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What is a profinite group?
– a compact Hausdorff totally disconnected group
– a closed subgroup of a Cartesian product of finite groups
– an inverse limit of finite groups
– a Galois group of an algebraic field extension.

(Totally disconnected: no connected subset has more than one element.

A [locally] compact Hausdorff space is totally disconnected iff its compact open subsets form a base for its topology.)
What is a profinite group a pro-$p$ group?
– a compact Hausdorff totally disconnected group in which $g^{p^n} \to 1$ as $n \to \infty$, for all $g \in G$
– a closed subgroup of a Cartesian product of finite groups of finite $p$-groups
– an inverse limit of finite groups of finite $p$-groups
– a Galois group of an algebraic field extension such that each finite subextension has $p$-power degree.
Where do profinite groups arise?

Profinite groups arise in:

- Analysis, general topological group theory
  (quotients of compact top. groups modulo connected component of the identity)

- Finite group theory
  (encode info. about infinite families, asymptotic properties)

- Infinite group theory
  (examples, completions, etc.)

- Algebraic number theory
  (Galois groups, etc.)

- Model theory, combinatorics
  (groups of automs. of rooted trees & other structures, small index property)
If $X$, $Y$ are groups, so is $X \times Y$: $(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$.

If $X$, $Y$ are top. spaces so is $X \times Y$: base of open sets

\[ \{U \times V \mid U \text{ open in } X, V \text{ open in } Y\} \]

Let $G$ be both a group and a top. space: $G$ is a topological group if both $x \mapsto x^{-1}$ and $(x, y) \mapsto xy$ (from $G \times G$ to $G$) are continuous.

If $X$, $Y$ are top. groups, so is $X \times Y$.

**Infinite Products.**  $C = \prod_{\lambda \in \Lambda} X_\lambda$ is the Cartesian product of $(X_\lambda \mid \lambda \in \Lambda)$.

Elements are vectors $(x_\lambda)_{\lambda \in \Lambda}$, entries indexed by $\Lambda$.

Projection maps $\pi_\lambda : C \to X_\lambda$.

If each $X_\lambda$ is a group so is $C$: $(x_\lambda)(y_\lambda) = (x_\lambda y_\lambda)$.

If each $X_\lambda$ is a top. space, so is $C$: subbase of open sets

\[ \{\pi_\lambda^{-1}(U_\lambda) \mid \lambda \in \Lambda, U_\lambda \text{ open in } X_\lambda\} \]

If each $X_\lambda$ is a top. group, so is $C$.

**Tychonoff’s theorem.**  If each $X_\lambda$ is compact then so is $C$.

So if each $X_\lambda$ is compact Hausdorff, so is each closed subset of $C$.

If $X_\lambda$ is a cpct. Haus. top. group, so is each closed subgroup of $C$.

(E.g., take $X_\lambda$ finite, with discrete top.)
A directed set is a poset $I$ such that for all $i_1, i_2 \in I$ there is an element $j \in I$ for which $i_1 \leq j$ and $i_2 \leq j$.

(1.1.1) Definition. An inverse system $(X_i, \varphi_{ij})_I$ of top. spaces indexed by a directed set $I$ is a family $(X_i | i \in I)$ of top. spaces with a family $(\varphi_{ij}: X_j \to X_i | i, j \in I, i \leq j)$ of conts. maps such that $\varphi_{ii} = \text{id}_{X_i}$ for each $i$ and $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ whenever $i \leq j \leq k$.

So for $i \leq j \leq k$ the diagram

\[
\begin{array}{ccc}
X_j & \xrightarrow{\varphi_{ij}} & X_i \\
\downarrow{\varphi_{jk}} & & \downarrow{\varphi_{ik}} \\
X_k & & \\
\end{array}
\]

commutes.

Sets with no specified topology are given the discrete topology. If $(X_i, \varphi_{ij})$ consists of top. groups and conts. homoms., $(X_i, \varphi_{ij})$ is an inverse system of top. groups; similarly for top. rings.
Inverse limits

Let \((X_i, \varphi_{ij})_I\) be an inverse system of top. spaces, \(Y\) a top. space. A family \((\psi_i: Y \to X_i \mid i \in I)\) of conts. maps is compatible if \(\varphi_{ij}\psi_j = \psi_i\) for \(i \leq j\). That is, each diagram

![Diagram](attachment://diagram.png)

is commutative.
(1.1.3) Definition. An inverse limit \((X, \varphi_i)\) of an inverse system \((X_i, \varphi_{ij})_i\) of top. spaces (resp. groups, rings) is a top. space (resp. group, ring) \(X\) together with a compatible family \((\varphi_i: X \to X_i)\) of conts. maps (resp. conts. homoms.) with the foll. universal property (UP):

for each compatible family \((\psi_i: Y \to X_i)\) of conts. maps from a space \(Y\) (resp. conts. homoms. from a group or ring \(Y\)), there is a unique conts. map (resp. conts. homom.) \(\psi: Y \to X\) such that \(\varphi_i \psi = \psi_i\) for each \(i\).
That is, there is a unique \(\psi\) such that the foll. diagrams are commutative.

\[
\begin{array}{ccc}
Y & \xrightarrow{\psi} & X \\
\downarrow{\psi_i} & & \downarrow{\varphi_i} \\
X_i & & X
\end{array}
\]
(1.1.4) Let \((X_i, \varphi_{ij})\) be an inverse system.

(a) If \((X^{(1)}, \varphi_i^{(1)})\) and \((X^{(2)}, \varphi_i^{(2)})\) are inverse limits, then \(\exists\) isom. \(\bar{\varphi}: X^{(1)} \to X^{(2)}\) such that \(\varphi_i^{(2)} \bar{\varphi} = \varphi_i^{(1)}\) for each \(i\). (If \((X_i, \varphi_{ij})\) is an inverse system of spaces then \(\bar{\varphi}\) is just a homeom.)

(b) Write \(C = \text{Cr}(X_i \mid i \in I)\) and \(\pi_i\) for the projection \(C \to X_i\). Define

\[
X = \{c \in C \mid \varphi_{ij} \pi_j(c) = \pi_i(c) \text{ for all } i, j \text{ with } j \geq i\}
\]

and \(\varphi_i = \pi_i|_X\) for each \(i\). Then \((X, \varphi_i)\) is an inverse limit of \((X_i, \varphi_{ij})\).

(c) If \((X_i, \varphi_{ij})\) is an inverse system of top. groups and conts. homoms., then \(X\) is a top. group and the maps \(\varphi_i\) are conts. homoms.
So the inverse limit of \((X_i, \varphi_{ij})_I\) exists and is unique up to isom. Call it the inverse limit, write \(\lim\leftarrow \leftarrow (X_i, \varphi_{ij})\), or just \(\lim X_i\).
Write \(s \lim X_i\) for the inverse limit constructed above as subgroup of \(C\).

**Note.** If \((X_i, \varphi_{ij})_I\) is an inverse system of top. groups, then from (b) its inverse limit, as a set, is also the inverse limit of \((X_i, \varphi_{ij})\) as an inverse system of sets.

**(0.3.4)** Let \((G_\lambda | \lambda \in \Lambda)\) be top. groups, and
\[ C = \text{Cr}(G_\lambda | \lambda \in \Lambda). \]
Define multiplication in \(C\) pointwise \((x_\lambda)(y_\lambda) = (x_\lambda y_\lambda)\). Then \(C\) is a topological group, i.e., mult. and inversion are conts. maps.

Proof: definition of product topology plus

**Claim.** A map \(f: Z \rightarrow C\) is conts. \(\iff\) each \(\pi_\lambda f\) is continuous.
\(\Rightarrow\) klar.
Suppose each \(\pi_\lambda f\) conts. If \(U_i\) is open in \(X_{\lambda_i}\) for \(i = 1, \ldots, n\) then each \((\pi_{\lambda_i} f)^{-1}(U_i)\) is open in \(Z\) so
\[ f^{-1}(\cap_{i=1}^n \pi_{\lambda_i}^{-1}(U_i)) = \cap_{i=1}^n (\pi_{\lambda_i} f)^{-1}(U_i) \text{ is open in } Z. \]
If $f : X \to Y$ and $g : Y \to Z$ are conts. so is $gf : X \to Z$.

A bijection $f : X \to Y$ is a homeomorphism iff $f, f^{-1}$ are both conts.

(0.1.2) 
(a) Each closed subset of a compact space is compact.
(b) Each compact subset of a Hausdorff space is closed.
(c) If $f : X \to Y$ is continuous and $X$ is compact then $f(X)$ is compact.
(d) If $f : X \to Y$ is continuous and bijective and if $X$ is compact and $Y$ is Hausdorff then $f$ is a homeomorphism.
(e) If $f : X \to Y$ and $g : X \to Y$ are continuous and $Y$ is Hausdorff then $\{x \in X \mid f(x) = g(x)\}$ is closed in $X$.

(0.1.3) Let $X$ be a totally disconnected space. Then $\{x\}$ is closed in $X$, for each $x \in X$. 
Let \( C = \text{Cr}(X_\lambda \mid \lambda \in \Lambda) \) be the product of spaces \( X_\lambda \). The projection map \( \pi_\lambda \) takes \( (x_\lambda) \) to \( x_\lambda \). The open sets in the product top. are the unions of sets

\[
\pi_{\lambda_1}^{-1}(U_1) \cap \cdots \cap \pi_{\lambda_n}^{-1}(U_n)
\]

with \( n \) finite, each \( \lambda_i \) in \( \Lambda \) and \( U_i \) open in \( X_{\lambda_i} \). So each projection \( \pi_\lambda \) is continuous (prod. top. is minimal with this property).

**Remark.** Let \( a = (a_\lambda) \in C \) and let \( N \) be an open neighbourhood of \( a \) in \( C \). Thus \( N \) contains a set \( S \) of form (\( \ast \)) with \( a \in S \), so there exist \( \lambda_1, \ldots, \lambda_n \in \Lambda \) and open \( U_i \subseteq X_{\lambda_i} \) with \( a_{\lambda_i} \in U_i \) for each \( i \) and with \( \pi_{\lambda_1}^{-1}(U_1) \cap \cdots \cap \pi_{\lambda_n}^{-1}(U_n) \subseteq N \). In particular, \( N \) contains

\[
\{ x \in C \mid \pi_{\lambda_1}(x) = a_{\lambda_1}, \ldots, \pi_{\lambda_n}(x) = a_{\lambda_n} \}.
\]
(0.2.1) Let \((X_\lambda \mid \lambda \in \Lambda)\) be a family of top. spaces, \(C\) their Cartesian product.

(a) If each \(X_\lambda\) is Hausdorff, so is \(C\).
(b) If each \(X_\lambda\) is totally disconnected, so is \(C\).
(c) If each \(X_\lambda\) is compact, so is \(C\).

*Proof.* (a) and (b) are elementary; in (b), since a conts. image of a connected set is connected, the projection in each \(X_\lambda\) of a non-empty connected set has one element.

(c) is Tychonoff’s Theorem, postponed.
(1.1.5) Let $(X_i, \varphi_{ij})_I$ be an inverse system and write $X = \lim \leftarrow X_i$.

(a) If each $X_i$ is Hausdorff, so is $X$.
(b) If each $X_i$ is totally disconnected, so is $X$.
(c) If each $X_i$ is Hausdorff, then $\lim \leftarrow X_i$ is closed in $C = \text{Cr}(X_i \mid i \in I)$.
(d) If each $X_i$ is compact and Hausdorff, so is $X$.
(e) If each $X_i$ is a non-empty compact Hausdorff space, then $X \neq \emptyset$. 

Topology of inverse limits
(1.1.6) Let \((X, \varphi_i)\) be an inverse limit of an inverse system \((X_i, \varphi_{ij})_I\) of non-empty compact Hausdorff spaces. Then
(a) \(\varphi_i(X) = \bigcap_{j \geq i} \varphi_{ij}(X_j)\) for each \(i \in I\).
(b) the sets \(\varphi_i^{-1}(U)\) with \(i \in I\) and \(U\) open in \(X_i\) form a base for the topology on \(X\).
(c) if \(Y \subseteq X\) and \(\varphi_i(Y) = X_i\) for each \(i\) then \(Y\) is dense in \(X\).
(d) If \(Y\) is a top. space and \(\theta\) is a map \(Y \rightarrow X\) then \(\theta\) is conts. iff each \(\varphi_i\theta\) is conts.
(e) If \(A\) is discrete and \(f: X \rightarrow A\) is a conts., then \(f\) factors through some \(X_i\), i.e., \(\exists i\) and conts. \(g: X_i \rightarrow A\) with \(f = g\varphi_i\).
Let \( \rho \) be an equivalence relation on a top. space \( X \). \( X/\rho \) is the quotient set (elements the equiv. classes) \( q \) the quotient map \( X \rightarrow X/\rho \) (maps \( x \in X \) to its equiv. class). The quotient topology on \( X/\rho \) has as open sets all \( V \subseteq X/\rho \) such that \( q^{-1}(V) \) is open in \( X \). Then \( q \) is continuous. Easy to check the following:

if \( f: X \rightarrow Z \) is a continuous map to a space \( Z \) such that \( \rho \)-equiv. elements have the same image under \( f \), then there is a unique continuous map \( f^*: X/\rho \rightarrow Z \) such that \( f = f^*q \).

Call a subspace of a top. clopen if it is closed and open.

(1.1.7) Let \( X \) be a compact Hausdorff totally disconnected space. Then \( X \) is the inverse limit of its discrete quotient spaces.
(1.4.1) Let $G$ be a group, $\mathcal{L}$ a non-empty family of normal subgroups such that if $K_1, K_2 \in \mathcal{L}$ and $K_1 \cap K_2 \leq K_3 \triangleleft G$ then $K_3 \in \mathcal{L}$. Let $\mathcal{T}$ be the family of all unions of sets of cosets $Kg$ with $K \in \mathcal{L}$, $g \in G$. Then

(a) $\mathcal{T}$ is a topology on $G$ and $G$ is a top. group;
(b) $\mathcal{L}$ is the set of open normal subgroups of $G$ wrt this topology.