Exercise list 2

**General small cancellation**

**Definition 1.** Let $p$ be a natural number. We say that a 2-complex $X$ satisfies the $C(p)$ *small cancellation condition* provided that for each 2-cell $R \to X$ its attaching map $\partial R \to X$ is not a concatenation of fewer than $p$ pieces in $X$.

1. Prove that if a 2-complex $X$ satisfies $C'(\frac{1}{p})$, then it satisfies $C(p+1)$. Is the converse true?

2. Let $X$ be a 2-complex satisfying $C'(\frac{1}{2})$. Prove that every reduced disc diagram over $X$ satisfies $C'(\frac{1}{2})$.

3. Show the converse of the previous exercise. That is, if every reduced disc diagram over a 2-complex $X$ satisfies $C'(\frac{1}{2})$, then $X$ satisfies $C'(\frac{1}{2})$.

4. Show that the presentation complex of a surface group given by the presentation
   
   $\langle a_1, b_1, a_2, b_2, \ldots a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$

   satisfies condition $C'(\frac{1}{4g-1})$.

5. Prove that if a 2-complex $X$ has a piece of length 2, then there exist a reduced disc diagram over $X$ with internal vertex of degree 4.

6. Let $\tilde{X} \to X$ be a covering map. Prove that $X$ satisfies $C'(\frac{1}{p})$ if and only if $\tilde{X}$ does.

7. Let $X$ be a $C'(\frac{1}{6})$ and let $D \to X$ be a reduced disc diagram. Let $\gamma \subseteq \partial D$ be a path mapped to a geodesic in $X$. Show that there are no 1-shells, 2-shells or 3-shells along $\gamma$. What changes if $X$ is $C'(\frac{1}{5})$? And what about $C'(7)$?

**Greendlinger Lemma:**

8. Let $X$ be a $C(6)$ 2-complex, and let $\varphi : D \to X$ be a reduced disc diagram. Assume that $D'$ is a non-singular component of $D$ consisting of more than a single 2-cell. Let

   $$K = \sum_{1 \text{-shells of } D'} 1 + \sum_{2 \text{-shells of } D'} \frac{2}{3} + \sum_{3 \text{-shells of } D'} \frac{1}{3}.$$  

   Show that $K \geq 2$.

   Hint: compare to baby case.

9. Let $X$ be a $C(6)$ 2-complex. Let $R_1$ and $R_2$ be distinct 2-cells of $X$. Show that $\partial R_1 \cap \partial R_2$ is connected.

10. A finite presentation $\langle S \mid R \rangle$ is Dehn if every word $w \in F(S)$ representing the trivial element in the presented group has a cyclic subword $u$ such that

    (a) $u$ is a cyclic subword of some relator $r \in R$;

    (b) $|u| > \frac{1}{2}|r|$.

    Show that $C'(\frac{1}{5})$ presentations are Dehn.

11. (Linear isoperimetric inequality) Let $X$ be a $C'(\frac{1}{6})$ complex. Show that there exists a constant $M$ such that for any disc diagram $D \to X$ it holds that:

    $$\#\{2 \text{-cells of } D\} \leq M \#\{1 \text{-cells in } \partial D\}.$$  

   Comment: this implies that $X$ (with an appropriate metric) is hyperbolic.