Algebraische Topologie

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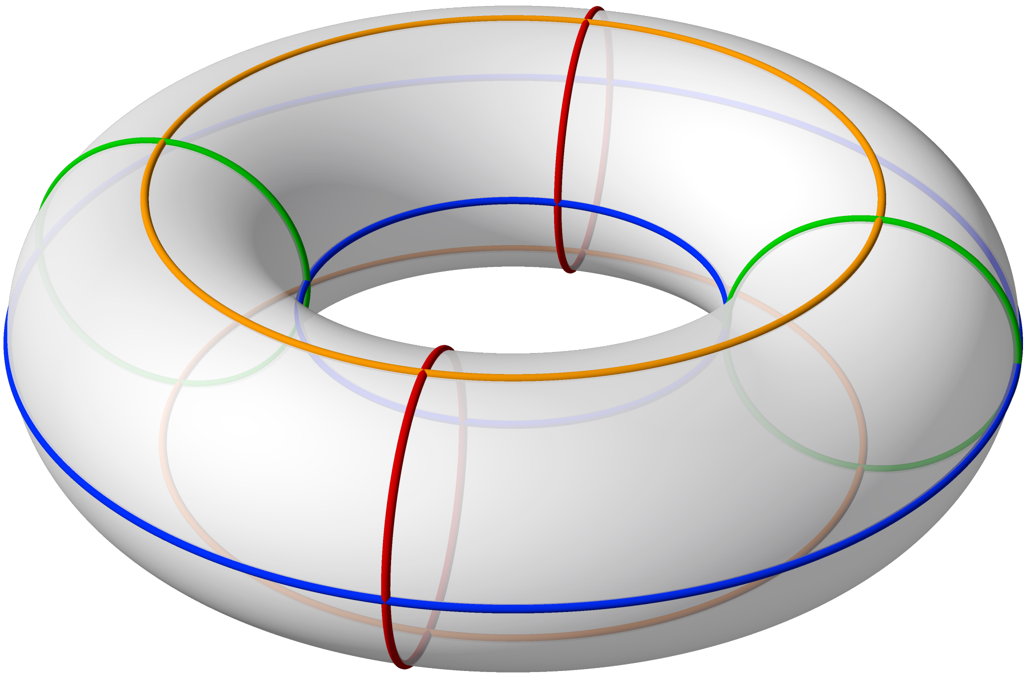


Figure 1: 2-dimensional torus

Lecture 1: Fundamental group functor

1.1 Some general point-set topology

1.1.1 Topological spaces and continuous maps.

If you need a refresher on set theory or general topology, see e.g. [3]. Let us briefly recall that topological spaces are a generalisation of metric spaces. Given a metric space (X, d), $x \in X$ and $r > 0$ we let

$$B(x, r) := \{y \in X : d(x, y) < r)$$

be the *open ball around x of radius r*. Then we proceed to define that a set $U \subset X$ is *open* if for every $u \in U$ there exists $r > 0$ such that $B(u, r) \subset U$. The collection of all open sets is called the *topology*, and X together with this collection is called a *topological space*.

While there exist topological spaces which are not *metrisable*, i.e. do not arise from a metric in the way described above, in this course we will only be interested in metrisable topological spaces.

We say that a continuous function $f : X \to Y$ between two topological spaces is *continuous* iff for any open set $U \subset Y$ the set $f^{-1}(U)$ is open (if you have not encountered this definition of continuity, then check as an exercise that our definition is equivalent to any of the ones you know). We say that f is a *homeomorphism* if there exists a continuous function $g : Y \to X$ such that $fg = \mathbf{id}_Y$ and $gf = \mathbf{id}_X$.

> **Example 1.1.1.** The spaces $(0, 1)$ and $\mathbb{R}$ are homeomorphic. Indeed first we check that $(0, 1)$ and $(-1, 1)$ are homeomorphic, for example using the map $s : (0, 1) \to (-1, 1)$ given by $s(x) := 2x - 2$. A homeomorphism $t : (-1, 1) \to \mathbb{R}$ is given for example by $t(x) = \tan(\frac{\pi}{2} \cdot x)$.

Given a subset $A \subset X$, we say that A is a *retract* of X if there exists a continuous map $f : X \to A$ such that $f_{|A} = \mathbf{id}_A$. Such an f is called a *retraction*.

Two typical questions which we try to answer in any topology course are as follows.

(a) Given two topological spaces X and Y, are they homeomorphic?

(b) Given an inclusion $A \subset X$ of topological spaces, is A a retract of X?

1.1.2 Compact, connected and path-connected spaces

We can answer these questions in some cases using some notions from general topology which the reader already knows. Recall that a topological space A is *compact* if the following holds. Suppose that there exists a family U_i, $i \in K$, of open sets such that $A = \bigcup_{i \in K} U_i$. Then there exists a finite set of indices $L \subset K$ such that $A = \bigcup_{i \in L} U_i$. We phrase this property frequently as "every open cover has a finite subcover".

> **Exercise 1.1.2.** *If X is a separable metric space then X is compact if and only if every sequence of points of X has a convergent subsequence.*

We will use the notation $I := [0, 1]$. A *path* in a topological space X is a continuous map $\alpha \colon I \to X$. We will use the notation $\alpha \colon x \overset{p}{\rightsquigarrow} y$ as a shorthand for saying that α is a path which connects x with y, i.e. $\alpha(0) = x$ and $\alpha(1) = y$. If $x = y$ then we say that α is a *loop at x.* Sometimes it is convenient to informally denote the constant loop at $x \in X$ by the same letter, i.e. we use the letter x also to denote the map $\mathrm{const}_x \colon I \to X$ defined by $\mathrm{const}_x(t) := x$ for all t.

We say that X is *connected* if the following holds. If $A, B \subset X$ are disjoint open sets and $X = A \cup B$ then either $A = X$ or $B = X$. In this course we are mostly interested in the slightly stronger notion of path-connectedness: we say that X is *path-connected* if for any $x, y \in X$ there exists a path which connects x with y.

> **Exercise 1.1.3.** *Show that if X is path-connected then it is connected. Show that the reverse implication does not always hold.*

The following exercise allows us to answer the questions from the previous subsection in some cases.

> **Exercise 1.1.4.** *Let $f \colon X \to Y$ be a continuous map between topological spaces. Show that if X is connected, respectively path-connected, respectively compact, then Y also has the respective property.*

Thus for example we see that $[0, 1]$ and $(0, 1)$ are not homeomorphic, since the first space is compact and the second is not, so in fact there are no surjective continuous maps from $[0, 1]$ to $(0, 1)$. Similarly we see that $[0, 1] \cup [2, 3]$ is not a retract of $[0, 3]$ since the first space is not path-connected and the second is.

1.2 Functors

Using only general-topological notions it would be hard to answer the questions such as whether $S^1 := \{z \in \mathbb{C} \colon |z| = 1\}$ is a retract of $D := \{z \in \mathbb{C} \colon |z| \leq 1\}$, or whether

$S^2 = \{x \in \mathbb{R}^3 \colon \|x\| = 1\}$ is homeomorphic with the torus $T^2 := S^1 \times S^1$.

We need better invariants to answer such questions. A convenient way to talk about invariants is with the langauge of functors. We introduce them here only informally, and more formal definitions will not be necessary to follow this course. However, an interested reader may consult e.g. [2].

We first fix a *category*, i.e. a collection of *objects* such as "all topological spaces" or "all metrisable topological spaces" or "all pairs (X, x) where X is a topological space and $x \in X$". Furthermore we fix what kind of *morphisms* we look at between those objects, e.g. "all continuous maps", "all continuous maps which are homeomorphisms", "Lipschitz-continuous maps", etc. Now a functor F is a way to associate to each object X a group $F(X)$, and to each morphism $f \colon X \to Y$ a homomorphism $F(f) \colon F(X) \to F(Y)$, in such a way that $F(\mathbf{id}_X) = \mathbf{id}_{F(X)}$ for any object X and $F(s \circ t) = F(s) \circ F(t)$ for all composable morphisms s, t.

It follows that if $f \colon X \to Y$ is a homeomorphism then $F(f) \colon F(X) \to F(Y)$ is an isomorphism, since if $gf = \mathbf{id}_X$ implies $F(g)F(f) = F(gf) = F(\mathbf{id}_X) = \mathbf{id}_{F(X)}$ and similarly $F(f)F(g) = \mathbf{id}_{F(Y)}$. This means that functors can be used to show that two spaces X and Y are not homeomorphic - a necessary condition for X and Y to be homeomorphic is that the groups $F(X)$ and $F(Y)$ should be isomorphic.

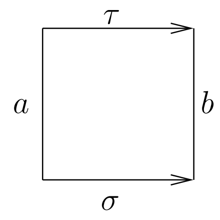
Functors can be also used to show that $A \subset X$ is not a retract of X. Indeed, if we have for example $F(X) = \{0\}$ and $F(A) \neq \{0\}$ then there is no retraction from X to A: indeed consider $i \colon A \to X$ to be the embedding and $f \colon X \to A$ to be a retraction, then $F(\mathbf{id}_A) = \mathbf{id}_{F(A)}$, so $F(\mathbf{id}_A)$ is not the 0 endomorphism of $F(A)$. However we have $F(\mathbf{id}_A) = F(f \circ i) = F(f) \circ F(i)$. Since $F(i) \colon F(A) \to F(X) = \{0\}$ we see that $F(i)$ is the 0-homomorphism and hence $F(\mathbf{id}_A)$ is in fact the 0 endomorphism of $F(A)$, which is a contradiction.

Remark 1.2.1. We could also demand that $F(X)$ is a e.g. a ring, a field, etc. In this case it is natural to demand that $F(f)$ should be a ring homomorphism, a field homomorphism, etc. However throughout the majority of this course we will not need this greater generality.

1.3 Definition of the fundamental group

1.3.1 Homotopy of paths

Given two paths $\sigma, \tau \colon a \overset{p}{\rightsquigarrow} b$ in a topological space X, we say they are *homotopic to each other relative to their ends*, written $\sigma \simeq \tau \,\mathrm{rel}\{0, 1\}$, if we can find a map $F \colon I \times I \to X$ such that for all $x \in I$ we have $F(x, 0) = \sigma(x)$, $F(x, 1) = \tau(x)$, $F(0, x) = a$, $F(x, 0) = b$. We express this frequently using the following diagram.



This diagram represents the domain of F, i.e. $I \times I$, and it shows what F does on the edges of the square. We say that F is a *homotopy between σ and τ*

We say that $\sigma\colon a \overset{p}{\rightsquigarrow} a$ is a *contractible loop* or a *homotopically trivial loop* if $\sigma \simeq a \operatorname{rel}\{0,1\}$ (note that by our convention we use the letter a here to denote the constant loop at a).

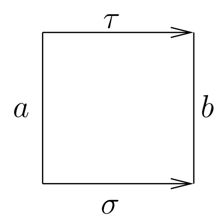
Lemma 1.3.1. *Homotopy relative to ends is an equivalence relation, i.e. if we have $\sigma, \tau, \rho\colon a \overset{p}{\rightsquigarrow} b$ then*

(a) $\sigma \simeq \sigma \, rel\{0,1\}$

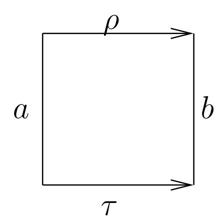
(b) $\sigma \simeq \tau \, rel\{0,1\} \implies \tau \simeq \sigma \, rel\{0,1\}$

(c) $\sigma \simeq \tau \, rel\{0,1\}, \tau \simeq \rho \, rel\{0,1\} \implies \sigma \simeq \rho \, rel\{0,1\}$

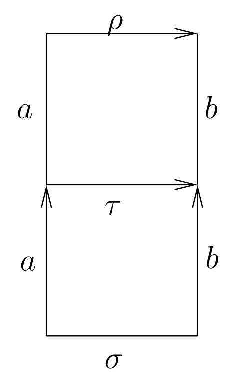
Proof. We leave the first two properties as exercises. We show how to prove the third property to illustrate how we will use the homotopy diagrams in the proofs in the future. The fact that $\sigma \simeq \tau \operatorname{rel}\{0,1\}$ is illustrated by the diagram



The fact that $\tau \simeq \rho \operatorname{rel}\{0,1\}$ is illustrated by the diagram



As such we can form the diagram



which shows that indeed $\sigma \simeq \rho \,\mathrm{rel}\{0,1\}$, as claimed. In symbols, we have

(a) a map $F\colon I \times I \to X$ such that for all $x \in X$ we have $F(x,0) = \sigma(x)$, $F(x,1) = \tau(x)$, $F(0,x) = a$, $F(x,0) = b$.

(b) a map $G\colon I \times I \to X$ such that for all $x \in X$ we have $F(x,0) = \tau(x)$, $F(x,1) = \rho(x)$, $G(0,x) = a$, $G(x,0) = b$.

As such we can form a map $\bar{H}\colon I \times [0,2] \to X$ by setting

$$\begin{aligned} \bar{H}(x,y) &:= F(x,y) \quad \text{when } y \leq 1 \\ \bar{H}(x,y) &:= G(x,y-1) \quad \text{when } y \geq 1, \end{aligned}$$

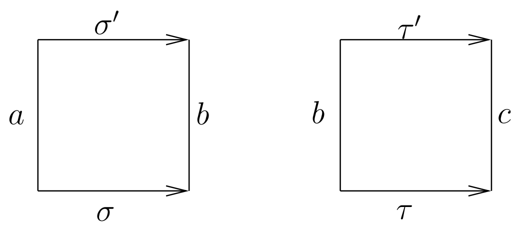
and finally we let $H(x,y) := \bar{H}(x,2y)$. □

1.3.2 Composition of paths

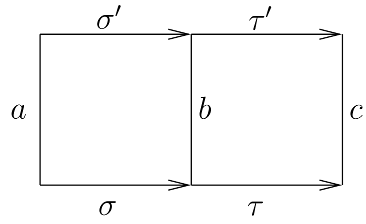
Given $\sigma\colon a \overset{p}{\rightsquigarrow} b$ and $\tau\colon b \overset{p}{\rightsquigarrow} c$ we can form the *concatenation* $\sigma\tau\colon a \overset{p}{\rightsquigarrow} c$ by first following σ and then τ, i.e. we let $\sigma\tau(x) := \sigma(2x)$ for $x \leq \frac{1}{2}$ and $\sigma\tau(x) := \tau(2x-1)$ for $x \geq \frac{1}{2}$. This operation is compatible with homotopies in the following sense.

Lemma 1.3.2. *Suppose $\sigma, \sigma'\colon a \overset{p}{\rightsquigarrow} b$, $\tau, \tau'\colon b \overset{p}{\rightsquigarrow} c$, and suppose also that $\sigma \simeq \sigma' \, rel\{0,1\}$ and $\tau \simeq \tau' \, rel\{0,1\}$. Then $\sigma\tau \simeq \sigma'\tau' \, rel\{0,1\}$.*

Proof. We have the diagrams



and so we can form the diagram



which witnesses the fact that $\sigma\tau$ and $\sigma'\tau'$ are homotopic relative to their ends. $\square$

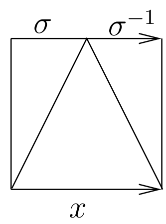
Now we are ready to define the fundamental group $\pi_1(X, x)$ of a pair (X, x), where X is a topological space and $x \in X$, as follows: we let $\pi(X, x)$ to be the set of all equivalence classes of loops at x in X. The homotopy class of a loop σ will be denoted by $[\sigma]$.

We define a binary operation on $\pi_1(X, x)$ by setting $[\sigma] \cdot [\tau] := [\sigma\tau]$. The previous lemma shows that this binary operation is well-defined on the elements of $\pi_1(X, x)$. We define the neutral element in $\pi_1(X, x)$ to be the class of the constant loop $[x]$, and the inverse is defined as $[\sigma]^{-1} := [\sigma^{-1}]$, where $\sigma^{-1}(x) := \sigma(1 - x)$.

Theorem 1.3.3. $\pi_1(X, x)$ *with the operations defined above is a group.*

Proof. Let us check for example that $[\sigma] \cdot [\sigma]^{-1} = [x]$. Checking the other properties is left as an exercise.

We need to show that $\sigma\sigma^{-1} \simeq x \operatorname{rel}\{0, 1\}$. This is witnessed by the following diagram:



In symbols, we define a homotopy $F\colon I \times I \to X$ between $\sigma\sigma^{-1}$ and x as follows.

$$\begin{aligned} F(s,t) &:= \sigma(2s) && \text{when } 2s \le t \\ &:= \sigma(t) && \text{when } t \le 2s \le 2 - t, \\ &:= \sigma^{-1}(2s-1) && \text{when } 2 - t \le 2s. \end{aligned}$$

$\square$

Remark 1.3.4. Note that the above argument shows that if $\sigma\colon a \overset{p}{\rightsquigarrow} b$ then $\sigma^{-1}\sigma$ is a loop at a which is contractible.

1.3.3 Change of basepoint

If X is not path-connected then clearly the isomorphism class of $\pi_1(X, x)$ might depend on the choice of $x \in X$. However when x and y can be connected by a path then we have the following lemma.

Lemma 1.3.5. *Let $\alpha\colon a \overset{p}{\rightsquigarrow} b$. Then we have an isomorphism $\alpha_*\colon \pi_1(X, a) \to \pi_1(X, b)$ given by*

$$\alpha_*\colon [\sigma] \mapsto [\alpha^{-1}\sigma\alpha].$$

Proof. The fact that α_* is well defined, i.e. if $\sigma \simeq \sigma' \operatorname{rel}\{0,1\}$ then $\alpha^{-1}\sigma\alpha \simeq \alpha^{-1}\sigma'\alpha \operatorname{rel}\{0,1\}$, follows from Lemma 1.3.2. Thus we need to check that α_* is a group homomorphism, i.e. $\alpha_*([a]) = [b]$ and $\alpha_*([\sigma\tau]) = \alpha_*([\sigma])\alpha_*([\tau])$, and that α_* is a bijection.

The fact that α_* is a bijection follows from directly checking that $(\alpha^{-1})_*$ is the inverse. Indeed, we have

$$\alpha_*(\alpha^{-1})_*\colon [\sigma] \mapsto [\alpha\alpha^{-1}\sigma\alpha\alpha^{-1}].$$

By Remark 1.3.4 we see that $\alpha\alpha^{-1}$ is contractible, and so $[\alpha\alpha^{-1}\sigma\alpha\alpha^{-1}] = [\sigma]$.

Let us check that $\alpha_*([a]) = [b]$. Indeed $\alpha_*([a]) = [\alpha^{-1}a\alpha]$. We need to find a homotopy from $\alpha^{-1}a\alpha$ to b. But it is clear that $\alpha^{-1}a\alpha$ is homotopic to $\alpha^{-1}\alpha$, so the statement follows from Remark 1.3.4 again.

Checking the property $\alpha_*([\sigma\tau]) = \alpha_*([\sigma])\alpha_*([\tau])$ is left as an exercise. □

Corollary 1.3.6. *If X is path-connected then the isomorphism class of $\pi_1(X, x)$ does not depend on the choice of $x \in X$*

Remark 1.3.7. This corollary allows us to somewhat informally talk about about the fundamental group $\pi_1(X)$ of X, without referring to a chosen point of X, whenever X is path-connected.

1.4 Extending π_1 to a functor

We are interested in the category whose objects are pairs (X, a), where $a \in X$, since π_1 is well-defined on (X, a). Between two such objects (X, a) and (Y, b) we look at all continuous maps $f\colon X \to Y$ such that $f(a) = f(b)$.

In order to extend π_1 to a functor on this category, we need to define $\pi_1(f)$ as some homomorphism between the groups $\pi_1(X, a)$ and $\pi_1(Y, b)$. By convention $\pi_1(f)$ will be usually denoted by f_*.

For a loop σ in X at a we define $f_*([\sigma]) := [f \circ \sigma]$. Let us check that f_* is well defined: if $F\colon I \times I \to X$ is a homotopy between σ and τ then $f \circ F$ is a homotopy between $f \circ \sigma$ and $f \circ \tau$, which is exactly what we need.

The fact that f_* is a group homomorphism boils down to the facts that i) $f \circ \mathrm{const}_a = \mathrm{const}_b$ and ii) $f \circ (\sigma\tau) = (f \circ \sigma)(f \circ \tau)$, which is clear by the definition of concatenation.

This finishes the definition of the fundamental group functor.

Lecture 2: Homotopy of maps

Let's start by generalising the notion of homotopy between paths to homotopies between arbitrary continuous maps. Whenever we consider a map $F\colon Y \times I \to X$, we denote with F_t the map $Y \to X$ given as $F_t(y) := F(y, t)$.

Definition 2.0.1. Let $f, g\colon Y \to X$ be continuous maps between topological spaces, and let $A \subset Y$ be such that $f_{|A} = g_{|A}$. We say that f and g are *homotopic relative to* A, written $f \simeq g \operatorname{rel} A$, if there exists a continuous map $F\colon Y \times I \to X$ such that $F_0 = f$, $F_1 = g$, and for all t we have $F_{t|A} = f_{|A}$.

If $A = \emptyset$ then we write $f \simeq g$. The map F is called a *homotopy* between f and g.

Exercise 2.0.2. *Show that $f \simeq g \operatorname{rel} A$ is an equivalence relation.*

Example 2.0.3. Suppose that $X, Y = \mathbb{R}^n$, $f(y) = y$, $g(y) = 0$ for all $y \in Y$. By considering $F(x, t) := tx$ we see that f and g are homotopic to each other.

If for some topological space X we have that $\mathbf{id}_X \simeq \mathrm{const}_x$ for some $x \in X$ then we say that X is *contractible*. Thus the previous example shows that $\mathbb{R}^n$ is contractible. More generally we have the following example.

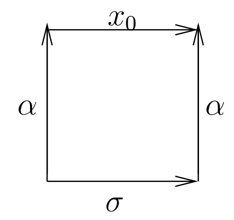
Example 2.0.4. Any convex subset $Y \subset \mathbb{R}^n$ is contractible. Indeed, we may fix $y_0 \in Y$ and define F by the formula $F(y, t) = (1 - t)y + ty_0$. In particular, the unit disk $D^n := \{x \in \mathbb{R}^n \colon \|x\| \leq 1\}$ is contractible.

If a space X is path-connected and $\pi_1(X) = \{0\}$ then we say that X is *simply connected.*

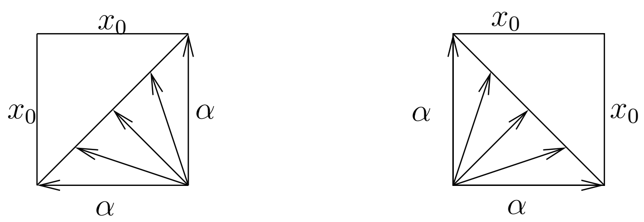
Proposition 2.0.5. *If X is contractible then it is simply connected.*

Proof. If $F\colon \mathbf{id}_X \simeq \mathrm{const}_{x_0}$ is a homotopy then for any $y \in X$ we can consider the path $\sigma\colon y \overset{p}{\rightsquigarrow} x$ given by $\sigma(t) := F(y, t)$. This shows that X is path-connected.

Arguing about $\pi_1(X)$ requires us to show that any loop at x_0 can be contracted to the constant loop at x_0 *while keeping the end points fixed.* Let us fix a loop $\sigma\colon I \to X$ at x_0. As the first step, we can consider the map $I \times I \ni (s,t) \mapsto F(\sigma(s),t) \in X$. In diagrammatic terms this gives us



where α is the loop at x_0 given by $\alpha(t) := F(x_0,t)$. This is not quite enough because this homotopy does not fix the end points. However we can consider the following two diagrams:



For example the left one represents the map $G\colon I\times I \to X$ defined as follows: $G(s,t) := x_0$ if $t \geq s$, $G(s,t) := \alpha(1+t-s)$ if $t < s$. Putting all three together gives us an end-preserving homotopy between the loop $\alpha^{-1}\sigma\alpha$ and the constant loop at x_0. This means that $[\alpha^{-1}\sigma\alpha] = [\alpha]^{-1}[\sigma][\alpha]$ is the trivial element in $\pi_1(X)$, and hence also $[\sigma]$ is the trivial element of $\pi_1(X)$. $\square$

The following exercises give us very important equivalent ways of thinking about contractible loops.

> **Exercise 2.0.6.** *Let σ be a loop at $x \in X$. Since $\sigma(0) = x = \sigma(1)$, we can consider σ as a map whose domain is $\mathbb{S}^1$. Show that $\sigma \simeq x$ rel$\{0,1\}$ if and only if σ can be extended to a map $D^2 \to X$.*

> **Exercise 2.0.7.** *Let X be a path-connected topological space. Show that the following conditions are equivalent.*
>
> *(a) $\pi_1(X) = \{0\}$ (i.e. X is simply connected).*
>
> *(b) $\forall f\colon \mathbb{S}^1 \to X$, we have that f can be extended to $\bar{f}\colon D^2 \to X$.*
>
> *(c) if $\sigma,\tau\colon a \overset{p}{\rightsquigarrow} b$ then $\sigma \simeq \tau$ rel$\{0,1\}$*

2.1 Behaviour of the induced map under homotopies

Now we proceed to investigate how does the homomorphism between fundamental groups induced by a continuous map changes under the homotopy. This is the content of the following proposition.

Proposition 2.1.1. *Let $f, g\colon Y \to X$, and suppose that $F\colon Y \times I \to X$ is a homotopy between f and g. Let $y_0 \in Y$, let $x_0 := f(y_0)$, $x_1 := g(y_0)$ and let $\alpha\colon x_0 \overset{p}{\rightsquigarrow} x_1$ be the path $\alpha(t) := F(y_0, t)$. Then we we have*

$$g_* = \alpha_* \circ f_*.$$

Proof. We need to show that for every loop σ in Y there is a homotopy which looks like this on the edges:

$$\begin{tikzcd} \cdot \arrow[r, "g \circ \sigma"] & \cdot \\ \cdot \arrow[u, "\alpha"] \arrow[r, "f \circ \sigma"'] & \cdot \arrow[u, "\alpha"'] \end{tikzcd}$$

It is clear that $F(\sigma(s), t)\colon I \times I \to X$ is exactly such a homotopy. □

Corollary 2.1.2. *(a) If $f, g\colon Y \to X$ are homotopic then f_* is an isomorphism if and only if g_* is an isomorphism.*

(b) Let $f, g\colon Y \to X$, and suppose that $F\colon Y \times I \to X$ is a homotopy between f and g relative to some $y_0 \in Y$ and let $x_0 := f(y_0)$. Then we we have

$$g_* = f_*.$$

□

Definition 2.1.3. (a) A continuous map $f\colon Y \to X$ is a homotopy equivalence if there exists a continuous map $g\colon X \to Y$ such that $fg \simeq \mathbf{id}_X$, $gf \simeq \mathbf{id}_Y$.

(b) (Originally introduced in Lecture 3, but fits here better) Given $A \subset X$, we say that A is a deformation retract of X if there exists a continuous map $r\colon X \to A$ such that $r_{|A} = \mathbf{id}_A$ and $i \circ r \simeq \mathbf{id}_X \operatorname{rel}_A$, where $i\colon A \to X$ is the natural embedding. Such a map r is called a deformation retraction.

We note that a deformation retraction is in particular a homotopy equivalence, since we can take $i\colon A \to X$ to be natural embedding of A into X, and clearly we have $r \circ i = \mathbf{id}_A$ and $i \circ r \simeq \mathbf{id}_X \operatorname{rel} A$.

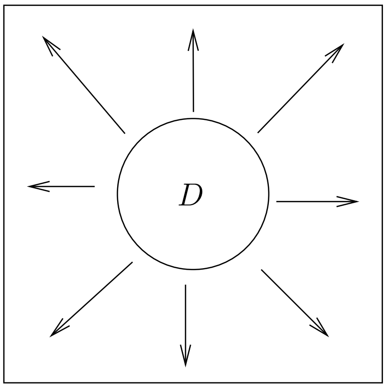
Corollary 2.1.4. *If $f\colon X \to Y$ is a homotopy equivalence then $f_*\colon \pi_1(X,x) \to \pi(Y,f(x))$ is an isomorphism.*

Proof. We have $f_*g_* = (fg)_* = (\mathbf{id}_Y)_* = \mathbf{id}_{\pi_1(Y)}$ and similarly $g_*f_* = \mathbf{id}_{\pi_1(X)}$. $\square$

Definition 2.1.5. If (X,x_0) and (Y,y_0) are two topological spaces then we can define their *wedge* $X \vee Y$ as the space $X \sqcup Y/\sim$, where the equivalence relation $\sim$ identifies the point $x_0 \in X$ with $y_0 \in Y$.

If we consider $X \vee Y$ as a space with a basepoint then typically we choose the point which arises from identifying x_0 with y_0.

Example 2.1.6. Let D be a disk (open or closed) in the torus $T^2 := \mathbb{S}^1 \times \mathbb{S}^1$. Then $T^2 \setminus D$ deformation retracts onto $S^1 \vee S^1$. Indeed, the deformation retraction flows along the vector field in the following figure.



(This picture shows the fundamental domain in $\mathbb{R}^2$. To obtain $\mathbb{R}^2/\mathbb{Z}^2$ we need to identify horizontal and vertical edges, respectively.)

2.2 Computation of $\pi_1(S^1)$

In order to compute $\pi_1(S^1)$, we identify it with $\mathbb{R}/\mathbb{Z}$, and we use the fact that the quotient map $\varphi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is a group homomorphism which is a homeomorphism when we restrict it to the interval $(-\frac{1}{2}, \frac{1}{2})$.

Let $\psi\colon \mathbb{R}/\mathbb{Z} \setminus \{\frac{1}{2} + \mathbb{Z}\} \to (-\frac{1}{2}, \frac{1}{2})$ be the inverse to φ, so that $\varphi\psi(x) = x$ for all $x \in \mathbb{R}/\mathbb{Z} \setminus \{\frac{1}{2} + \mathbb{Z}\}$.

We start with the following lemma.

Lemma 2.2.1 ("Path lifting property")**.** *Let $\sigma\colon 0 \overset{p}{\rightsquigarrow} a$ be a path in $\mathbb{R}/\mathbb{Z}$. Then there exists a unique path $\bar\sigma$ in $\mathbb{R}$ which starts at 0 and such that $\varphi \circ \bar\sigma = \sigma$.*

Proof. We have $\sigma\colon I \to \mathbb{R}/\mathbb{Z}$. Since I is compact, the function σ is uniformly continuous, and so we can find $\varepsilon > 0$ such that $|y - y'| < \varepsilon$ implies $|\sigma(y) - \sigma(y')| < \frac{1}{2}$. Thus for such y, y' we can consider $\psi(\sigma(y) - \sigma(y'))$.

Now let N be such that $\frac{1}{N} < \varepsilon$. We define

$$\bar\sigma(y) := 0 + \psi\left(\sigma(\frac{1}{N}y) - \sigma(0)\right) + \psi\left(\sigma(\frac{2}{N}y) - \sigma(\frac{1}{N}y)\right) + \ldots + \psi\left(\sigma(y) - \sigma(\frac{N-1}{N}y)\right)$$

It is clear that $\bar\sigma$ is continuous, $\bar\sigma(0) = 0$, and $\varphi\bar\sigma = \sigma$. The uniqueness of $\bar\sigma$ can be argued as follows: if $\bar\tau\colon I \to \mathbb{R}$ is another path with the same properties then $\bar\sigma - \bar\tau$ is a continuous map $I \to \mathbb{R}$ whose image lies in $\ker(\varphi) = \mathbb{Z}$. Since $\bar\sigma - \bar\tau$ is continuous, we see that its image contains just a single point and therefore this point is $\bar\sigma(0) - \bar\tau(0) = 0$. But this means that $\bar\sigma - \bar\tau = 0$, so $\bar\sigma = \bar\tau$. $\square$

We also need the following lemma.

Lemma 2.2.2 ("Homotopy lifting property")**.** *Let $a \in \mathbb{R}/\mathbb{Z}$ and let $\sigma, \tau\colon 0 \overset{p}{\rightsquigarrow} a$. Let $\bar\sigma, \bar\tau$ be the paths in $\mathbb{R}$ given by the previous lemma. Suppose that $F\colon \sigma \simeq \tau$ rel$\{0,1\}$ is a homotopy between σ and τ. Then there exists a unique homotopy $\bar F\colon \bar\sigma \simeq \bar\tau$ rel$\{0,1\}$ such that $\varphi \circ \bar F = F$.*

Proof. The proof is very similar to the proof of the previous lemma. We have that $F\colon I \times I \to \mathbb{R}/\mathbb{Z}$ is uniformly continuous, so we can find $\varepsilon > 0$ such that if $|y - y'| < \varepsilon$ then $F(y) - F(y') < 1$. We let N be such that $N\varepsilon > \sqrt{2}$ (since the distance between y and y' in $I \times I$ is bounded by $\sqrt{2}$). We define

$$\bar F(y) := 0 + \psi\left(F(\frac{1}{N}y) - F(0)\right) + \psi\left(F(\frac{2}{N}y) - F(\frac{1}{N}y)\right) + \ldots + \psi\left(F(y) - F(\frac{N-1}{N}y)\right).$$

It is clear that $\bar F$ is continuous, $\bar F(0,0) = 0$, and $\varphi\bar F = F$. Uniqueness of $\bar F$ with such properties follows as in the previous proof. Let us finally check that $\bar F$ indeed is a homotopy between $\bar\sigma$ and $\bar\tau$, i.e. it can be described by the diagram

$$\begin{tikzcd}
\cdot \arrow[r, "\bar\tau"] \arrow[d, dash, "0"'] & \cdot \arrow[d, dash, "\bar a"] \\
\cdot \arrow[r, "\bar\sigma"'] & \cdot
\end{tikzcd}$$

where $\bar{a}$ is some point in $\mathbb{R}$ such that $\varphi(\bar{a}) = a$.

The fact that the bottom edge is as claimed follows from the uniqueness part of the previous lemma. The fact that the vertical edges are constant follows by noting that $\varphi\bar{F} = F$, and the edges in the diagram describing F are constant. This is indeed enough, since the preimage of a single point under φ is a discrete set and $\bar{F}$ is continuous. Now in particular we know that the top-left corner is mapped to 0, and hence the fact that the top edge is as claimed follows again from the uniqueness part of the previous lemma. □

With the previous two lemmas established, we are ready to prove the following theorem.

Theorem 2.2.3. *We have $\pi_1(S^1) \cong \mathbb{Z}$. The isomorphism sends $[\sigma]$ to the endpoint of $\bar{\sigma}$.*

Proof. Let us denote the map which sends $[\sigma]$ to $\bar{\sigma}(1)$ by χ. By the homotopy lifting property we have that χ is well-defined. It is clear that the class of the constant loop is sent to 0, thus we only need to check that

$$\chi([\sigma]) + \chi([\tau]) = \chi([\sigma\tau]).$$

Let us denote $\chi([\sigma]) = \bar{\sigma}(1)$ by m and $\chi([\tau]) = \bar{\tau}(1)$ by n. Consider the path $\bar{\tau}' \colon I \to \mathbb{R}$ defined as $\bar{\tau}'(t) := m + \tau(t)$, and we note that $\bar{\sigma}\bar{\tau}'$ is a lift of $\sigma\tau$. Therefore we deduce that $\chi([\sigma\tau]) = \bar{\sigma}\bar{\tau}'(1)$, which is clearly equal to $m + n$. This finishes the proof. □

In a very similar fashion we can prove the following more general theorem.

Theorem 2.2.4. *Let G be a simply connected topological group and let $H \lhd G$ be a normal discrete subgroup (i.e. for some open set $U \subset G$ we have $U \cap H = \{e\}$.). Then $\pi_1(G/H) \cong H$*

Proof. Left as an exercise. □

This allows us to deduce for example that $\pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$, since $S^1 \times S^1$ is homeomorphic to $\mathbb{R}^2/\mathbb{Z}^2$. However, this particular result can be also deduced from the following theorem.

Theorem 2.2.5. *Let X, Y be topological spaces and let $x \in X$, $y \in Y$. Then we have an isomorphism*

$$\pi_1(X, x) \times \pi_1(Y, y) \to \pi_1(X \times Y, (x, y))$$

given as follows: the pair $([\sigma], [\tau])$ is sent to the class of the path (σ, τ).

Proof. Left as an exercise. □

Example 2.2.6. (a) We can now show that $\mathbb{S}^1$ is not a retract of D^2. Indeed if there was a retraction $r\colon D^2 \to \mathbb{S}^1$ then the composition $\mathbb{S}^1 \to D^2 \xrightarrow{r} \mathbb{S}^1$ would be the identity map. However the induced map on the fundamental groups is the 0 map, since it factors through $\pi_1(D^2) = \{0\}$.

(b) Using the previous example we can show the *Brauer fixed point theorem in dimension* 2, i.e. show that if $f\colon D^2 \to D^2$ is a continuous map then for some $x \in D^2$ we have $f(x) = x$. Indeed if f is a map without fixed points then we define a new map $r\colon D^2 \to \mathbb{S}^1$ as follows: we let $r(x)$ to be the point on $\mathbb{S}^1 = \partial D^2$ which lies on the infinite ray starting from $f(x)$ and passing through x. Since f is assumed not to have any fixed points, we deduce that r is well-defined. It is straightforward to check that r is a retraction from D^2 to $\mathbb{S}^1$, which is a contradiction.

Lectures 3-4: Van Kampen's theorem

3.1 Baby case for simply-connected spaces

Let us start with the following proposition, which is a special case of van Kampen's theorem.

Proposition 3.1.1. *Suppose that $X = U \cup V$, where both U and V are simply connected open sets, and $U \cap V$ is a non-empty path-connected set. Then X is simply connected.*

Proof. Let us fix a base point $x \in U \cap V$. First let us argue about path-connectedness, so let $u \in U$ and $v \in V$. Then there exists $\sigma\colon u \overset{p}{\rightsquigarrow} x$ and $\tau\colon x \overset{p}{\rightsquigarrow} v$, and so $\sigma\tau\colon u \overset{p}{\rightsquigarrow} v$.

Now let us show that $\pi_1(X) = \{[x]\}$. Let us fix a loop $\sigma\colon I \to X$ with $\sigma(0) = \sigma(1) = x$. Clearly it is enough to show that σ is homotopic to a product of loops, each of which is entirely contained either in U or in V. For this we start by expressing σ as a product of paths, each of which is in either U or V, as follows.

Consider the covering of I by the open sets $\sigma^{-1}(U)$ and $\sigma^{-1}(V)$. Since both these sets are open we can express them as unions of relatively open intervals. Since I is compact we deduce that there exist finitely many relatively open intervals $I_1, I_2, \ldots, I_n \subset I$ such that $I = \bigcup I_j$ and for each j we have either $\sigma(I_j) \subset U$ or $\sigma(I_j) \subset V$. For each two of those intervals which intersect, let us choose a point in the intersection, and let us order those points. If necessary, let us add 0 and 1 to these points, to obtain a sequence $0 = x_0 < x_1 < \ldots < x_m = 1$.

It is clear by construction that for each $j < m$ we have that σ restricted to $[x_j, x_{j+1}]$ is a path which is entirely contained either in U or in V. Let us call this path σ_j.

Now for each $j < m$ let us fix a path $\tau_j\colon x_j \overset{p}{\rightsquigarrow} x$. By Remark 1.3.4, the loop $\sigma = \sigma_0\sigma_1\ldots\sigma_{m-1}$ is homotopic to the loop

$$\sigma_0\tau_1\tau_1^{-1}\sigma_1\tau_2\tau_2^{-1}\ldots\sigma_{m-2}\tau_{m-1}\tau_{m-1}^{-1}\sigma_{m-1}.$$

Now we can write

$$[\sigma] = [\sigma_0\tau_1][\tau_1^{-1}\sigma_1\tau_2]\ldots[\tau_{m-1}^{-1}\sigma_{m-1}],$$

which finishes the proof. □

Example 3.1.2. For $n \geq 2$ we have $\pi(\mathbb{S}^n) = \{0\}$. Indeed let $p, q \in \mathbb{S}^n$ be the north and south poles, respectively. Then we have $\mathbb{S}^n = U \cup V$, where $U = \mathbb{S}^n \setminus \{p\}$ and $V = \mathbb{S}^n \setminus \{q\}$. Now U and V are contractible, hence simply connected, and their intersection is path connected (it deformation retracts to $\mathbb{S}^{n-1}$). As such we can use the previous proposition to deduce that $\pi(\mathbb{S}^n) = \{0\}$.

3.2 Some group theoretic constructions

3.2.1 Free groups

In order to present the general version of van Kampen's theorem, we need to introduce some group theoretic constructions. Let us start with the *free groups*. The *free group F_2 on two symbols a and b* is defined as follows. As a set, we consider F_2 to consist of all reduced words in the four letters a, a^{-1}, b, b^{-1}, where a given word is *reduced* if it does not contain a substring of the form $aa^{-1}, bb^{-1}, b^{-1}b, a^{-1}a$. The neutral element is defined to be the empty string. The inverse is defined by

$$(s_1 \dots s_k)^{-1} := s_k^{-1} s_{k-1}^{-1} \dots s_1^{-1} \tag{3.1}$$

The binary operation is defined as follows: given $\omega_1, \omega_2 \in F_2$ we define $\omega_1 \cdot \omega_2$ by first concatenating these two words, and then reducing them (i.e. successively removing all occurrences of $aa^{-1}, bb^{-1}, b^{-1}b, a^{-1}a$). We note that here there is only one possibel order of applying the reductions, and so this binary operation is well-defined.

Lemma 3.2.1. *The above definition makes F_2 into a group.*

Proof. Let e denote the empty string. The fact that $w \cdot e = e \cdot w = w$ for every $w \in F_2$ is clear. Similarly it is clear that $w \cdot w^{-1} = e$. The only non-trivial part is to check associativity. Associativity clearly follows from the following claim: if we start from an unreduced word w, there is a unique reduced word $\bar{w}$ such that if we start reducing w then we arrive at $\bar{w}$. (In other words, "the order of reductions does not matter".)

In order to check this claim we consider the rooted 4-regular tree T with oriented edges labelled by the symbols a and b, such that at each vertex we have exactly one incoming edge a, one outgoing edge a, one incoming edge b and one outgoing edge b. Let us denote the root vertex of T with e.

Now every word in the letters a, a^{-1}, b, b^{-1} can be traced on T starting from e in the obvious way. We note that reduced words are exactly those such that the traced path is a *geodesic path*, i.e. a path of the shortest possible length. Furthermore we note that reducing a given word corresponds to shortening the traced path, without altering its endpoints. As such, the desired claim follows from the fact that T has *unique geodesics*, i.e. any two vertices can be joined by a unique geodesic path. □

Similarly we can define the free group on an arbitrary (finite or infinite) set of symbols.

3.2.2 Free products

Suppose that A and B are two groups. Then the *free product* $A * B$ of A and B is defined as follows. As a set, we define $A * B$ as the set of all reduced words in the alphabet $(A \setminus \{e_A\}) \sqcup (B \setminus \{e_B\})$. In this case we say that a word is reduced if for every two consecutive letters s_1, s_2, we have that either $s_1 \in A, s_2 \in B$ or $s_2 \in B, s_1 \in A$. We define the neutral element to be the empty string, and the inverses are defined by the formula (3.1). The binary operation is defined as follows: given ω_1 and ω_2 we define $\omega_1 \cdot \omega_2$ by first concatenating these two words, and then reducing them.

Lemma 3.2.2. *The above definition makes $A * B$ into a group.*

Proof. As before, the nontrivial part of the proof is to check associativity, which again boils down to checking that the order of reductions does not matter. Let K_A and K_B be the complete graphs with vertex sets A and B, respectively. We consider them rooted with the respective neutral elements as the root vertices. Let us build inductively a sequence of graphs $G_0, G_1, \ldots$, with $V(G_i) \subset V(G_{i+1})$. The elements of $V(G_i) \setminus V(G_{i-1})$ will be called "stage i vertices". We set $G_0 = K_A$, and to construct G_1 we attach a copy of K_B at every vertex G_0. To construct G_2 we attach a copy of K_B at every vertex of stage 1, to construct G_3 we attach a copy of K_A at every vertex of stage 2, and we continue inductively. We define G as the union of all these graphs.

Now G has all the properties of T from the previous proof: the words in the alphabet $(A \setminus e_A) \sqcup (B \setminus e_B)$ can be traced on G starting at its root, and reduced words are exactly those such that the traced path is a geodesic path. Furthermore we note that reducing a given word corresponds to shortening the traced path, without altering its endpoints. As such the desired claim follows from the fact that the graph G has unique geodesics. □

Remark 3.2.3. It is clear that there is a natural isomorphism between the free group F_2 defined in the previous subsection and the free product $\mathbb{Z} * \mathbb{Z}$, and we often consider these two groups as the same.

3.2.3 Amalgamated free products

Suppose now that A, B, C are groups and $\alpha \colon C \to A$, $\beta \colon C \to B$ are group homomorphisms. Then we define the *amalgamated free product* $A *_C B$ as the quotient of $A * B$

by the normal subgroup generated by the reduced words $\alpha(c)\beta(c^{-1})$, $c \in C \setminus \{e_C\}$. We note that the notation $A *_C B$ is ambiguous, since the construction depends on α and β in the crucial way, nevertheless in all cases it will be clear what α and β to take.

Example 3.2.4. (a) For any group G we have $G *_0 0 \cong G$.

(b) For any group G and a subgroup H we have $G *_H 0 \cong G/\langle\langle H \rangle\rangle$. This shows that in general the maps $A \to A *_C B$ and $B \to A *_C B$ are not injections. However, if both $\alpha\colon C \to A$ and $\beta\colon C \to B$ are injections then $A \to A *_C B$ and $B \to A *_C B$ are injections - this can be shown by constructing a graph similar to the graph used in the proof of Lemma 3.2.2.

Shortly we will express fundamental groups of surfaces as certain amalgamated free products. Another interesting example, which we however don't have time to analyse, is the group $SL(2, \mathbb{Z})$ which is isomorphic to the amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$.

3.2.4 Presentations

When dealing with amalgams (i.e. amalgamated free products), and in many other situations, it is convenient to use the following notation. A *presentation* is an expression of the form

$$\langle g_1, g_2, \ldots \mid r_1, r_2, \ldots \rangle,$$

where r_i are words in the letters $g_1, g_1^{-1}, g_2, g_2^{-1}, \ldots$. We say that a presentation is *finite* if there are only finitely many g_i's and finitely many r_i's. Given a presentation, we associate to it the group defined as the quotient of the free group on $g_1, g_2, \ldots$ by the normal subgroup generated by the elements $r_1, r_2, \ldots$.

Remark 3.2.5. (a) Let G be the group given by a presentation $\langle g_1, g_2, \ldots \mid r_1, r_2, \ldots \rangle$. Suppose that H is another group generated by elements $h_1, h_2, \ldots$, and suppose the words $s_1, s_2, \ldots$,which arise from $r_1, r_2, \ldots$ by replacing g_i's with h_i's, are all equal to e_H in H. Then there exists a unique surjective homomorphism $G \to H$ which sends g_i to h_i for all i.

(b) Similarly suppose that $G = A *_C B$, where $\alpha\colon C \to A$, $\beta\colon C \to B$ and suppose that $\varphi\colon A \to H$, $\psi\colon B \to H$ are group homomorphisms such that $\varphi \circ \alpha = \psi \circ \beta$. Then there exists a unique homomorphism $\zeta\colon G \to H$ such that $\zeta \circ \alpha$ and $\zeta \circ \beta$ extend $\varphi \circ \alpha$ and $\psi \circ \beta$ respectively.

Example 3.2.6. $\langle a, b \mid [a, b] \rangle$, where $[a, b] := aba^{-1}b^{-1}$, is a presentation of $\mathbb{Z}^2$. We have a homomorphism

$$\varphi\colon F_2 \to \mathbb{Z}^2$$

which sends a and b to the standard generators of $\mathbb{Z}^2$. We have $\langle\langle [a, b] \rangle\rangle \subset \ker \varphi$, and we only need to check that $\ker(\varphi) = \langle\langle [a, b] \rangle\rangle$. Let us assume that for some

$w \in F_2$ we have $\varphi(w) = (0,0)$. We can move all the b's to the front in w, at the cost of introducing some commutators. For example

$$abab = abba[a^{-1}, b^{-1}] = bba[a^{-1}, b^{-1}]^2 a[a^{-1}, b^{-1}].$$

Now we can move all the a's to the front using conjugation relation $wa = aw^a$, where $w^a := a^{-1}wa$. Thus for example the above word is equal to

$$bbaa[a^{-1}, b^{-1}]^a [a^{-1}, b^{-1}]^a [a^{-1}, b^{-1}].$$

After these operations we get a word with some $b^k a^l$ at the front followed by a product of conjugates of $[a^{\pm 1}, b^{\pm 1}]^{\pm 1}$. Let us argue that all elements of the form $[a^{\pm 1}, b^{\pm 1}]^{\pm 1}$ are in $\langle\langle [a,b] \rangle\rangle$. Indeed, in the quotient group $F_2 / \langle\langle [a,b] \rangle\rangle$ the images of a and b commute, and so also images of a^s and b^t commute for any $s, t \in \mathbb{Z}$. Thus $[a^k, b^l]^{\pm 1} \in \langle\langle [a,b] \rangle\rangle$. This shows that if $w \in \ker(\varphi)$ then it is in $\langle\langle [a,b] \rangle\rangle$, which shows that φ is indeed injective.

3.3 Van Kampen's theorem

We are now ready to state and prove van Kampen's theorem.

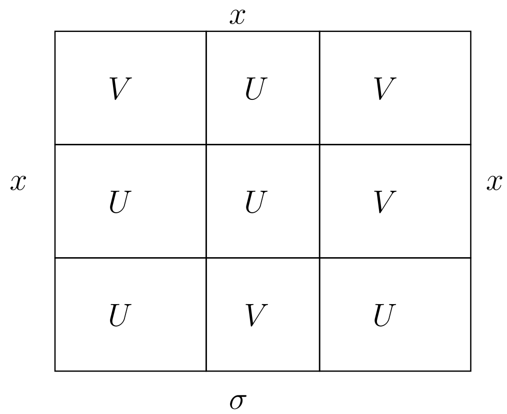
Theorem 3.3.1. *Suppose that $X = U \cup V$, where both U and V are path-connected open sets, and $U \cap V$ is a non-empty path-connected set. Then $\pi_1(X)$ is isomorphic to*

$$\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

Proof. Let us fix a base point $x \in U \cap V$. We have the map $\varphi \colon \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \to \pi_1(X)$ given by defining $\varphi([\sigma_1][\sigma_2]\dots[\sigma_k])$ to be the class of the loop $\sigma_1\sigma_2\dots\sigma_k$. The fact that φ is well-defined follows directly from Remark 3.2.5. The fact that φ is surjective is shown just like in the proof of Proposition 3.1.1.

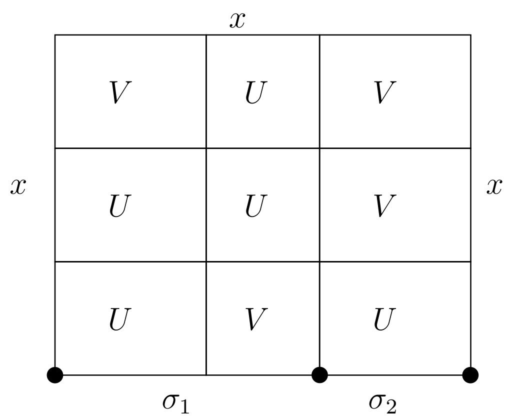
The most interesting part of this proof is to show that φ is injective. Let us assume that $[\sigma_1][\sigma_2]\dots[\sigma_k] \in \ker\varphi$, where each σ_i is either a loop entirely in U or entirely in V. Our aim is to show that $[\sigma_1][\sigma_2]\dots[\sigma_k]$ is the trivial element of the group $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$. By assumption we know that $\sigma := \sigma_1\sigma_2\dots\sigma_k$ is homotopic to the constant loop at x. Thus let $F : I \times I \to X$ be a homotopy between σ and const_x.

We start by finding a division of $I \times I$ into a rectangular grid such that each rectangle is mapped entirely to U or entirely to V, e.g.



This can be done as in the proof Proposition 3.1.1: we consider the covering of $I \times I$ by the two open sets $F^{-1}(U)$ and $F^{-1}(V)$, then we refine this partition so that it consists entirely of open squares whose edges are parallel to the edges of $I \times I$, and we use compactness to choose finitely many open squares. It is a simple exercise to show that we can pass to slightly smaller *closed* squares whose union is still $I \times I$. Now to obtain the desired rectangular grid we take all vertical and horizontal lines which contain an edge belonging to one of these finitely many closed squares.

After subdividing further, we may assume that the end-points corresponding to each σ_i are vertices of the grid, e.g.

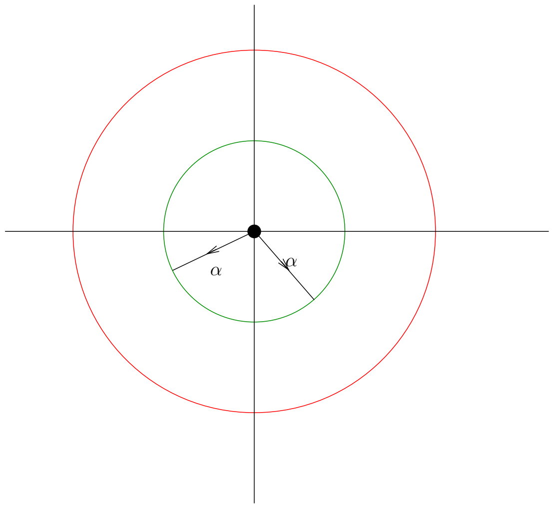


In the second step, we modify F so that each vertex is mapped to x, making sure that after the modification the top and side edges are as before. Furthermore we want to assure that the bottom edge of $I \times I$ is a concatanation of some loops $\sigma_1'\sigma_2' \dots \sigma_l'$ each of which is in either in U or V and such that we have the equality $[\sigma_1][\sigma_2] \dots [\sigma_k] = [\sigma_1'][\sigma_2'] \dots [\sigma_l']$ in $\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$.

We first focus on the internal vertices, and then discuss the vertices at the bottom edge. Suppose that some internal vertex $v \in I \times I$ is mapped to y. Let us fix a path $\alpha \colon x \overset{p}{\rightsquigarrow} y$, and we may assume that if $y \in U \cap V$ then α is a path in $U \cap V$. Let us take $R > 0$ such that the closed ball $\bar{B}(v, R)$ does not contain any vertices other than v. Let us fix a homeomorphism $h \colon \bar{B}(v, R) \setminus \bar{B}(v, R/2) \to \bar{B}(v, R) \setminus \{v\}$ which is identity on the boundary of $\bar{B}(v,R)$. Now we define a homotopy F' as follows.

$$\begin{aligned} F'(z) &:= F(z) \quad \text{when } z \notin \bar{B}(v,R) \\ &:= F \circ h(z) \quad \text{when } z \in \bar{B}(v,R) \setminus \bar{B}(v,R/2) \\ &:= \alpha\left(\frac{2\|z-v\|}{R}\right) \quad \text{when } z \in \bar{B}(v,R/2) \end{aligned}$$

The described operation is schematically depicted on the following figure:

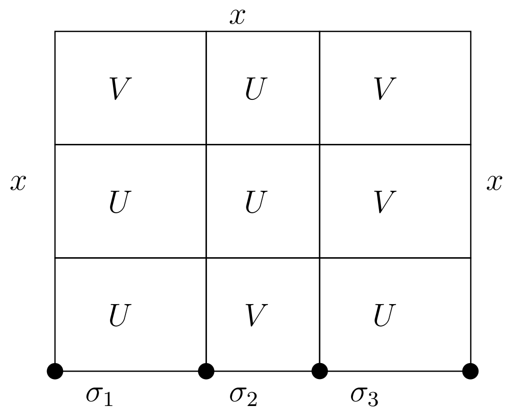


The vertex in the middle is the vertex v. Outside of the red circle we do not modify F at all. Between the red and the green circle we have "compressed F", and this compression gives us the freedom inside the green circle to radially use the path α, so that the vertex v gets mapped to x.

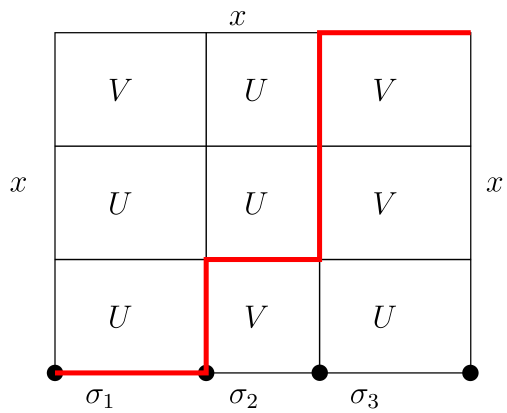
Of course this operation does not change anything on the boundary of $I \times I$, as long as we change only internal vertices.

Now we apply the same operation to the vertices at the bottom edge (the formulas for F' are the same). Let us consider how it changes the σ_i's: we replace each σ_i by a loop of the form $\tau_1\alpha_1^{-1}\alpha_1\tau_2\alpha_2{-}1\alpha_2 \ldots \tau_m$. By construction, if σ_i is a loop in, say, U, then also $\tau_i\alpha_i^{-1}$ and $\alpha_i\tau_{i+1}$ are loops which lie entirely in U. Thus in $\pi_1(U)$ the element $[\sigma_i]$ is equal to the product $[\tau_1\alpha_1^{-1}][\alpha_1\tau_2]\ldots[\alpha_{m-1}\tau_m]$. As such we deduce the following: if we denote with $\sigma_1', \sigma_2', \ldots, \sigma_l'$ the loops given by the restriction of F' to the intervals between consecutive vertices of the bottom edge, then the elements $[\sigma_1][\sigma_2]\ldots[\sigma_k]$ and $[\sigma_1'][\sigma_2']\ldots[\sigma_l']$ are equal in $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$. (The argument shows that they are in fact equal in $\pi_1(U) * \pi_1(V)$.)

Thus without loss of generality we may assume that every edge of the grid represents a loop at x, and that σ_i's go between consecutive vertices of the bottom edge, like so:



The final step of the proof consists of proving the following statement: consider any path from the bottom left corner to the top right corner which goes along the edges of the grid, always either up or to the right, e.g. the red path here:

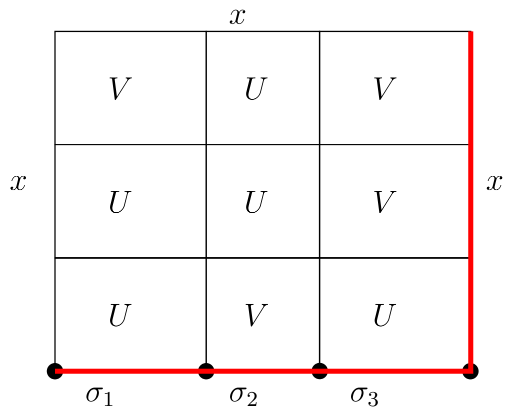


Let the loops at the edges of this path be $\tau_1, \tau_2, \ldots, \tau_m$. We would like to show that

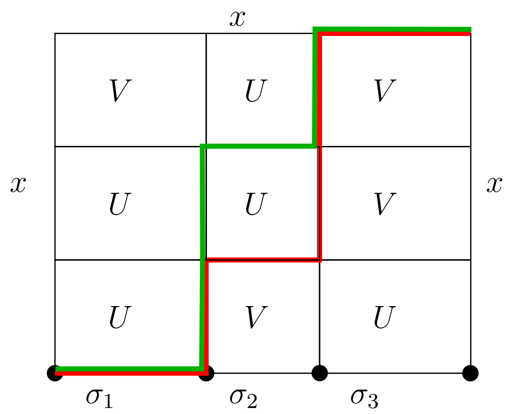
$$[\sigma_1][\sigma_2]\ldots[\sigma_k] = [\tau_1][\tau_2]\ldots[\tau_m].$$

in $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$. This is enough to finish the proof since following the bottom and right edge gives the element $[\sigma_1][\sigma_2]\ldots[\sigma_k][x]\ldots[x] = [\sigma_1][\sigma_2]\ldots[\sigma_k]$ (equality in $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$) and following the left edge and the top edge gives the element $[x]\ldots[x] = [x]$.

We prove the statement by induction on the number of rectangles under the path. The case 0 is the path



The corresponding word is $[\sigma_1][\sigma_2]\dots[\sigma_k][x][x]\dots[x]$, so the statement is true. For the inductive statement we only need to "pass one square", for example we assume that we know the statement for the red path, and we need to prove it for the green path:



Let us assume that the word read from the red path is $[\tau_1]\dots[\tau_l]$. We note that if the square we are passing is for example in U, then in the inductive step we replace $[\tau_i][\tau_{i+1}]$ with $[\tau'_i][\tau'_{i+1}]$, such that $[\tau_i][\tau_{i+1}] = [\tau'_i][\tau'_{i+1}]$ in $\pi_1(U)$, by Exercise 2.0.7. This shows that

$$[\tau_1]\dots[\tau_i][\tau_{i+1}]\dots[\tau_l] = [\tau_1]\dots[\tau'_i][\tau'_{i+1}]\dots[\tau_l]$$

in $\pi_1(U)*_{\pi_1(U\cap V)}\pi_1(V)$, which finishes the proof. (We remark that the need to amalgamate over $\pi_1(U\cap V)$ stems from the following fact: given an edge of the grid which lies in both U and V in one step of the induction it can be considered as an element of $\pi_1(U)$ and in another as an element of $\pi_1(V)$.) □

Exercise 3.3.2. *Modify the proof of van Kampen's theorem to show the following more general version: Suppose that $X = \bigcup_{i\in I} U_i$ is path-connected, and U_i are path-connected open sets, such that intersection of any two of them and any three of them is path-connected. Then $\pi_1(X)$ is the quotient of the free product $\ast_{i\in I}\pi_1(U_i)$ by the normal subgroup generated by all the words of the form $s_*(c)t_*(c^{-1})$, where for some $k,l \in I$ we have $c \in \pi_1(U_k\cap U_l)$, and $s\colon U_k\cap U_l \to U_k, t\colon U_k\cap U_l \to U_l$ are the inclusions.*

(Hint: the main modification which needs to be done concerns the choice of the paths α *connecting vertices to the base point. The assumption of the theorem is that triple intersections of open sets are path-connected, but in the proof as presented above a single vertex might typically lie in 4 distinct open sets. As such the rectangles in the proof should be defined in such a way that every iinternal vertex lies in only 3 rectangles instead of 4.)*

Lecture 5: Examples for van Kampen's theorem

4.1 Abelianisation

In order to distinguish spaces using fundamental groups it is convenient to introduce one more group-theoretic construction. If G is a group then we let $[G, G]$ be the *commutator subgroup*, i.e. the subgroup generated by all the elements of the form $[g, h]$, $g, h \in G$.

Exercise 4.1.1. *Show that $[G, G]$ is a normal subgroup.*

We let $G^{ab} := G/[G, G]$ be the *abelianisation* of G. It is the "largest" abelian group onto which G surjects, which the following exercise makes precise.

Exercise 4.1.2. *Show that if A is an abelian group and $\varphi\colon G \to A$ is a homomorphism then there exists a unique $\psi\colon G^{ab} \to A$ such that φ is equal to the composition $G \to G^{ab} \xrightarrow{\psi} A$, where $G \to G^{ab}$ is the standard quotient map.*

Example 4.1.3. It follows from Example 3.2.6 that $F_2^{ab} \cong \mathbb{Z}^2$. More generally, we have $(\ast_{i\in I}\,\mathbb{Z})^{ab} \cong \oplus_{i\in I}\mathbb{Z}$

Exercise 4.1.4. *Let G be a group generated by some set S of elements of G. Show that $[G, G]$ is equal to the normal group generated by all the elements of the form $[s, t]$ where $s, t \in S$. Deduce that if $\langle g_1, g_2, \ldots | r_1, r_2, \ldots\rangle$ is a presentation of a group G then G^{ab} has a presentation $\langle g_1, g_2, \ldots | r_1, r_2, \ldots, [g_i, g_j],\ i, j = 1, 2, \ldots\rangle$*

Exercise 4.1.5. *Suppose that a group G has a presentation $\langle g_1, \ldots g_k | r_1, r_2, \ldots\rangle$, such that for all i we have $r_i \in [G, G]$. Then G^{ab} is isomorphic to $\mathbb{Z}^k$ and the natural map $G \to G^{ab}$ maps g_i's to the generators.*

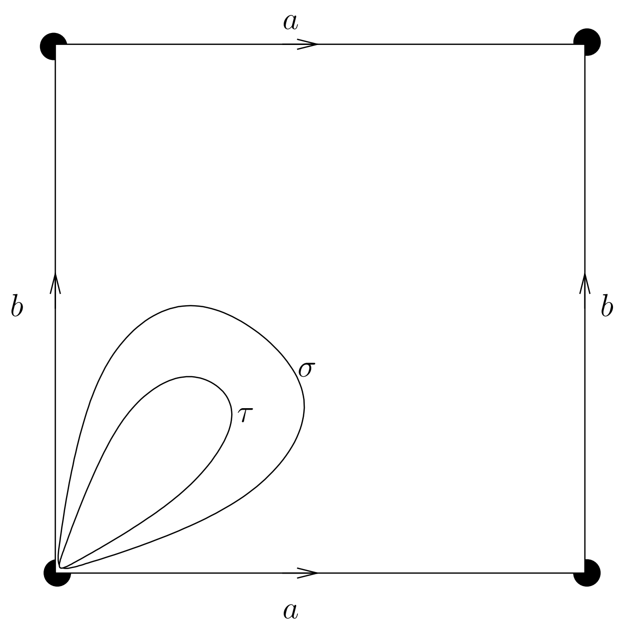
4.2 Ad-hoc examples

The fundamental groups of most of the spaces which we will encouter in this course can be computed using van Kampen's theorem. Let's start with some explicit examples.

Example 4.2.1. (a) Suppose X and Y are topological spaces with base points x, y such that some neighbourhood of x and some neighbourhood of y are contractible. Then $\pi_1(X \vee Y, (x, y)) \cong \pi(X, x) * \pi_1(Y, y)$. Indeed, we can write $X \vee Y = U \cup V$ where U is the union of X together with a contractible neighbourhood of $y \in Y$, and similarly V is the union of Y and a contractible neighbourhood of $x \in X$. It is left as an exercise to check that U and V are homotopy equivalent to X and Y respectively, and $U \cap V$ is contractible. With this in mind the claim follows.

(b) A particular case of the previous example is $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) \cong \mathbb{Z} * \mathbb{Z}$. Similar argument shows that $\pi_1(\bigvee_{i \in I} \mathbb{S}^1) \cong \ast_{i \in I} \mathbb{Z}$.

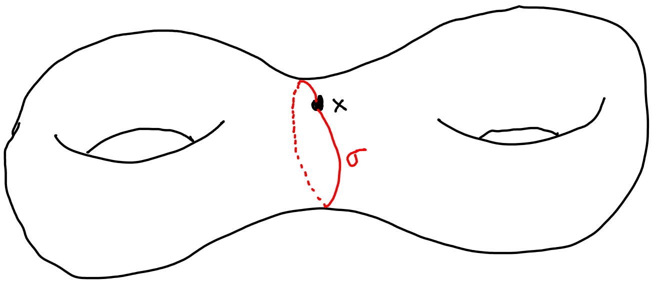
Example 4.2.2. We can compute $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ again, using van Kampen's theorem this time. Consider the contractible loops σ and τ at the basepoint as in the following figure:



We let U be everything inside the loop σ, and V be everything outside the loop τ. Then U is contractible, V deformation retracts onto $\mathbb{S}^1 \vee \mathbb{S}^1$, and $U \cap V$ deformation retracts onto the loop σ. Thus in order to use van Kampen's theorem we need to understand the map $\varphi \colon \mathbb{Z} \cong \pi_1(U \cap V) \to \pi_1(V)$ induced by the inclusion $U \cap V \subset V$. Clearly the loop σ is homotopic to the loop $aba^{-1}b^{-1}$, and so φ maps $[\sigma]$ to $[aba^{-1}b^{-1}] \in \pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$. Thus van Kampen's theorem shows that $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$

is isomorphic to $\langle a, b \mid [a, b]\rangle$. We have seen in Example 3.2.6 that this group is isomorphic to $\mathbb{Z}^2$.

Example 4.2.3. In a similar fashion we can compute $\pi_1(M_2)$, where M_2 is the surface of genus 2:



We can cut it into two parts along the loop σ to obtain the pieces U' and V', each of which can be deformation retracted into $\mathbb{S}^1 * \mathbb{S}^1$. We enlarge U' and V' slightly to get open sets U and V in M_2, such that the intersection $U \cap V$ deformation retracts onto σ. If we denote the generators of $\pi_1(U)$ by a, b and the generators of $\pi_1(V)$ by c, d then we see that $[\sigma] \in \pi_1(U \cap V)$ is mapped to $[a, b]$ and to $[c, d]$ under the respective inclusions. It follows that the presentation of $\pi_1(M_2)$ is $\langle a, b, c, d \mid [a, b] = [c, d]\rangle$, which is isomorphic to

$$\langle a, b, c, d \mid [a, b][c, d]\rangle.$$

By Exercise 4.1.5, we see that $\pi_1(M_2)^{ab} \cong \mathbb{Z}^4$, which shows that M_2 is not homotopy equivalent to $M_1 = \mathbb{S}^1 \times \mathbb{S}^1$ (in particular M_2 is not homeomorphic to M_1). Furthermore, we can show that the loop σ is not a retract of M_2. Indeed, we have $[\sigma] = [a, b]$ in $\pi_1(M_2)$, so the composition $\pi_1(\sigma) \to \pi_1(M_2) \to \pi_1(\sigma)$ cannot be equal to identity, since the second map sends all commutators to 0.

4.3 Cell complexes

Definition 4.3.1. A *finite-dimensional cell complex* C is given by a sequence of sets $C_0, C_1, C_2, \ldots, C_d$, and for every $c \in C_i$, $i = 1, \ldots, n$, we have a "gluing map" $\gamma_c \colon \mathbb{S}^{i-1} = \partial D^i \to X_{i-1}$, where X_i, the i-dimensional skeleton, is defined as follows. We let $X_0 := C_0$ with the discrete topology, and for $i > 0$ we let

$$X_i = \left(X_{i-1} \sqcup \bigsqcup_{c \in C_i} D^i_c \right) / \sim,$$

where the relation $\sim$ is the smallest equivalence relation which identifies $x \in \partial D^i_c$ with $\gamma_c(x) \in X_{i-1}$.

The number d is called the *dimension* of the cell complex. The elements of C_0 are called vertices, the elements of C_1 are called edges, and in general the elements of C_i are called i-dimensional faces. The same names can be applied to the images of the corresponding disks D^i in X_d.

We frequently informally consider C to be the space X_d, forgetting about the cell structure. Given a topological space X, we may ask whether it *admits a cell structure*, which means finding a cell complex C homeomorphic with X.

Example 4.3.2. The standard cell structure of $\mathbb{S}^1 \times \mathbb{S}^1$ is as follows: we let $C_0 = \{p\}$, $C_1\{a, b\}$, $C_2 = \{f\}$, The gluing maps $\gamma_a, \gamma_b \colon \mathbb{S}^0 \to X_0$ are the only ones possible, i.e. they send both points of $\mathbb{S}^0$ to p. Thus X_1 is the wedge of two circles which we label with a and b. The gluing map $\gamma_f \colon \mathbb{S}^1 \to X_1$ is given by the loop $aba^{-1}b^{-1}$.

Usually we only care about specifying the gluing maps up to homotopy, because of the following lemma.

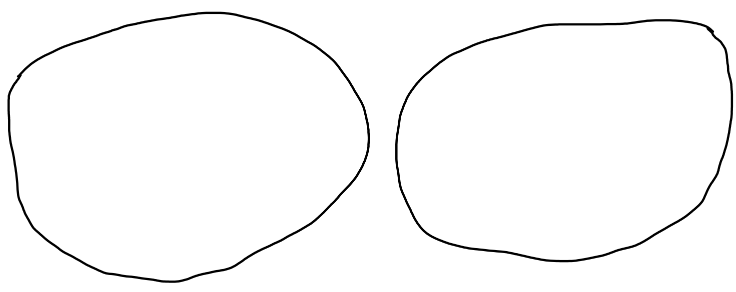
Lemma 4.3.3. *Suppose X is a cell complex and suppose $\delta, \gamma \colon \mathbb{S}^{n-1} \to X$ are homotopic. Let $\sim_\gamma$ and $\sim_\delta$ be the equivalence relations on $X \sqcup D^n$ which identify $x \in \mathbb{S}^{n-1} = \partial D^n$ with $\delta(x) \in X$ and $\gamma(x) \in X$, respectively. Then the spaces $(X \sqcup D^n)/{\sim_\delta}$ and $(X \sqcup D^n)/{\sim_\gamma}$ are homotopy equivalent.*

Proof. $\square$

4.4 Links

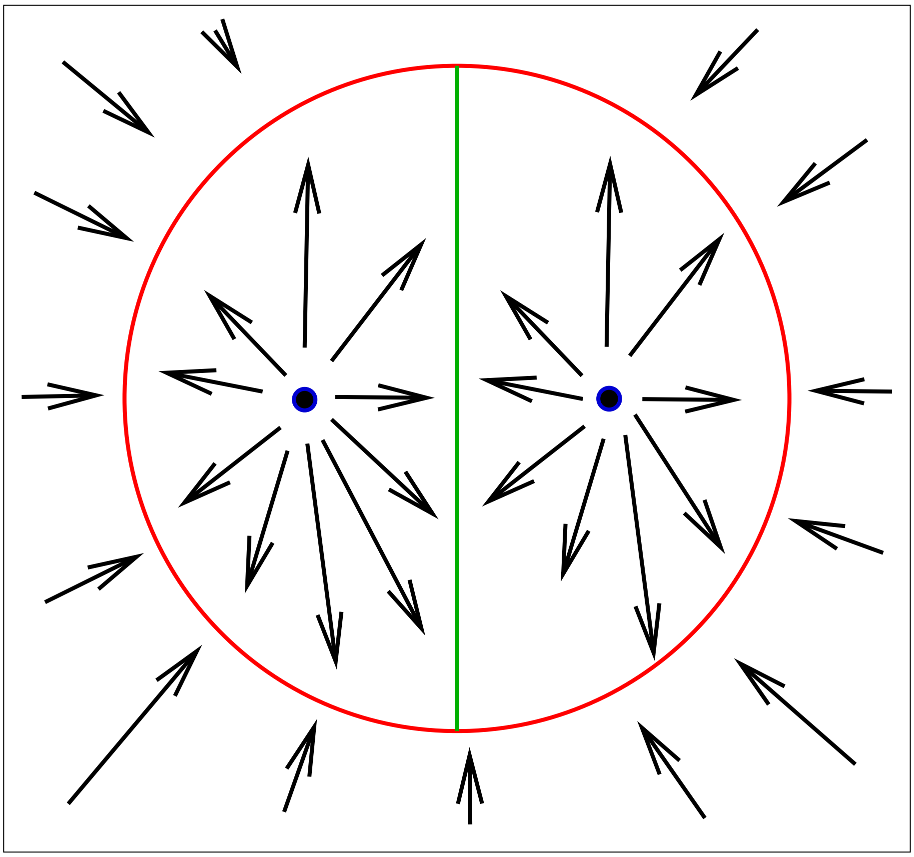
A *link* is an embedding of disjoint circles in the 3-dimensional Euclidean space. Two links L_1 and L_2 are *equivalent* if there exists a homeomorphism $h \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(L_1) = h(L_2)$. As such we see that if L_1 and L_2 are equivalent then $\pi_1(\mathbb{R}^3 \setminus L_1) \cong \pi_1(\mathbb{R}^3 \setminus L_2)$. Thus the fundamental group of the complement of a link has a potential to be used to distinguish non-equivalent links.

Example 4.4.1. The *unlink* L consists of two unknots which can be separated by a plane:



In this example we would like to compute $\pi_1(L^c)$.

(a) Let U be an unknot (i.e. $\mathbb{S}^1$ embedded in $\mathbb{R}^3$ in the unknotted fashion). We start by showing that $\mathbb{R}^3 \setminus U$ is homotopy equivalent to $\mathbb{S}^2 \vee \mathbb{S}^1$. First we show that $\mathbb{R}^3 \setminus U$ deformation retracts onto $\mathbb{S}^2 \cup D$, where D is a fixed diameter of $\mathbb{S}^2$ which goes through U. We describe the deformation retraction by describing the vector field on $\mathbb{R}^3 \setminus U$ along which the retraction should flow. Consider any plane P which contains D. The vector field on P should be as follows:



In this figure, $\mathbb{S}^2 \cap P$ is in red, $D = D \cap P$ is in green, and the two blue points are $U \cap P$.

Thus it remains to show that $\mathbb{S}^2 \cup D$ is homotopy equivalent to $\mathbb{S}^2 \vee \mathbb{S}^1$. To see this consider any geodesci path between the ends of the circle.

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