Problem Set 7 – Iterative Solution of the Two-Dimensional Laplace Equation

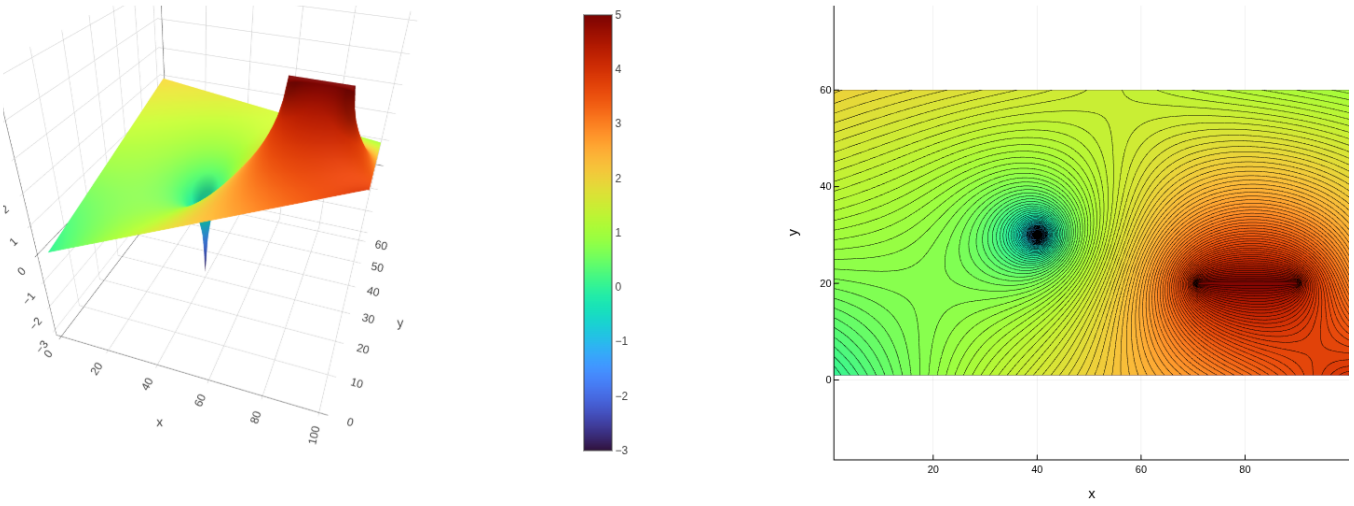


Figure 1: A Solution of the 2D Laplace equation.

1 The Laplace Equation

1.1 Harmonic functions

Let Ω be an open subset of $\mathbb{R}^n$. A function $u : \Omega \to \mathbb{R}$ is called **harmonic** if it satisfies the *Laplace equation*

$$\Delta u = 0 \qquad \text{in } \Omega, \tag{1}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \tag{2}$$

denotes the *Laplace operator*.

1.2 Physics

The Laplace equation arises throughout physics, typically as an equation for a steady state: heat diffusion (where u is temperature), electrostatics (where u is the electric potential), gravitation, and potential flow.

Throughout this problem set we will stick to the most concrete of these interpretations and think of u as a temperature.

1.3 Boundary value problems

Two standard problems for the Laplace equation are the **Dirichlet** and the **Neumann** problem.

Dirichlet problem. We seek a function u on a bounded open region Ω with boundary $\partial\Omega$ such that

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $g : \partial\Omega \to \mathbb{R}$ is a prescribed function on $\partial\Omega$.

Intuition. The boundary is held at prescribed temperatures, and after a sufficiently long time, the interior temperature settles to the unique harmonic extension of those boundary values.

Note the distinction between *steady* and *static*: unless g is constant on $\partial\Omega$, heat keeps flowing through Ω from the hotter to the colder parts of the boundary. What no longer changes in a steady state is the temperature profile u itself; that is precisely what the Laplace equation describes.

Neumann problem. Here one prescribes not the value of u but its normal derivative along the boundary $\partial\Omega$

$$\frac{\partial u}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla u = f, \tag{4}$$

i.e., the heat flux f through the boundary. For instance $\frac{\partial u}{\partial \mathbf{n}} = 0$ describes a perfectly insulating boundary.

In this problem set you will study iterative solvers for the Dirichlet problem.

1.4 The mean value property

If $B(x, r)$ is a ball with centre x and radius r, and if $\overline{B(x,r)} \subset \Omega$, then harmonic functions u satisfies

$$u(x) = \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} u \, d\sigma = \frac{1}{|B(x,r)|} \int_{B(x,r)} u \, dV. \tag{5}$$

Here $|B(x,r)|$ is the volume of the ball, and $|\partial B(x,r)|$ is the surface area of its boundary.

In words: the value of a harmonic function at a point is both the average of its values on the surrounding sphere and the average over the ball. Consequently, a non-constant harmonic function has no interior local maximum or local minimum.

1.5 Harmonic functions in two dimensions

If $w(z) = u(x, y) + iv(x, y)$ is holomorphic, then its real part u and imaginary part v are harmonic. This follows from the Cauchy–Riemann equations.

Conversely, every harmonic function on an open subset of $\mathbb{R}^2$ is locally the real part of a holomorphic function. If the domain is simply connected, a global holomorphic function exists.

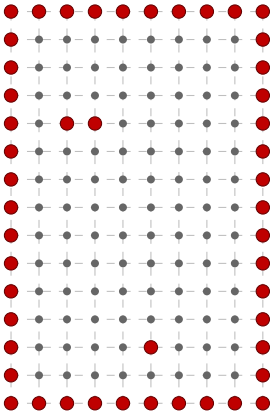


Figure 2: A rectangular grid with prescribed boundary (thick red) and interior (small grey) nodes.

2 A Discrete Grid Approximation

Let Ω be approximated by a rectangular grid with spacing h. Denote the grid value at node (i, j) by $u_{i,j}$. For an interior node, approximate the Laplacian by centred differences:

$$\Delta u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2} + O(h^2). \tag{6}$$

Neglecting the $O(h^2)$ term, the Laplace equation $\Delta u = 0$ becomes

$$u_{i,j} = \frac{1}{4}\left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right). \tag{7}$$

Thus each interior value is the average of its four neighbours. This is a discrete version of the mean value property.

The region Ω may also contain holes: small subregions removed from the domain, on whose boundaries Dirichlet data are prescribed (figure 2).

3 *Numerics: Iterative Solvers*

The discretized Laplace equation (7) with Dirichlet boundary conditions can be written as a sparse linear system $A\mathbf{u} = \mathbf{b}$ where $\mathbf{u}$ is a vector containing all interior $u_{i,j}$. The square matrix A is the discretized Laplace operator, a 5-point stencil (block tridiagonal), symmetric negative definite, of size $n \times n$, where n is the number of interior nodes; $\mathbf{b}$ is nonzero only in rows adjacent to the boundary.

You will study three iterative solvers: the **Jacobi** method, the **Gauß-Seidel** method, and **Successive over-relaxation (SOR)**.

The basic idea of iterative algorithms. One starts from an arbitrary initial vector $\mathbf{u}^{(0)}$, applies an update map $\mathbf{\Phi}$, and sets

$$\mathbf{u}^{(k+1)} = \mathbf{\Phi}\big(\mathbf{u}^{(k)}\big). \tag{8}$$

The map is designed so that the solution of $A\mathbf{u} = \mathbf{b}$ is a *fixed point* of $\mathbf{\Phi}$, i.e. $\mathbf{\Phi}(\mathbf{u}^*) = \mathbf{u}^*$; if, in addition, $\mathbf{\Phi}$ brings the error closer to zero in each step, the iterates converge to the solution.

One stops as soon as the change $\|\mathbf{u}^{(k+1)} - \mathbf{u}^{(k)}\|$ or the *residual* $\|\mathbf{b} - A\mathbf{u}^{(k)}\|$ drops below a prescribed tolerance. Note that both are computable without knowing the exact solution.

3.1 *Jacobi method*

For our problem, Jacobi reduces to a very simple and intuitive algorithm[1]: *Sweep through all interior points and set their value to the average of their nearest neighbours:*

[1] Proposal for your report: derive this!

$$u_{i,j}^{(k+1)} = \Phi(\mathbf{u}^{(k)}) = \frac{1}{4}\left(u_{i+1,j}^{(k)} + u_{i-1,j}^{(k)} + u_{i,j+1}^{(k)} + u_{i,j-1}^{(k)}\right). \tag{9}$$

Repeat until the residual norm (we ignore a harmless factor h^{-2})

$$\|\mathbf{u}\|_r^2 = \sum_{\substack{i,j \\ \text{interior points}}} \left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\right)^2 \tag{10}$$

is sufficiently small relative to that of the initial configuration:

$$\|\mathbf{u}^{(k)}\|_r \le \varepsilon \cdot \|\mathbf{u}^{(0)}\|_r \tag{11}$$

3.2 *Gauß–Seidel method*

This method is a small variation on Jacobi: we modify the Jacobi update (9) so that already-updated neighbours are used immediately (in-place update).

3.3 Successive Over-relaxation (SOR)

This method modifies Gauß–Seidel: the updates are again performed in place, but each proposed value is deliberately overshot.
We change

$$\mathbf{u}^{k+1} = \mathbf{u}^k + \mathbf{\Phi}(\mathbf{u}^k) - \mathbf{u}^k \tag{12}$$

$$\text{to} \quad \mathbf{u}^{k+1} = \mathbf{u}^k + \omega\left(\mathbf{\Phi}(\mathbf{u}^k) - \mathbf{u}^k\right) \tag{13}$$

with some $\omega > 1$. Thus we make larger steps in the calculated direction in $\mathbf{u}$-space.
A theoretical analysis shows that SOR converges for the model problem if and only if $0 < \omega < 2$; the value $\omega = 1$ reproduces Gauß–Seidel, $\omega > 1$ is over-relaxation, $0 < \omega < 1$ under-relaxation.

4 Coding Tasks

- The following Dirichlet boundary conditions were used to produce figure 1:
  - 60×100 grid points (including boundary)
  - Outer boundary: The 4 corners are set (clockwise, see figure 1 to the temperature 0,4,1 and 2 and the edges interpolate linearly between these values.
  - Inner boundary:

    ```
    u[30, 40]  = -3.0
    u[20, 70:90] .= 5.0
    ```

  You may use this as a starting point. There is ample room for experimentation.
- Start with all interior nodes at zero temperature.
- Iterate until the stopping criterion (11) is satisfied with $\varepsilon = 10^{-6}$, or until at most 5000 sweeps have been performed.
- Implement Jacobi, Gauß–Seidel and SOR with parameter $\omega \in \{1.5, 1.7, 1.9, 1.92, 1.94\}$.
- Record the residual norm as a function of sweep number k, plot and compare the convergence histories of the different methods.
- Plot the final temperature distribution similar to figure 1.