Problem Set 6 – Simple Hamiltonian Integrators for the Sitnikov system

Our goal is to write Julia code that solves numerically the equations of motion for the Sitnikov problem. and visualizes the solution similar to the following figures:

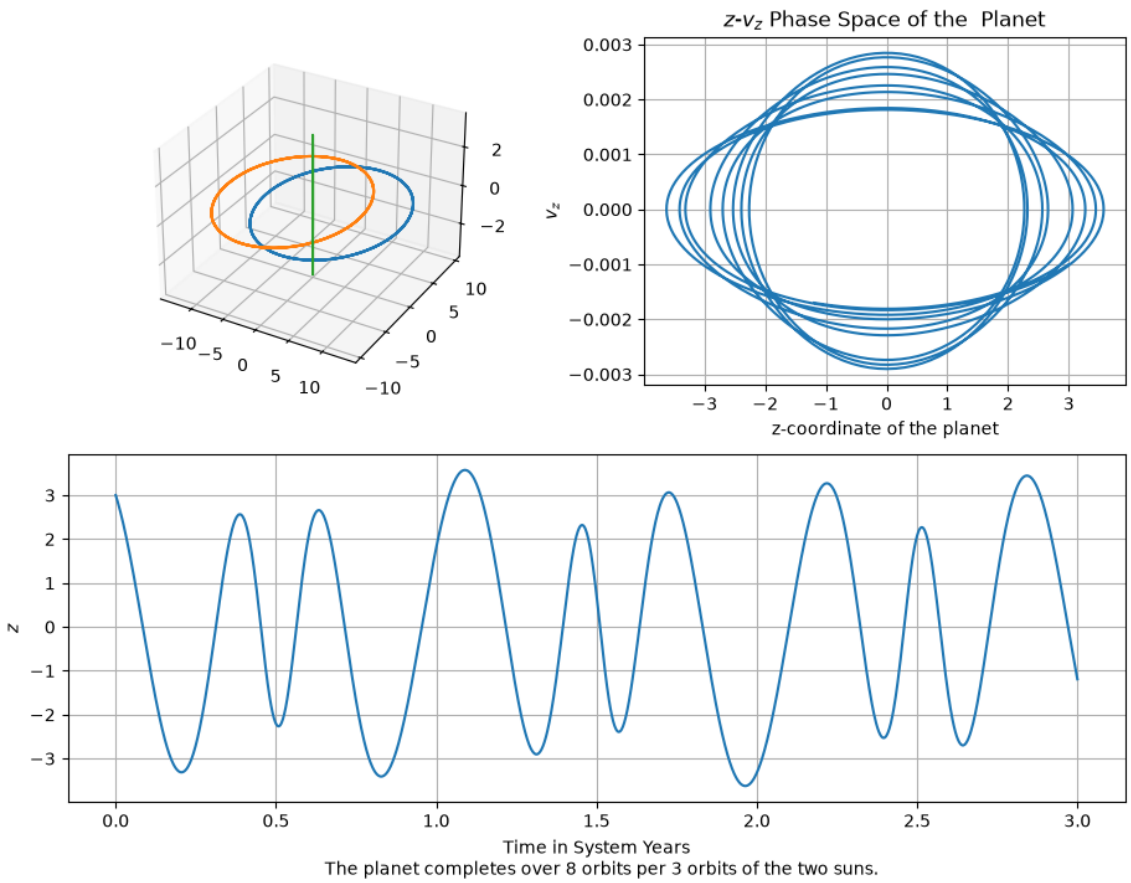


Figure 1: A Sitnikov System

A Sitnikov system is a highly symmetric special case of the gravitational three-body problem in which two suns with equal masses orbit their common center of mass on fixed Keplerian ellipses, while a planet with negligible mass moves along the line perpendicular to their orbital plane, passing through the barycenter. Due to the high symmetry of the problem, the planet's orbit is one-dimensional, it moves up and down along the z-axis.

1 Hamilton mechanics in phase space

- The mechanical state of N bodies moving in space is described by $3N$ coordinates q_i and $3N$ momenta[1] p_i where $i = 1, ..., 3N$.
- The *phase space* of a N-body system ist $\mathbb{R}^{6N}$ with coordinates $\mathbf{z} = (\mathbf{q}, \mathbf{p})$.
- Time evolution is a curve $\mathbf{z}(t) = (\mathbf{q}(t), \mathbf{p}(t))$ in phase space.
- The *Hamiltonian* $H(\mathbf{q}, \mathbf{p})$ is the total mechanical energy of the system written as function of $\mathbf{q}$ and $\mathbf{p}$. Given an initial point in phase space, the future evolution of

[1] In our case the momentum is just velocity multiplied by mass, $\vec{p} = m\vec{v}$. So you can picture the momentum vector as a velocity vector, scaled by the mass of the body.

the system is completely determined by Hamilton's equations:

$$\frac{d}{dt}q_i(t) = \frac{\partial H(\mathbf{q},\mathbf{p})}{\partial p_i} \tag{1}$$

$$\frac{d}{dt}p_i(t) = -\frac{\partial H(\mathbf{q},\mathbf{p})}{\partial q_i} \tag{2}$$

This is a system of $6N$ first-order ordinary differential equations[2] (ODEs) defining the flow of time in phase space.

[2] The partial derivatives on the RHS only tell us how to calculate the RHS from the function H. They don't make it a PDE.

Notation. *We will denote vectors in $\mathbb{R}^3$ (position, momentum of one body) by an arrow, $\vec{q}$. Higher-dimensional vectors are denoted by bold symbols like $\mathbf{p}$ and $\mathbf{q}$. Indices a, b will enumerate the N bodies and indices i, j denote the components of a vector. The index n will always enumerate time steps.*

The total energy is the sum of kinetic energy and potential energy due to gravitational attraction:

$$H(\mathbf{p},\mathbf{q}) = T(\mathbf{p}) + V(\mathbf{q}) \tag{3}$$

$$H(\mathbf{p},\mathbf{q}) = \sum_{a=1}^{N} \frac{\vec{p}_a^2}{2m_a} - \frac{G}{2}\sum_{a=1}^{N}\sum_{b\neq a}^{N} \frac{m_a m_b}{|\vec{q}_a - \vec{q}_b|} \tag{4}$$

Here, m_a is the mass of the a-th body, G is the gravitational constant and $|\vec{q}_a - \vec{q}_b|$ is the distance between bodies a and b. The resulting Hamilton's equations are

$$\frac{d}{dt}q_{a,i}(t) = \frac{\partial T(\mathbf{p})}{\partial p_{a,i}} = \frac{p_{a,i}}{m_a}; \qquad a = 1,\dots,N; \quad i = 1,\dots,3; \tag{5}$$

$$\frac{d}{dt}p_{a,i}(t) = -\frac{\partial V(\mathbf{q})}{\partial q_{a,i}} = G\sum_{b\neq a} \frac{m_a m_b}{|\vec{q}_a - \vec{q}_b|^3}\left(q_{b,i} - q_{a,i}\right). \tag{6}$$

2 Simple numerical solver for ODEs

Our equations of motion (1,2) are first order, autonomous[3] in a higher-dimensional linear space $\mathbb{R}^n$ (with $n = 6N$ in our case):

[3] Autonomous: the f_i do not depend explicitely on t

$$\frac{d}{dt}z_1(t) = f_1(z_1(t),\dots,z_n(t)) \tag{7}$$

$$\frac{d}{dt}z_2(t) = f_2(z_1(t),\dots,z_n(t)) \tag{8}$$

$$\vdots$$

$$\frac{d}{dt}z_n(t) = f_n(z_1(t),\dots,z_n(t)). \tag{9}$$

with initial conditions $z_i(0) = z_{0,i}$ We express this concisely using vector notation as:

$$\frac{d}{dt}\mathbf{z}(\mathbf{t}) = \mathbf{f}(\mathbf{z}(\mathbf{t})); \qquad \mathbf{z}(0) = \mathbf{z}_0. \tag{10}$$

A numerical method generates approximations $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \dots$ to the solution $\mathbf{z}(t)$ at various values of time t. We will only consider integrators with fixed step size h; so $\mathbf{z}_n$ is an approximation to $z(t = nh)$.

2.1 Explicit Euler method (Euler 1786)

Choose a small enough value h for your time steps.

$$t_{n+1} = t_n + h$$
$$\mathbf{z}_{n+1} = \mathbf{z}_n + h\mathbf{f}(\mathbf{z}_n)$$

2.2 Heun's method

Heun's is the simplest of a family called predictor-corrector methods. First we make a preliminary Euler step (prediction), calculate the slope $\mathbf{f}$ at this point and then make our (corrected) step using the average of the slopes at $\mathbf{z}_n$ and at $\tilde{\mathbf{z}}_{n+1}$:

$$\begin{aligned} t_{n+1} &= t_n + h \\ \tilde{\mathbf{z}}_{n+1} &= \mathbf{z}_n + h\mathbf{f}(\mathbf{z}_n) \\ \mathbf{z}_{n+1} &= \mathbf{z}_n + \frac{h}{2}\left(\mathbf{f}(\mathbf{z}_n) + \mathbf{f}(\tilde{\mathbf{z}}_{n+1})\right) \end{aligned}$$

2.3 Størmer-Verlet method

Euler and Heun are general methods. Symplectic integrators are adapted to the special structure of Hamiltonian equations. One of the simplest symplectic integrators is the Størmer-Verlet or leapfrog method:

$$t_{n+1} = t_n + h \tag{11}$$

$$p_{i,n+\frac{1}{2}} = p_{i,n} - \frac{1}{2}h\frac{\partial V}{\partial q_i}(\mathbf{q}_n) \tag{12}$$

$$q_{i,n+1} = q_{i,n} + h\frac{\partial T}{\partial p_i}\left(\mathbf{p}_{n+\frac{1}{2}}\right) \tag{13}$$

$$p_{i,n+1} = p_{i,n+\frac{1}{2}} - \frac{1}{2}h\frac{\partial V}{\partial q_i}(\mathbf{q}_{n+1}), \qquad i = 1, \ldots, 3N \tag{14}$$

3 Initial conditions

3.1 Units

Let's measure distances in astronomical units (1 au = 149 597 870.7km, the average Earth-Sun distance), times in Earth days (1d = 86 400s), and masses as multiples of the solar mass $\mathrm{M}_\odot$. In these units, the gravitational constant is

$$G = 2.95912208286 \times 10^{-4}\,\frac{\mathrm{au}^3}{\mathrm{d}^2\mathrm{M}_\odot} \tag{15}$$

3.2 Initial positions

The two suns of equal mass move on elliptical orbits with equal semi-major axis a and eccentricity ε about their common focus. Let the semi-major axis a lie along the x-axis, with the common focus F_2 at the origin of the system. The simulation begins at the moment when the suns are at their respective aphelion. At this point, the velocity vector of one sun points in the positive y-direction and that of the other in the negative y-direction. The magnitude of the velocity vectors can be calculated using the vis-viva equation as

$$v = \frac{1}{2}\sqrt{G\frac{m_1 + m_2}{2a}\,\frac{1-\varepsilon}{1+\varepsilon}} \tag{16}$$

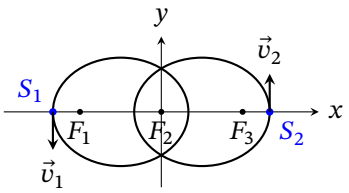


Figure 2: Initial positions and velocities of the two Suns S_1 and S_2

The planet has a very small mass relative to the solar masses and starts at an arbitrary point on the z-axis with a velocity along the z-axis.

The system in Fig. 1 used

- Semi-major axis: $a = 10$ au
- Eccentricity: $\varepsilon = 0.3$

- Solar masses: $m_1 = m_2 = 1\ \mathrm{M}_\odot$
- Planet mass: $m_p = 1 \times 10^{-8}\ \mathrm{M}_\odot$
- Initial position of the planet: $z = 3$ au
- Initial velocity of the planet: $v_z = -1 \times 10^{-3}$ au/d

3.3 Correct total momentum

Since the total momentum $\mathbf{p}_t = \sum_{\text{objects } a} m_a \mathbf{v}_a$ of this initial data is non-zero, the system as a whole will move slowly through space. Therefore it is advisable to correct every initial velocity vector $\mathbf{v}_a$ according to $\mathbf{v}_a = \mathbf{v}_a - \frac{\mathbf{p}_t}{M}$ where $M = \sum_a m_a$ is the total mass. This ensures vanishing total momentum.

4 Implementation in Julia

Your task is to write Julia code that numerically integrates the equations of motions for the Sitnikov system using different integrators.

- It is recommended to use the package `Distances.jl` for the fast calculation of all pairwise distances. If, e.g, the coordinates are available in a $3 \times N$-matrix `q`, then

  ```
  using Distances
  ...
  D = pairwise(Euclidean(), q, dims=2)
  ```

  returns a symmetric $N \times N$ matrix where `D[i,j]` is the distance between body `i` and body `j`.
- Do not use packages like `DifferentialEquations.jl`; write your own numerical integrator! (You may use them to compare and check your code.)
- Use a plot package, e.g., `Plots.jl` or `GLMakie.jl` to visualize the orbits similar to Figure 1.
- Calculate the total energy H of the system every 20 steps. Check the conservation of energy for the different methods (Euler, Heun, Størmer-Verlet). Plot the energy over time.
- Suggested parameters are between step size $h = 10d$, 2×10^4 time steps and $h = 100d$, 2×10^3 time steps.

4.1 Tasks

- Study the influence of the eccentricity of the solar system on the movement of the planet.
- Study the influence of the planet mass. A large planet with, e.g., $m_p = 0.03\ \mathrm{M}_\odot$ will influence the movements of the two suns. They no longer stay in the xy plane.