Exercise list 1

Coarse geometry and hyperbolicity

1. Compute the number of geodesics connecting two points in the Cayley graph of $\mathbb{Z}^n$, with the standard generating set.

2. Draw a part of the Cayley graph of the Baumslag-Solitar group $BS(1,n)$ corresponding to the following presentation: $(a,b \mid ab^n a^{-1} b^{-1})$.

   **Definition 1.** Let $X, Y$ be metric spaces. A map $f : X \to Y$ is called a quasi isometric embedding if there exist constants $C, D > 0$ such that
   \[ C^{-1}d_X(x, y) - D \leq d_Y(f(x), f(y)) \leq Cd_X(x, y) + D \]

3. Show that a composition of quasi-isometric embeddings is again a quasi-isometric embedding. Similarly, prove that a composition of coarse embeddings is again a coarse embedding.

4. Are the metric spaces $\mathbb{N}$ and $\mathbb{Z}$ (with the standard metrics) quasi-isometric?

5. Let $T_n$ denote the infinite tree in which every vertex has degree $n = 2, 3, \ldots$. Let $n \neq m \geq 2$. Are $T_n$ and $T_m$ quasi-isometric?

   **Definition 3.** Let $\delta > 0$. A geodesic metric space $X$ is $\delta$-hyperbolic if any geodesic triangle in $X$ is $\delta$-thin, i.e. the $\delta$-neighbourhood of any two sides of the triangle contains the third side.

6. Prove that trees are $0$-hyperbolic, and that $0$-hyperbolic graphs are trees.

   **Baby case**

   Consider the graph $\Gamma$ from the picture above, induced by a tiling of the hyperbolic plane by heptagons where every vertex has degree three.

   6. Consider the CW-complex $P$ obtained by gluing a 2-cell to each heptagon of $\Gamma$. Let $D$ be a subcomplex of $P$ homeomorphic to a disc. Show that $D$ has at least one 2-cell with four or more edges in $\partial D$.

      Suggestion: play with the Euler characteristic formula.

6. Let $D$ be a subcomplex of $P$ whose boundary is a geodesic triangle of $\Gamma$. Refine the statement of the previous exercise and show that $D$ has no internal 2-cells (meaning 2-cells whose intersection with $\partial D$ is empty).

8. Prove that $\Gamma$ is hyperbolic.