blobs to backbone (2-bone)
  red $\bigcup$ blue = 2-bone
- current flow picture
- n-block can be decomposed into $n+1$-blocks, connecting bonds and rest.
- Hausdorff dimensions d_k of k-blocks and k-bones are equal (identify the k endpoints of a k-bone $\Rightarrow$ k-block)
- generalized Fisher exponents τ_{kn}: number of k-blocks of size s inside n-block scales as $N(s) \sim s^{-\tau_{kn}}$

percolation: $d_1 = \frac{91}{48}$; $d_2 = 1.6431(2)$; $d_3 = 1.20(5)$ (Paul & Stanley 2002)

Path Crossing Exponents



annulus geometry

k disjoint crossing paths

a) all in percolating cluster:
monochromatic

b) k_1 paths in cluster, k_2 in complement: polychromatic

scaling of probability for path crossing:

$$P_k(r, R) \sim \left(\frac{r}{R}\right)^{\tilde{x}_k}$$

monochromatic exponents:

$r \to 0$: paths in k-bone

$$\tilde{x}_k + d_k = D$$

transfer matrix simulations *Jacobsen, Zinn-Justin 2002*

$$\tilde{x}_2 = 0.3569(6)$$

polychromatic path crossing exponents:

Aizenman, Duplantier, Aharony 1999

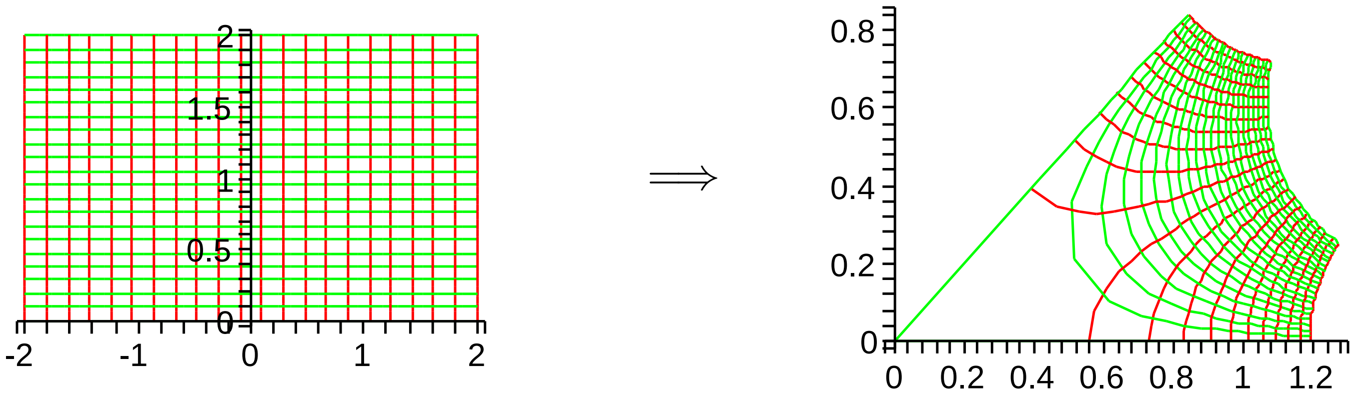
- x_{k_1,k_2} depends only on $k = k_1 + k_2$, as long as both k_1, k_2 are nonzero
- $x_k = \frac{1}{12}(k^2 - 1)$
- $x_k < \tilde{x}_k < x_{k+1}$

- generalization to Potts cluster?

Fractal dimensions

	fractal dimension of	codimension
d_1	clusters	$\tilde{x}_1$
d_n	n-bones or n-blocks	monochromatic path crossing exponents $\tilde{x}_n$
d_H	cluster hull	polychromatic exponent x_2
d_{EP}	external perimeter	x_3
d_R	red bonds	x_4

- complex map $z \mapsto w(z)$ where $\frac{\partial w}{\partial \bar{z}} = 0$
- angle-preserving bijection
- Riemann mapping theorem: arbitrary domain $\mathcal{D}$ has conformal map to upper half plane $\mathbb{H}$
- singularities (not angle preserving) at discrete points on boundary



- electrostatic picture: potential of equilibrium charge distribution on conducting boundary

Example: Percolation crossing probability

Probability that percolating cluster connects two segments $\overline{AB}$ and $\overline{CD}$ of a domain:

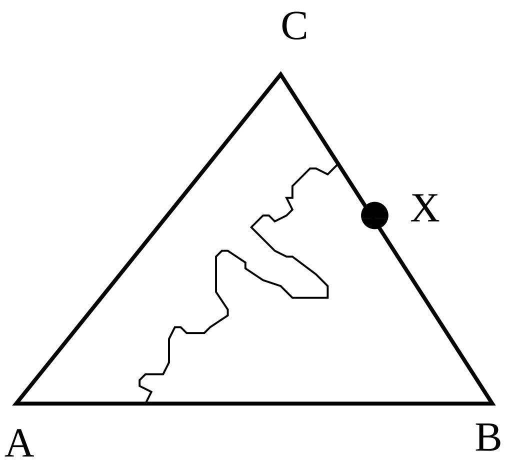
Cardy 1992

$$P = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})\Gamma(\frac{1}{3})}\eta^{1/3}{}_2F_1(\frac{1}{3},\frac{2}{3},\frac{4}{3};\eta)$$

with $\eta = \frac{(A-B)(C-D)}{(A-C)(B-D)}$.

Carleson 2001: This is inverse of conformal map to equilateral triangle:

$$P(AB, XC) = \frac{|XC|}{|AB|}$$



Conformal maps and random paths: SLE

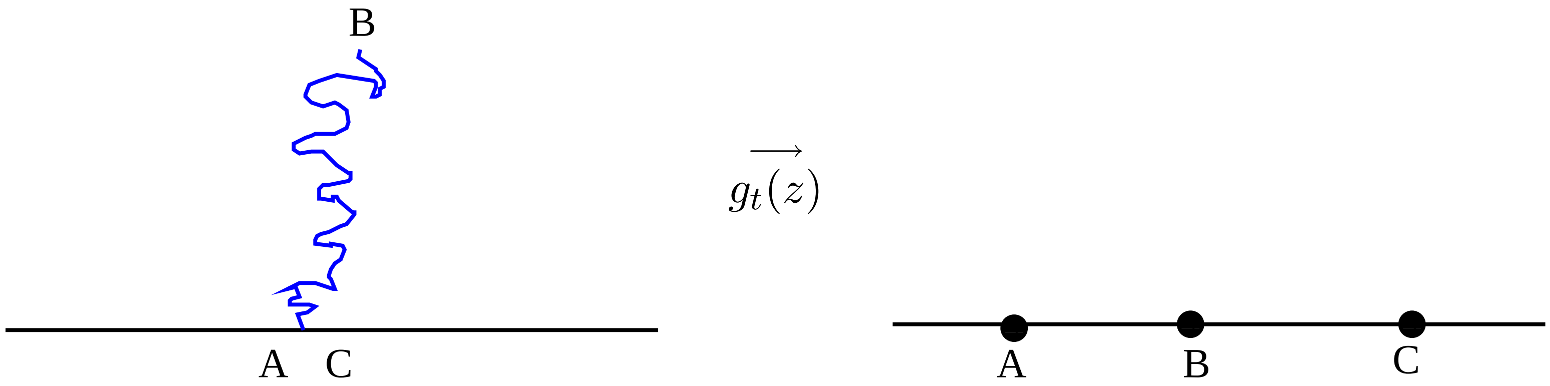
Measure $\mu(\gamma;\mathcal{D})$ on random curves γ connecting 2 points on domain $\mathcal{D}$

Critical interfaces of models with local interactions should have this property:

- conformal invariance: $(\Phi \star \mu)(\gamma;\mathcal{D}) = \mu(\Phi(\gamma),\Phi(\mathcal{D}))$
- Markov property: $\mu(\gamma_2|\gamma_1;\mathcal{D}) = \mu(\gamma_2;\mathcal{D}\setminus\gamma_1)$

$\Longrightarrow$ It exists a one-parameter family of such measures: SLE_k

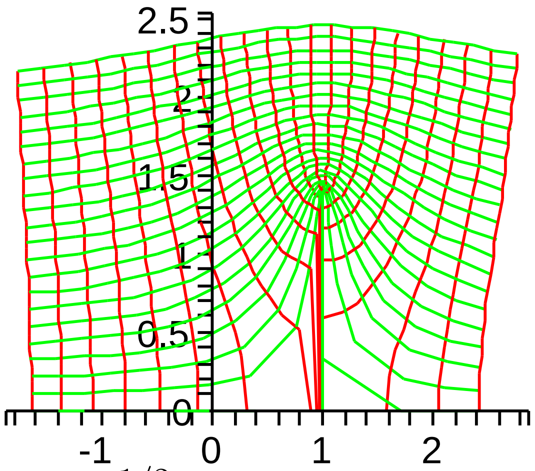
- "grow process" of a random curve $\gamma(t)$ on $\mathbb{H}$
- At every time t, study the conformal map $g_t(z)$ which "unzips the zipper":



K. Löwner 1923: *Untersuchungen über schlichte konforme Abbildungen des Einheitskreises*

- Map $g(z)$ is not unique: impose $g(z) = z + O(1/z)$ for large z.
- Reparametrization of time t: $g_t(z) = z + \frac{2t}{z} + O(1/z^2)$
- $\gamma(t)$ is called the trace of $g_t(z)$.

- simplest example: $g_t(z) = c + \sqrt{(z-c)^2 + 4t}$



trace goes from c vertically upwards, height $2t^{1/2}$

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - c}$$

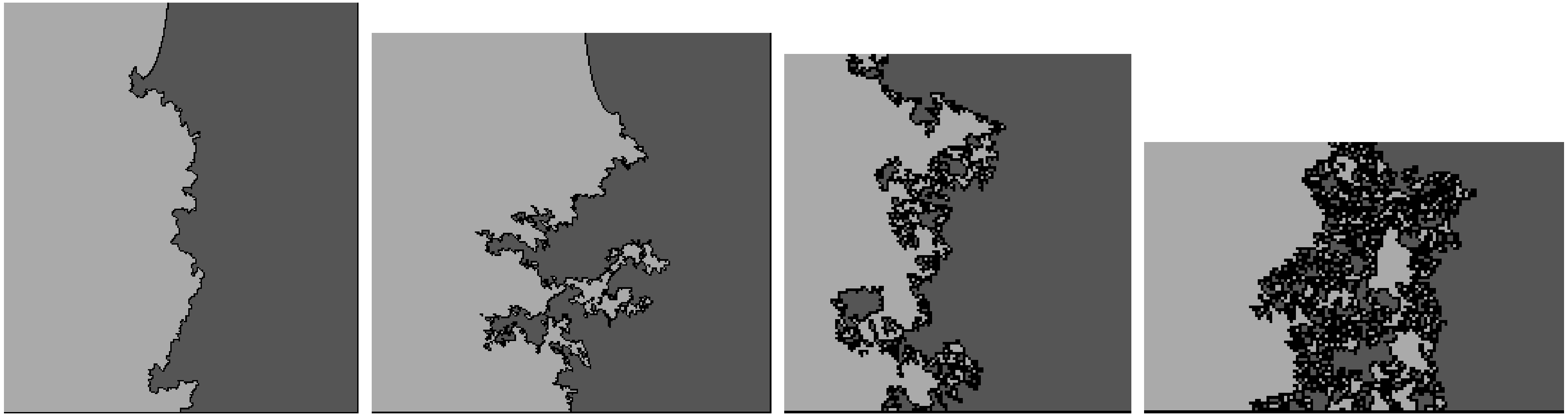
- generalization: $c \Rightarrow c(t)$

Theorem (O. Schramm 2000): **S**tochastic **L**öwner **e**volution (SLE)

Markov property + conformal invariance $\Longrightarrow$ $c(t)$ is proportional to a standard one-dimensional Brownian motion $c(t) = \sqrt{\kappa} B(t)$

$$\frac{dg_t(z)}{dt} = \frac{2}{g_t(z) - \sqrt{\kappa} B(t)}$$

- $\kappa = 0$: trace is straight line upwards
- larger κ: trace turns randomly to L and R more frequently



$\kappa = 1.3$ $\quad$ $\kappa = 4$ $\quad$ $\kappa = 6$ $\quad$ $\kappa = 12$

$\kappa \leq 4$	simple curve	$d_H = 1 + \frac{\kappa}{8}$
$4 < \kappa \leq 8$	infinite many self-touchings	
$8 < \kappa$	space-filling	$d_H = 2$

SLE duality: For $\kappa > 4$, the frontier of the curve is an SLE curve with

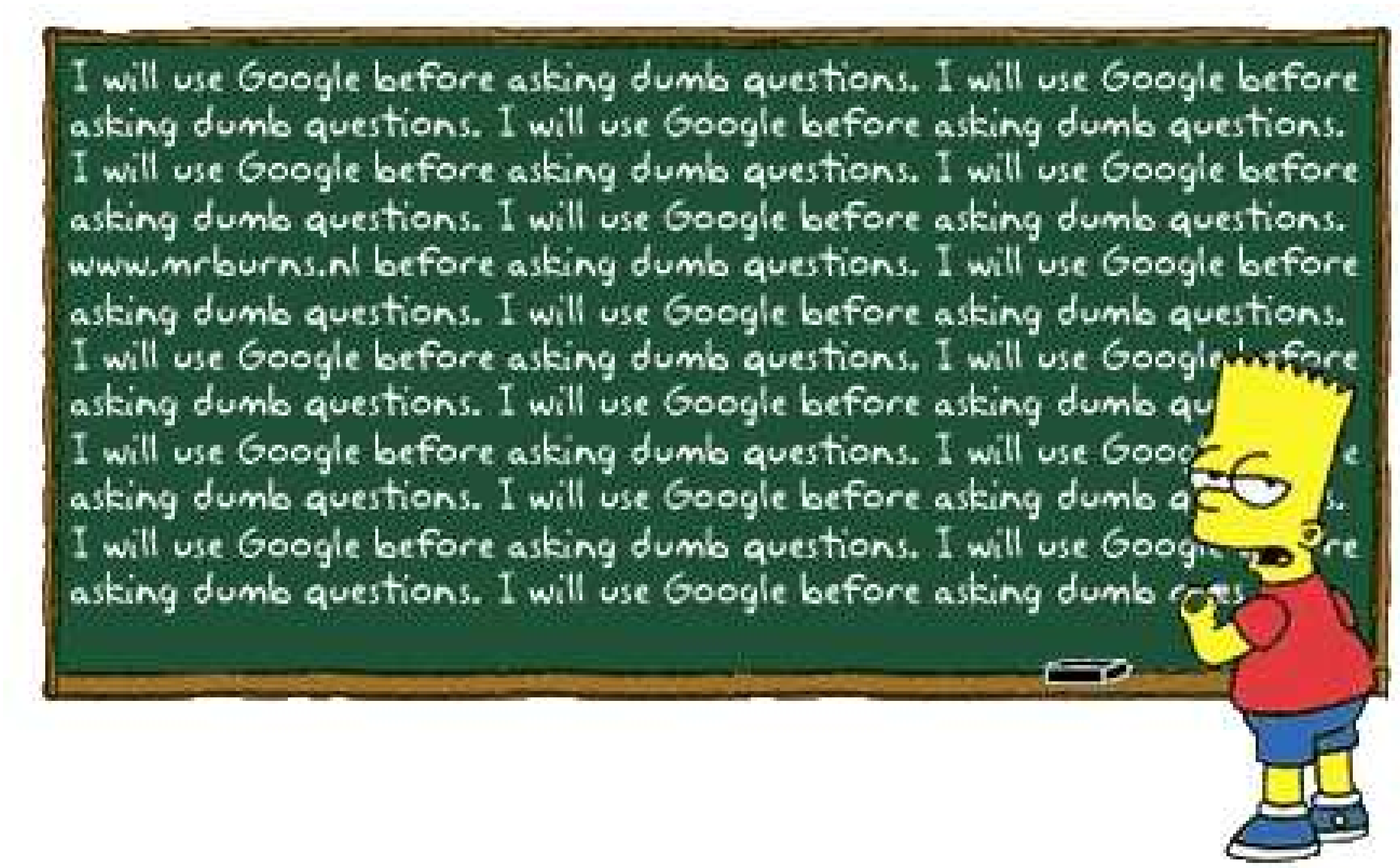
$$\tilde{\kappa} = 16/\kappa \qquad \text{trace-hull duality}$$

SLE examples

		d	
$\kappa = 2$	loop-erased random walks	1.25	
$\kappa = \frac{8}{3}$	self-avoiding walks	1.33	
$\kappa = 4$	$q = 4$ Potts cluster boundaries	1.5	
$\kappa = \frac{16}{3}$	$q = 2$ Ising cluster boundaries	1.66	
$\kappa = 6$	$q = 1$ percolation cluster boundaries	1.75	$q = 4\cos^2\left(\frac{4\pi}{\kappa}\right)$
$\kappa = 8$	$q = 0$ boundaries of spanning trees	2	

Outlook

- many things not mentioned:
  - relation SLE $\leftrightarrow$ CFT
  - random surfaces (aka "2D quantum gravity" aka "annealed disorder")
  - generalizations: $SLE(\kappa, \rho)$ (addition of drift forces)
    related to Coulomb gas picture
  - properties of transfer matrix
  - tricritical models
- geometric understanding of phase transitions and quantum fields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