Exercise list 2

General small cancellation

Definition 1. Let p be a natural number. We say that a 2-complex X satisfies the $C(p)$ *small cancellation condition* provided that for each 2-cell $R \to X$ its attaching map $\partial R \to X$ is not a concatenation of fewer than p pieces in X.

1. Prove that if a 2-complex X satisfies $C'(\frac{1}{p})$, then it satisfies $C(p+1)$. Is the converse true?
2. Let X be a 2-complex satisfying $C'(\frac{1}{p})$. Prove that every reduced disc diagram over X satisfies $C'(\frac{1}{p})$.
3. Show the converse of the previous exercise. That is, if every reduced disc diagram over a 2-complex X satisfies $C'(\frac{1}{p})$, then X satisfies $C'(\frac{1}{p})$.
4. Show that the presentation complex of a surface group given by the presentation

$$\langle a_1, b_1, a_2, b_2, \dots a_g, b_g | [a_1, b_1] \cdots [a_g, b_g] \rangle$$

   satisfies condition $C'(\frac{1}{4g-1})$.
5. Prove that if a 2-complex X has a piece of length 2, then there exist a reduced disc diagram over X with internal vertex of degree 4.
6. Let $\tilde{X} \to X$ be a covering map. Prove that X satisfies $C'(\frac{1}{p})$ if and only if $\tilde{X}$ does.
7. Let X be a $C'(\frac{1}{6})$ and let $D \to X$ be a reduced disc diagram. Let $\gamma \subseteq \partial D$ be a path mapped to a geodesic in X. Show that there are no 1-shells, 2-shells or 3-shells along γ. What changes if X is $C'(\frac{1}{5})$? And what about $C(7)$?

Greendlinger Lemma:

8. Let X be a $C(6)$ 2-complex, and let $\varphi : D \to X$ be a reduced disc diagram. Assume that D' is a non-singular component of D consisting of more than a single 2-cell. Let

$$K = \sum_{\text{1-shells of } D'} 1 + \sum_{\text{2-shells of } D'} \frac{2}{3} + \sum_{\text{3-shells of } D'} \frac{1}{3}.$$

   Show that $K \geq 2$.

   Hint: compare to baby case.
9. Let X be a $C(6)$ 2-complex. Let R_1 and R_2 be distinct 2-cells of X. Show that $\partial R_1 \cap \partial R_2$ is connected.
10. A finite presentation $\langle S \mid R \rangle$ is *Dehn* if every word $w \in F(S)$ representing the trivial element in the presented group has a cyclic subword u such that
    (a) u is a cyclic subword of some relator $r \in R$;
    (b) $|u| > \frac{1}{2}|r|$.

    Show that $C'(\frac{1}{6})$ presentations are Dehn.
11. (Linear isoperimetric inequality) Let X be a $C'(\frac{1}{6})$ complex. Show that there exists a constant M such that for any disc diagram $D \to X$ it holds that:

$$\#\{\text{2-cells of } D\} \leq M \#\{\text{1-cells in } \partial D\}.$$

    Comment: this implies that X (with an appropriate metric) is hyperbolic.