Exercise list 4

Famous examples (aside from Damian's)

1. Prove the following theorem by Rips:

   Let $\lambda > 0$, and let G be a group with presentation $\langle a_1, \dots, a_n \mid \{R_i\}_{i \in \mathcal{I}} \rangle$. Then there exists a short exact sequence of groups
   $$1 \to K \to H \xrightarrow{\varphi} G \to 1$$
   such that

   (a) H is a finitely presented group which has a presentation satisfying condition $C'(\lambda)$.

   (b) K is finitely generated.

   Hint: Consider H generated by $a_1, \dots, a_n$ and two extra generators b_1, b_2, and let φ be the projection map. For each R_i add one relation to H that ensures that φ is an epimorphism, and add other relations so that $\langle b_1, b_2 \rangle$ is a normal subgroup. All of this while making H $C'(\frac{1}{6})$.

2. Apply Rips construction to find an example of a hyperbolic group without solvable subgroup membership problem.

   Hint: use the fact that there exist groups without solvable word problem

3. Apply Rips construction to find an incoherent hyperbolic group (an incoherent group is a group that has a finitely generated subgroup that is not finitely presented).

   Hint: $F_2 \times F_2$ is incoherent.

4. The following example is due to Pride. Let
   $$G = \langle x, y \mid xU_1, yV_1, xU_2, yV_2 \dots \rangle.$$
   Choose U_i, V_i such that G is $C'(\frac{1}{6})$ and not residually finite (actually, show that there are no proper normal subgroups of finite index).

   Hint: show that a normal subgroup of finite index has to contain the normal closure of a^n and b^n for some $n \in \mathbb{N}$.