Exercise list 1

Coarse geometry and hyperbolicity

1. Compute the number of geodesics connecting two points in the Cayley graph of $\mathbb{Z}^n$, with the standard generating set.

2. Draw a part of the Cayley graph of the Baumslag-Solitar group $BS(1,n)$ corresponding to the following presentation: $\langle a, b \mid ab^n a^{-1} b^{-1} \rangle$.

   Definition 1. Let X, Y be metric spaces. A map $f : X \to Y$ is called a *quasi isometric embedding* if there exist constants $C, D > 0$ such that

   $$C^{-1} d_X(x,y) - D \leq d_Y(f(x), f(y)) \leq C d_X(x,y) + D$$

   Definition 2. Let X, Y be metric spaces. A map $f : X \to Y$ is a *coarse embedding* if there exist non decreasing functions $\psi_-, \psi_+ : [0,\infty) \to [0,\infty)$ such that

   $$\lim_{t \to \infty} \psi_-(t) = \infty$$

   and the inequality

   $$\psi_-(d_X(x,y)) \leq d_Y(f(x), f(y)) \leq \psi_+(d_X(x,y))$$

3. Show that a composition of quasi-isometric embeddings is again a quasi-isometric embedding. Similarly, prove that a composition of coarse embeddings is again a coarse embedding.

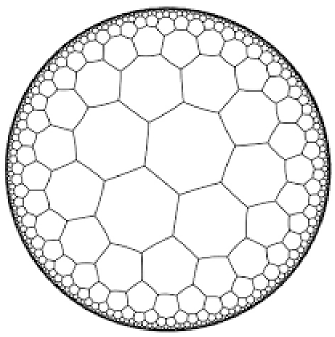
4. Are the metric spaces $\mathbb{N}$ and $\mathbb{Z}$ (with the standard metrics) quasi-isometric?

5. Let T_n denote the infinite tree in which every vertex has degree $n = 2, 3, \ldots$. Let $n \neq m \geq 2$. Are T_n and T_m quasi-isometric?

   Definition 3. Let $\delta > 0$. A geodesic metric space X is δ-*hyperbolic* if any geodesic triangle in X is δ-thin, i.e. the δ-neighbourhood of any two sides of the triangle contains the third side.

6. Prove that trees are 0-hyperbolic, and that 0-hyperbolic graphs are trees.

Baby case



Consider the graph Γ from the picture above, induced by a tiling of the hyperbolic plane by heptagons where every vertex has degree three.

6. Consider the CW-complex P obtained by gluing a 2-cell to each heptagon of Γ. Let D be a subcomplex of P homeomorphic to a disc. Show that D has at least one 2-cell with four or more edges in ∂D.

   Suggestion: play with the Euler characteristic formula.

7. Let D be a subcomplex of P whose boundary is a geodesic triangle of Γ. Refine the statement of the previous exercise and show that D has no internal 2-cells (meaning 2-cells whose intersection with ∂D is empty).

8. Prove that Γ is hyperbolic.